Symplectic Plücker Transformations

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Dedicated to Hans Vogler on the occasion of his 60th birthday

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Abstract: Plücker transformations of symplectic spaces with
dimensions other than three are induced by orthogonality-preserving
collineations. For three-dimensional symplectic spaces all Plücker
transformations can be obtained - up to orthogonality-preserving
collineations - by replacing some but not necessarily all non-
isotropic lines by their absolute polar lines.

1. Introduction

In this paper we discuss bijections of the set \( L \) of lines of a symplectic
space, i.e. a (not necessarily finite-dimensional) projective space with
orthogonality based upon an absolute symplectic\(^1\) quasipolarity. Following [1],
two lines are called related, if they are concurrent and orthogonal, or if
they are identical. A bijection of \( L \) that preserves this relation in both
directions is called a (symplectic\(^2\)) Plücker transformation. We shall show
that any bijection \( L \rightarrow \hat{L} \) taking related lines to related lines is already a
Plücker transformation. Moreover, a complete description of all Plücker trans-
formations (cf. the abstract above) will be given.

2. Symplectic spaces

Let \((P, \mathcal{L})\) be a projective space, \(3 \leq \dim(P, \mathcal{L}) \leq \infty\). Assume that \( \pi \) is a
symplectic quasipolarity [11], [12]. Thus \( \pi \) assigns to each point \( X \) of \( P \) a

\(^1\)Instead of ‘symplectic’ some authors are using the term ‘null’.

\(^2\)We shall omit the word ‘symplectic’, since we do not discuss other types of
Plücker transformations in this paper. Cf., however, [11], [2], [4], [5],
[8,p.80ff], [9], [10] for results on other Plücker transformations.
hyperplane $X^n$ with $X \in X^n$; furthermore $Y \in X^n$ implies $X \in Y^n$ for all $X, Y \in \mathcal{P}$. Cf. also [6] for an axiomatic description of projective spaces endowed with a quasipolarity.

We define a mapping from the lattice of subspaces of $(\mathcal{P},\mathcal{L})$ into itself by setting

$$\mathcal{J} \mapsto \bigcap \{X^n | X \in \mathcal{J} \} \text{ for all subspaces } \mathcal{J} \neq \emptyset \text{ and } \emptyset \mapsto \mathcal{P}. \quad (1)$$

This mapping is again written as $\pi$ and is also called a quasipolarity. If $(\mathcal{P},\mathcal{L})$ is finite-dimensional, then it is well known that $\pi$ is an antiautomorphism of the lattice of subspaces of $(\mathcal{P},\mathcal{L})$. In case of infinite dimension the mapping (1) still has the properties

$$(\mathcal{J}_1 \vee \mathcal{J}_2)^\pi = \mathcal{J}_1^n \cap \mathcal{J}_2^n, \quad (\mathcal{J}_1 \cap \mathcal{J}_2)^\pi \supset \mathcal{J}_1^n \vee \mathcal{J}_2^n, \quad \mathcal{J} \subset \mathcal{J}^{\pi^n}$$

for all subspaces $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J} \in \mathcal{P}$. Note that in the last formula strict inclusions are not necessarily preserved, if $\mathcal{J}_1$ and $\mathcal{J}_2$ both have infinite dimension. Moreover, it is an easy induction to show for all finite-dimensional subspaces $\mathcal{J} \in \mathcal{P}$ that $\mathcal{J}^{\pi^n} = \mathcal{J}$ and that every complement of $\mathcal{J}^n$ has the same finite dimension as $\mathcal{J}$.

$(\mathcal{P},\mathcal{L},\pi)$ is a **symplectic space** with **absolute quasipolarity** $\pi$ [7,p.384ff], [11]. In terms of an underlying vector space $V$ of $(\mathcal{P},\mathcal{L})$ the symplectic quasipolarity $\pi$ can be described by a non-degenerate alternating bilinear form of $V \times V$ into the (necessarily commutative) ground field of $V$. If $(\mathcal{P},\mathcal{L})$ is finite-dimensional, then it is well known that $\dim(\mathcal{P},\mathcal{L})$ is odd.

We are introducing two binary relations on $\mathcal{L}$: Given $a, b \in \mathcal{L}$ then define $a$ and $b$ to be **orthogonal** ($\perp$), if $a \cap b^n \neq \emptyset$. The lines $a$ and $b$ are called **related** ($\sim$), if $a \cap b$ and $a \cap b \neq \emptyset$, or if $a = b$. Given orthogonal lines $a, b$ there exists a point $R \in a \cap b^n$. Therefore

$$R^n \supset (a \cap b^n) \supset a^n \vee b^n = a^n \vee b.$$  

The line $b$ has a point in common with $a^n$, since $R^n$ is a hyperplane and $a^n$ is a co-line. Consequently, $\perp$ and $\sim$ are symmetric relations.

Each line $a \in \mathcal{L}$ either is contained in $a^n$ or is a complement of $a^n$, since $a \cap a^n$ being a single point would imply that $X^n = a \cap a^n$ for all points $X \in a \setminus a^n$, in contradiction to $\pi|\mathcal{P}$ being injective. A line $a \in \mathcal{L}$ is **isotropic** (self-orthogonal) if, and only if, $a$ is **totally isotropic**, i.e., $a \subset a^n$. We shall write $\mathcal{J}$ for the set of all isotropic lines.

If $Q$ is a point, then $\mathcal{L}[Q]$ stands for the star of lines with centre $Q$ and

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3There are, e.g., hyperplanes $H \subset \mathcal{P}$ with $H \neq X^n$ for all $X \in \mathcal{P}$. For all such hyperplanes $H^n = \mathcal{P}^n = \emptyset$, although $H \neq \mathcal{P}$.
\[ J(Q) := L(Q) \cap J \] for the set of all isotropic lines through \( Q \). In the following Lemma 1 we state two simple properties of isotropic lines that are well known in case of finite dimension [3,p.181ff], [7,p.384ff] but hold as well for infinite dimension:

**Lemma 1.** If \( Q \in \mathcal{P} \), then all isotropic lines through \( Q \) are given by

\[ J(Q) = \{ x \in L | Q \in x \subset Q^n \}. \]

Let \( a \in \mathcal{L} \setminus J \) be non-isotropic. The set of isotropic lines intersecting the line \( a \) equals the set of all lines intersecting both \( a \) and \( a^n \).

**Proof.** Let a line \( x \) with \( Q \in x \subset Q^n \) be given. This implies \( Q^n \cap x^n \) so that \( x \) and \( x^n \) are in the same hyperplane \( Q^n \). Since \( x^n \) is a co-line, \( x \) and \( x^n \) cannot be skew, i.e. \( x \in J(Q) \). On the other hand, from \( x \in J(Q) \) follows immediately that \( x \subset x^n \subset Q^n \).

Next let \( a \in \mathcal{L} \setminus J \). If \( b \in J \) intersects \( a \) at a point \( Q \), say, then \( Q \in b \subset b^n \) implies \( b^n = b \subset Q^n \), whereas \( Q \in a \) tells us \( a^n \subset Q^n \). Thus, as before, \( b \) and \( a^n \) are not skew. Conversely, given points \( Q \in a \) and \( R \in a^n \) then \( R \in a^n \subset Q^n \) and \( Q \in a \subset R^n \), whence \( Q \lor R \subset Q^n \cap R^n = (Q \lor R)^n \).

We apply this result to show

**Lemma 2.** Distinct lines \( a,b \in L \) with \( a \cap b \neq \emptyset \) are related if, and only if, \( a \in J \) or \( b \in J \).

**Proof.** If one of the given lines is isotropic, then \( a \sim b \). Conversely, if \( a \sim b \) and \( a \notin J \), say, then \( b \notin J \) by Lemma 1.

As an immediate consequence we obtain

**Lemma 3.** Let \( M \) be a set of mutually related lines. Then at most one line of \( M \) is non-isotropic.

Given lines \( a,b \in L \) then there is always a finite sequence

\[ a \sim a_1 \sim \ldots \sim a_n \sim b; \]

This is trivial when \( a = b \). If \( a \cap b =: Q \) is a point, then there exists a line \( a_1 \in J(Q) \) so that \( a \sim a_1 \sim b \) by Lemma 2. If \( a \) and \( b \) are skew then there exists a common transversal line of \( a \) and \( b \), say \( c \), whence repeating the previous construction for \( a,c \) and then for \( c,b \) gives the required sequence. Thus \((L,\sim)\)
is a Plücker space\(^4\) [1,p.199]. A (symplectic) Plücker transformation is a bijective mapping \(\varphi : \mathcal{L} \to \mathcal{L}\) preserving the relation \(\sim\) in both directions. We say that \(\varphi\) is induced by a mapping \(\kappa : \mathcal{P} \to \mathcal{P}\), if
\[(A \vee B)^\varphi = A^\kappa \vee B^\kappa\] for all \(A, B \in \mathcal{P}\) with \(A \neq B\).

The group \(\text{PGSp}(\mathcal{P}, \pi)\) consists of all collineations \(\mathcal{P} \to \mathcal{P}\) commuting with \(\pi\) [7,p.388ff], [8,p.19]. Obviously, each \(\kappa \in \text{PGSp}(\mathcal{P}, \pi)\) is inducing a Plücker transformation.

If \(\dim(\mathcal{P}, \mathcal{L}) = 3\), then for each duality \(\tau\) with \(\tau^\tau = \tau\) the restriction \(\tau|_{\mathcal{L}}: \mathcal{L} \to \mathcal{L}\) is a Plücker transformation. Moreover, in the three-dimensional case there are always Plücker transformations not arising from collineations or dualities: Let \(\mathcal{L}_1\) be any subset of \(\mathcal{L} \setminus \mathcal{J}\) such that \(\mathcal{L}_1^\pi = \mathcal{L}_1\). Then define
\[
\delta : \mathcal{L} \to \mathcal{L}, \quad \left\{\begin{array}{ll}
x \mapsto x & \text{if } x \in \mathcal{L} \setminus \mathcal{L}_1, \\
x \mapsto x^\pi & \text{if } x \in \mathcal{L}_1.
\end{array}\right.
\]

Such a bijection \(\delta\) will be called partial \(\pi\)-transformation (with respect to \(\mathcal{L}_1\)); it is a Plücker transformation of \((\mathcal{L}, \sim)\), since
\[a \sim b \iff a^\pi \sim b^\pi \iff a^{\pi \sim b}\] for all \(a, b \in \mathcal{L}, a \neq b\).

The identity on \(\mathcal{L}\) and the restriction of \(\pi\) to \(\mathcal{L}\) are partial \(\pi\)-transformations, as follows from setting \(\mathcal{L}_1 := \emptyset\) and \(\mathcal{L}_1 := \mathcal{L} \setminus \mathcal{J}\), respectively. For every other choice of \(\mathcal{L}_1\) (e.g., \(\mathcal{L}_1 := (a, a^\pi)\)) it is easily seen that there exist two non-isotropic concurrent lines \(x \in \mathcal{L} \setminus \mathcal{L}_1, y \in \mathcal{L}_1\). Then \(x^\delta = x\) and \(y^\delta = y^\pi\) are skew lines. Such a Plücker transformation cannot arise from a collineation or duality.

3. The three-dimensional case

**Theorem 1.** Let \((\mathcal{P}, \mathcal{L}, \pi)\) be a 3-dimensional symplectic space and let \(\beta : \mathcal{L} \to \mathcal{L}\) be a bijection such that
\[a \sim b \text{ implies } a^\beta \sim b^\beta\] for all \(a, b \in \mathcal{L}\).

Then there exists a partial \(\pi\)-transformation \(\delta : \mathcal{L} \to \mathcal{L}\) such that \(\delta \beta\) is induced by a collineation \(\kappa \in \text{PGSp}(\mathcal{P}, \pi)\).

Theorem 1 is a consequence of the subsequent Propositions 1.1 - 1.4 in which \(\beta\) and \((\mathcal{P}, \mathcal{L}, \pi)\) are given as above.

**Proposition 1.1.** There exists an injective mapping \(\kappa : \mathcal{P} \to \mathcal{P}\) such that

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\(^4\)Alternatively, \(\mathcal{L}\) may be seen as the set of vertices of a graph with two vertices joined by an edge if, and only if, the corresponding lines are distinct and related. We refrain, however, from using terminology of graph theory.
\[ J(Q)^\beta = J(Q') \text{ for all } Q \in \mathcal{P}. \]  

(3)

Moreover, \( \beta \) is a Plücker transformation, since

\[ J^\beta = J. \]  

(4)

Proof. By the invariance of \( \sim \) under \( \beta \), the elements of \( J(Q)^\beta \) are mutually related. We infer from Lemma 3 that \( J(Q)^\beta \) contains at most one non-isotropic line. Thus \( J(Q)^\beta \cap J \) has at least two distinct elements, whence it is a subset of a pencil of isotropic lines, say \( J(Q') \) with \( Q' \in \mathcal{P} \).

We show \( J(Q') \subset J(Q)^\beta \): Assume, to the contrary, that there exists a line \( x \notin J(Q) \) with \( x^\beta \notin J(Q') \). Recall that at most one line of \( J(Q)^\beta \) is non-isotropic. Therefore there is a point \( X' \in x^\beta \) that is not incident with any line of \( J(Q)^\beta \). Thus we can draw a line \( b' = b^\beta \) through \( X' \) that is not related to any line of \( J(Q)^\beta \). Hence \( b \) is not related to any line of \( J(Q) \). By \( \dim(\mathcal{P}, \mathcal{L}) = 3 \), \( b \) and the plane \( Q^\alpha \) have a common point lying on some line \( c \in J(Q) \), so that \( c \sim b \), a contradiction.

Next \( J(Q)^\beta \subset J(Q') \) will be established: Suppose there is a line \( a \in J(Q) \) such that \( a^\beta \notin J(Q') \). Then \( a^\beta \sim J(Q') \) forces that \( a^\beta \) is a non-isotropic line either through the point \( Q' \) or in the plane \( Q^\alpha \). Let \( d \in J(Q) \) be non-isotropic, whence \( J(Q)^\beta \cup \{ d^\beta \} \) is a set of mutually related lines containing the non-isotropic line \( a^\beta \). Since \( d^\beta \notin J(Q)^\beta \) and \( J(Q') \subset J(Q)^\beta \), the line \( d^\beta \neq a^\beta \) also has to be non-isotropic in contradiction to Lemma 3.

To sum up, there is a mapping \( \kappa \) satisfying formula (3). The injectivity of \( \kappa \) follows from the bijectivity of \( \beta \) together with (3).

Finally, we prove (4): \( J^\beta \subset J \) is a consequence of (3). Conversely, assume that \( e \in \mathcal{L} \setminus J \). Choose a point \( R \in e \). Then \( J(R)^\beta \cup \{ e^\beta \} = J(R^\beta) \cup \{ e^\beta \} \) is a set of mutually related lines. Therefore \( e^\beta \) is a non-isotropic line either through \( R^\alpha \) or in \( R^\beta \). Lemma 2 and \( J^\beta = J \) imply that \( \beta \) is a Plücker transformation.■

**Proposition 1.2.** Let \( a \in \mathcal{L} \). Then

\[ a^{\alpha \beta} = a^{\alpha \beta}, \]  

(5)

\[ Q^\kappa \in a^\beta \cup a^{\alpha \beta} \text{ for all } Q \in a. \]  

(6)

If \( a \in \mathcal{L} \setminus J \), then either

\[ Q^\kappa \in a^\beta \text{ for all } Q \in a, \]  

(7)

or

\[ Q^\kappa \in a^{\alpha \beta} \text{ for all } Q \in a. \]  

(8)

Proof. If \( a \in J \), then \( a^\beta \in J \), whence (5) follows from \( a = a^\alpha \) and \( a^\beta = a^{\alpha \beta} \). If \( a \in \mathcal{L} \setminus J \), then, by Lemma 1,
\[ C := \{ x \in L | x \neq a, x \sim a \} \]

is a hyperbolic linear congruence of lines with axes \( a \) and \( a^\pi \); moreover \( C \subset \mathcal{J} \).

We infer from \( \beta \) being a Plücker transformation and (4), that \( C^\beta \subset \mathcal{J} \) is also a hyperbolic linear congruence with \( a^\beta, a^{\pi\beta} \) being its axes. Obviously, only \( a^\beta \) and \( a^{\pi\beta} \) are meeting all lines of \( C^\beta \). On the other hand, by Lemma 1, the axes of \( C^\beta \) are \( a^\beta \) and \( a^{\pi\beta} \). This completes the proof of (5).

If \( a \in \mathcal{J} \), then (6) holds true, since \( \mathcal{Q}^\epsilon \in a^\beta = a^{\pi\beta} \). If \( a \in \mathcal{L} \setminus \mathcal{J} \) and \( \mathcal{Q}^\epsilon \not\in a \), then \( a^\beta = \mathcal{J}[Q]^\beta = \mathcal{J}[Q^\epsilon] \), whence \( Q^\pi \not\in a^\beta \) and therefore \( \mathcal{Q}^\epsilon \in a^{\pi\beta} = a^{\pi\beta} \), as required to establish (6).

Now let \( a \in \mathcal{L} \setminus \mathcal{J} \). Assume to the contrary that there exist points \( Q_0, Q_1 \in a \) such that \( Q^\epsilon_0 \in a^\beta \) and \( Q^\epsilon_1 \in a^{\pi\beta} \). Then \( a \not\in \mathcal{J} \) implies \( \mathcal{J}[Q_0] \cap \mathcal{J}[Q_1] = \emptyset \) whereas, by Lemma 1 and (3), \( Q^\epsilon_0 \cup Q^\epsilon_1 \in \mathcal{J}[Q_0]^\beta \cap \mathcal{J}[Q_1]^\beta \). This is a contradiction to \( \beta \) being injective.

**Proposition 1.3.** Write \( \mathcal{L}_1 \) for the set of all lines \( a \in \mathcal{L} \setminus \mathcal{J} \) satisfying (8). Then

\[
\delta : \mathcal{L} \to \mathcal{L}, \quad \left\{ \begin{array}{ll}
    x \mapsto x & \text{if } x \in \mathcal{L} \setminus \mathcal{L}_1, \\
    x \mapsto x^\pi & \text{if } x \in \mathcal{L}_1,
\end{array} \right. \tag{9}
\]

is a partial \( \pi \)-transformation. The Plücker transformation \( \delta \beta : \mathcal{L} \to \mathcal{L} \) takes intersecting lines to intersecting lines.

**Proof.** In order to show that \( \delta \) is a well-defined partial \( \pi \)-transformation, we just have to establish that \( a \in \mathcal{L}_1 \) implies \( a^\pi \in \mathcal{L}_1 \): Given \( Q_0 \in a \) and \( Q_1 \in a^\pi \) then \( Q_0 \cup Q_1 \) and \( (Q_0 \cup Q_1)^\beta = Q^\epsilon_0 \cup Q^\epsilon_1 \) are isotropic lines. Therefore

\[ Q^\epsilon_0 \cup Q^\epsilon_1 \neq a^{\pi\beta} = a^{\pi\beta} \in \mathcal{J} \]

so that \( Q^\epsilon_1 \not\in a^{\pi\beta} \). Now, by (8), \( a^\pi \in \mathcal{L}_1 \).

If distinct lines \( b \) and \( c \) intersect at a point \( R \), then \( b^{\beta} \cap c^{\beta} = R^\epsilon \) follows from (7), (8) and (9).

**Proposition 1.4.** The mapping \( \kappa : \mathcal{P} \to \mathcal{P} \) defined in (3) belongs to \( \text{PSp}(P,\pi) \).

The Plücker transformation \( \delta \beta \) is induced by this collineation \( \kappa \).

**Proof.** The bijection \( \delta \beta \) is taking intersecting lines to intersecting lines. Every star of lines is mapped under \( \delta \beta \) either onto a star of lines or onto a ruled plane \([4], [10, \text{Theorem 1}] \). The latter possibility does not occur, since \( \delta \beta \) is induced by \( \kappa \). Because of \( \dim(\mathcal{P}, \mathcal{L}) \) being finite, the mapping \( \kappa \) is a collineation \([10, \text{Theorem 3}] \). Finally, \( J^\beta = J \) implies \( \kappa \in \text{PSp}(P,\pi) \).
4. The higher-dimensional case

**Theorem 2.** Let \((\mathcal{P}, \mathcal{I}, \pi)\) be an \(n\)-dimensional symplectic space \((5 \leq n \leq \omega)\) and let \(\beta : \mathcal{I} \to \mathcal{I}\) be a bijection such that

\[
 a \sim b \implies a^\beta \sim b^\beta \text{ for all } a, b \in \mathcal{I}.
\]

Then \(\beta\) is induced by a collineation \(\kappa \in \text{PGSp}(\mathcal{P}, \pi)\).

As before, the Theorem will be split into several Propositions subject to the assumptions stated above.

**Proposition 2.1.** The bijection \(\beta\) takes intersecting lines to intersecting lines. There exists an injective mapping \(\kappa : \mathcal{P} \to \mathcal{P}\) inducing \(\beta\). This \(\kappa\) is preserving collinearity and non-collinearity of points. Moreover

\[
\mathcal{L}[Q]^\beta = \mathcal{L}[Q'] \text{ for all } Q \in \mathcal{P}.
\]

**Proof.** Suppose that \(a, b \in \mathcal{I}\) meet at a point \(Q\). If \(a \sim b\), then \(a^\beta\) and \(b^\beta\) are intersecting. Otherwise, by Lemma 2, \(a \not\sim b\) and \(b \not\sim a\). Then \(\mathcal{J}[Q] \cup \{a\}\) and \(\mathcal{J}[Q] \cup \{b\}\) are, respectively, sets of mutually related lines. Each line of \(\mathcal{L}\) is related to at least one line in \(\mathcal{J}[Q]\), since \(\mathcal{Q}^\omega\) is a hyperplane covered by \(\mathcal{J}[Q]\). If \(\mathcal{J}[Q]^\beta\) would be a set of coplanar lines, then all lines in \(\mathcal{L}\) would meet a fixed plane in contradiction to \(n \geq 5\). Thus \(\mathcal{J}[Q]^\beta\) is not contained in a plane, whence there exists a point \(Q'\) with \(\mathcal{J}[Q] \cup \mathcal{L}[Q']\). Since the elements of \(\mathcal{J}[Q] \cup \{a^\beta\}\) are mutually related, \(Q' \in a^\beta\). Repeating this for \(b\) yields \(Q' \in b^\beta\).

Now the assertions on \(\kappa\) follow from [10, Theorem 1].

**Proposition 2.2.** The bijection \(\beta\) is a Plücker transformation, since

\[
\mathcal{J}^\beta = \mathcal{J}.
\]

**Proof.** Given \(a \in \mathcal{J}\) then choose a point \(Q \in a\). We observe that \(a \sim \mathcal{L}[Q]\), whence \(a^\beta \sim \mathcal{L}[Q^\beta]\) by (10). Since \(\mathcal{L}[Q^\omega]\) contains more than one non-isotropic line, \(a^\beta \in \mathcal{J}\) follows from Lemma 2.

Given \(b \in \mathcal{L} \setminus \mathcal{J}\) then choose a point \(R \in b\). Assume to the contrary that \(b^\beta \in \mathcal{J}\). Then for each line

\[
\chi \in \mathcal{L}[R] \setminus (\mathcal{J}[R] \cup \{b\})
\]

there exists a line \(\chi^\beta \in \mathcal{J}[R]\) such that \(b, \chi, \chi^\beta\) are three distinct lines in one pencil. By the invariance of collinearity and non-collinearity of points under \(\kappa\), as is stated in Proposition 2.1, \(b^\beta, \chi^\beta, \chi^\beta\) are again three distinct lines in one pencil. However, \(b^\beta\) and \(\chi^\beta\) are isotropic, so that

\[
\chi^\beta \in \mathcal{J}[R^\beta].
\]
Hence $\mathcal{L}[R]^{\beta} \subset \mathcal{I}[R^{\infty}]$ which is impossible by (10).

Now (11) and Lemma 2 show that $\beta$ is a Plücker transformation.

**Proposition 2.3.** The mapping $\kappa : \mathcal{P} \to \mathcal{P}$, described in Proposition 2.1, is a collineation belonging to $\text{PTS} \mathcal{P}(\mathcal{P}, \pi)$.

**Proof.** Since $\beta$ is a Plücker transformation of $(\mathcal{L}, \sim)$, Proposition 2.1 can be applied to $\beta^{-1}$. Therefore $\beta$ and $\beta^{-1}$ are preserving intersection of lines. By [10, Theorem 2], the mapping $\kappa$ is a collineation and, by formula (11), $\kappa \in \text{PTS} \mathcal{P}(\mathcal{P}, \pi)$.

This completes the proof of Theorem 2.

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