A Three-Dimensional Laguerre Geometry and Its Visualization

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We describe and visualize the chains of the 3-dimensional chain geometry over the ring $\mathbb{R}(\varepsilon)$, $\varepsilon^3 = 0$.
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1 Introduction

The aim of the present paper is to discuss in some detail the *Laguerre geometry* (cf. [1], [6]) which arises from the 3-dimensional real algebra $\mathbb{L} := \mathbb{R}(\varepsilon)$, where $\varepsilon^3 = 0$. This algebra generalizes the algebra of real dual numbers $\mathbb{D} = \mathbb{R}(\varepsilon)$, where $\varepsilon^2 = 0$. The Laguerre geometry over $\mathbb{D}$ is the geometry on the so-called *Blaschke cylinder* (Figure 1); the non-degenerate conics on this cylinder are called *chains* (or *cycles*, *circles*). If one generator of the cylinder is removed then the remaining points of the cylinder are in one-one correspondence (via a *stereographic projection*) with the points of the plane of dual numbers, which is an isotropic plane; the chains go over to circles and non-isotropic lines. So the point space of the chain geometry over the real dual numbers can be considered as an affine plane with an extra "improper line".

The Laguerre geometry based on $\mathbb{L}$ has as point set the projective line $\mathbb{P}(\mathbb{L})$ over $\mathbb{L}$. It can be seen as the real affine 3-space on $\mathbb{L}$ together with an "improper affine plane". There is a point model for this geometry, like the Blaschke cylinder, but it is more complicated, and belongs to a 7-dimensional projective space ([6, p. 812]). We are not going to use it. Instead, we describe $\mathbb{P}(\mathbb{L})$ as an extension of the affine space on $\mathbb{L}$ by "improper points" which will be described via lines, parabolas, and cubic parabolas.

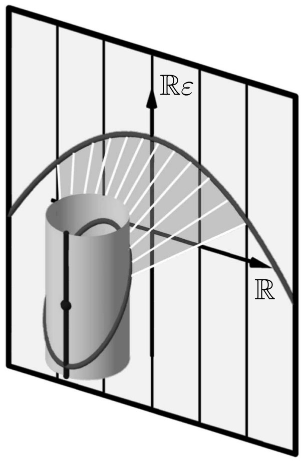

Figure 1

2 Higher order contact of twisted cubics

Here we present some results which will be needed in Section 3. We refer to [3], [5], and [9] for the basic properties of twisted cubics in the real projective space $\mathbb{P}_3(\mathbb{R})$.

Theorem 1 *Let C and $\widetilde{C}$ be twisted cubics of $\mathbb{P}_3(\mathbb{R})$ with a common point f, a common tangent line F at f, and a common osculating plane Φ at f. Then a collineation of $\mathbb{P}_3(\mathbb{R})$ taking C to $\widetilde{C}$ is uniquely determined by each of the following conditions:*

(I) *All lines of the pencil $\mathcal{L}(f,\Phi)$ are invariant.*
(II) *All points of the line F are invariant.*
(III) *All planes of the pencil with axis F are invariant.*

Proof. (I) We recall that distinct tangent lines of a twisted cubic are skew. The tangent surface of C intersects the osculating plane Φ in a curve which is the union of F and a conic K through f; the line F is tangent to K. Likewise the tangent surface of $\widetilde{C}$ yields a conic $\widetilde{K}$. Let r_i, $i \in \{1,2\}$, be distinct points of $C \setminus \{f\}$. The tangent lines of C at these points meet the plane Φ at points $k_i \in K \setminus \{f\}$, whence the lines $L_i := f \vee k_i$ are distinct. These lines meet $\widetilde{K}$ residually at points $\widetilde{k}_i$ which in turn are incident with tangent lines of $\widetilde{C}$ at distinct points $\widetilde{r}_i$. These points $\widetilde{r}_i$ are determined uniquely. So, every collineation of type (I) takes r_i to $\widetilde{r}_i$, and f to f. Conversely, there is a unique collineation κ of $\mathbb{P}_3(\mathbb{R})$ with $C^\kappa = \widetilde{C}$, $r_i^\kappa = \widetilde{r}_i$, and $f^\kappa = f$. Since F, $f \vee k_1$, and $f \vee k_2$ remain invariant under κ, all lines of the pencil $\mathcal{L}(f,\Phi)$ remain fixed. So this κ is the only collineation with the required properties.

(II) The proof runs in a similar manner. The osculating planes at r_i meet F at points $k_i \neq f$. Now $\widetilde{r}_i \in \widetilde{C} \setminus \{f\}$ can be chosen such that their osculating planes meet the line F at k_i.

(III) Each of the planes $F \vee r_i$ meets the twisted cubic $\widetilde{C}$ residually at a point $\widetilde{r}_i$. Now we can proceed as above. □

Let p_0, p_3, and p be three distinct points of C. Define the point p_1 as the intersection of the tangent line at p_0 with the osculating plane at p_3. Likewise, by changing the role of p_0 and p_3, a point p_2 is obtained. Then (p_0, p_1, p_2, p_3, p) is a frame of reference such that

$$C = \{\mathbb{R}(s^3, s^2t, st^2, t^3) \mid (0,0) \neq (s,t) \in \mathbb{R}^2\}. \tag{1}$$

We assume that $f = p_3 = \mathbb{R}(0,0,0,1)$, whence F is given by $x_0 = x_1 = 0$ and Φ has an equation $x_0 = 0$. A collineation of $\mathbb{P}_3(\mathbb{R})$ is of type (I), (II) or (III) if, and only if, it has a regular matrix with one of the following forms:

$$\text{(I):} \begin{pmatrix} 1 & a_{01} & a_{02} & a_{03} \\ 0 & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{11} & a_{23} \\ 0 & 0 & 0 & a_{33} \end{pmatrix}, \text{(II):} \begin{pmatrix} 1 & a_{01} & a_{02} & a_{03} \\ 0 & a_{11} & a_{12} & a_{13} \\ 0 & 0 & a_{22} & 0 \\ 0 & 0 & 0 & a_{22} \end{pmatrix}, \text{(III):} \begin{pmatrix} 1 & 0 & a_{02} & a_{03} \\ 0 & 1 & a_{12} & a_{13} \\ 0 & 0 & a_{22} & a_{23} \\ 0 & 0 & 0 & a_{33} \end{pmatrix} \tag{2}$$

Next we describe higher-order contact of twisted cubics; cf. also [3, pp. 211–219].

Theorem 2 *Let C be the twisted cubic (1) and let κ be a collineation of $\mathbb{P}_3(\mathbb{R})$ given by one of the matrices (2). Then the conditions stated in the first row, in the first and the second row, and in all rows of the table below are necessary and sufficient for the twisted cubics C and C^κ to have contact at the point $f = \mathbb{R}(0,0,0,1)$ of order 2, 3, and 4, respectively.*

	(I)		(II)		(III)	
1	$a_{33} = a_{11}$		$a_{22} = a_{11}$		$a_{33} = a_{22}^2$	
2	$a_{11} = 1,$	$a_{23} = -a_{01}$	$a_{11} = 1,$	$a_{01} = 2a_{12}$	$a_{22} = 1,$	$a_{23} = 2a_{12}$
3	$a_{01} = 0,$	$a_{13} = 2a_{02}$	$a_{12} = 0,$	$a_{13} = 2a_{02}$	$a_{12} = 0,$	$a_{13} = 2a_{02}$

(3)

Proof. The quadratic forms

$$G_1 : \mathbb{R}^4 \to \mathbb{R} : (x_0, x_1, x_2, x_3) \mapsto x_0x_3 - x_1x_2,$$
$$G_2 : \mathbb{R}^4 \to \mathbb{R} : (x_0, x_1, x_2, x_3) \mapsto x_1x_3 - x_2^2,$$

define a hyperbolic quadric $x_0x_3 - x_1x_2 = 0$ and a quadratic cone $x_1x_3 - x_2^2 = 0$ with vertex p_0. Their intersection is the twisted cubic C and the line $x_2 = x_3 = 0$. The tangent planes of the two surfaces at f are different. Let κ be given by a matrix A of type (I). The mapping

$$g : \mathbb{R} \to \mathbb{R}^4 : s \mapsto (s^3, s^2, s, 1) \cdot A$$

gives an arc of C^κ containing the point f, which has the parameter $s = 0$. The products of g with G_i are functions

$$s \mapsto (-a_{11}^2 + a_{33})s^3 + (-a_{01}a_{11} + a_{23})s^4 + (*),$$
$$s \mapsto (-a_{11}^2 + a_{11}a_{33})s^2 + (a_{01}a_{33} + a_{11}a_{23})s^3 + (a_{01}a_{23} - 2a_{11}a_{02} + a_{11}a_{13})s^4 + (*),$$

where $(*)$ denotes terms of higher order in s. The twisted cubics C and C^κ have contact of order m at f if, and only if, in both functions the coefficients at $s^0, s^1, \dots, s^m$ vanish [3, p. 147]. Now the assertions follow easily, taking into account that $a_{11} \neq 0$ and $a_{33} \neq 0$.

Similarly, if the matrix A is of type (II) then the functions

$$s \mapsto (-a_{11}a_{22} + a_{22})s^3 + (-a_{01}a_{22} - a_{11}a_{12})s^4 + (*),$$
$$s \mapsto (a_{11}a_{22} - a_{22}^2)s^2 + (a_{01}a_{22} - 2a_{12}a_{22})s^3 + (-2a_{02}a_{22} + a_{11}a_{13} - a_{12}^2)s^4 + (*),$$

are obtained, whereas for an A of type (III) we get

$$s \mapsto (-a_{22} + a_{33})s^3 + (-a_{12} + a_{23})s^4 + (*),$$
$$s \mapsto (-a_{22}^2 + a_{33})s^2 + (-2a_{12}a_{22} + a_{23})s^3 + (-2a_{02}a_{22} - a_{12}^2 + a_{13})s^4 + (*).$$

As above, the results are immediate. □

Let us now consider Φ as *plane at infinity*. Then our projective frame of reference determines an affine coordinate system in the usual way; a point $\mathbb{R}(1, x_1, x_2, x_3) \in \mathbb{P}_3(\mathbb{R})$ has affine coordinates (x_1, x_2, x_3). It is our aim to describe the results of Theorem 1 and Theorem 2 in affine terms. From the affine point of view the twisted cubics C and C^κ are *cubic parabolas*, projectively extended by the point $f = p_3$. So this point of higher order contact is *outside* the affine space. In what follows an *affine transformation* is understood to be a collineation fixing the plane Φ. We restrict ourselves to the description of higher order contact via regular matrices of type (I). Such a matrix A admits the following factorization:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{a_{33}}{a_{11}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{a_{13}}{a_{11}} \\ 0 & 0 & 1 & \frac{a_{23}}{a_{11}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{11} \end{pmatrix} \cdot \begin{pmatrix} 1 & a_{01} & a_{02} & a_{03} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

Conversely, if the entries a_{ij} in (4) are chosen arbitrarily, except for $a_{11}, a_{33} \neq 0$, then a regular matrix of type (I) is obtained. Formula (4) corresponds to a decomposition of κ into a *perspective affinity* with axis $x_3 = 0$ in the direction of p_3, a *shear* with an axis through the line $x_1 = x_2 = 0$ in the direction of p_3, a *stretching* fixing the origin p_0 with scale factor a_{11}, and a *translation* through the vector (a_{01}, a_{02}, a_{03}), respectively; this decomposition is uniquely determined.

The matrix A is of type (I.1) if, and only if, $a_{11} = a_{33}$, i.e., the first matrix in (4) is the unit matrix. The ultimate and the penultimate matrix in (4) together yield a *dilatation* and every dilatation arises in this way. Hence, up to dilatations, we obtain all twisted cubics which have second order contact with C at f by applying to C all shears with the properties mentioned above. Figure 2 shows the twisted cubic C and some of its images under a group Σ of shears in the direction of p_3 with the common axis $x_1 + x_2 = 0$. All these twisted cubics are on a parabolic cylinder Ψ ($x_1^2 - x_2 = 0$) which is invariant under the group Σ.

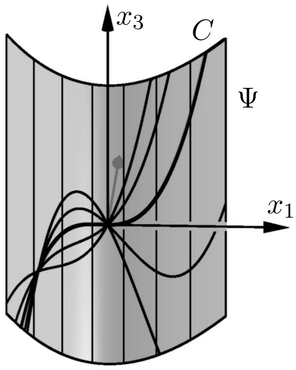

Figure 2

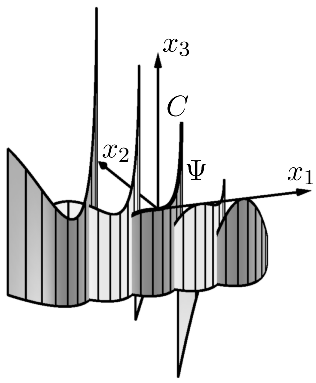

Figure 3

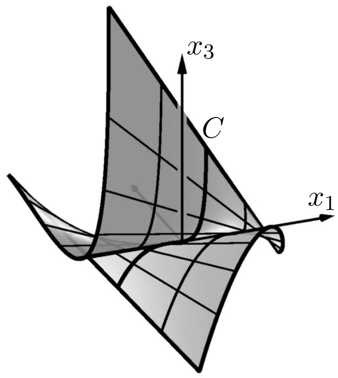

Figure 4

The matrix A is of type (I.1.2) if, and only if, it can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_{13} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a_{01} \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_{01} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a_{02} & a_{03} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (5)$$

As before, this factorization is unique and the coefficients can be chosen freely. The first (second) matrix gives a shear with axis $x_1 = 0$ ($x_2 = 0$) in the direction of p_3, whereas the remaining matrices yield a translation in the direction of p_1 and a translation parallel to the plane $x_1 = 0$. However, the second and the third matrix are linked via the common parameter a_{01}. As a_{01} varies in $\mathbb{R}$, their products comprise a one-parameter group Γ_1 of affine transformations. (See [4, I, p. 130], III 3, "*Nichtisotrope Cliffordschiebungen*": All points of the line $x_0 = x_2 = 0$ are invariant under Γ_1. All other point orbits are lines of a parabolic linear congruence with axis $x_0 = x_2 = 0$.) Hence, up to translations parallel to the plane $x_1 = 0$, we obtain all twisted cubics which have third order contact with C at f by applying to C all shears with axis $x_1 = 0$ and then all transformations of the group Γ_1. In Figure 3 the twisted cubic C and some of its images under affinities of Γ_1 are displayed. These curves lie on parabolic cylinders which are translates of Ψ. Figure 4 shows the ruled surface which arises by applying Γ_1 to the curve C. The illustrated lines are point orbits with respect to Γ_1. In particular, the x_1-axis of the coordinate system is the orbit of the origin; this line is an edge of regression of the surface.

The matrix A is of type (I.1.2.3) if, and only if, it can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2a_{02} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a_{02} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & a_{03} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

Again, this decomposition is unique and the coefficients can be chosen arbitrarily. The products of the first and the second matrix in (6) comprise a one-parameter subgroup Γ_2; cf. the remarks above. Hence, up to translations parallel to the line $x_1 = x_2 = 0$, we obtain all twisted cubics which have fourth order contact with C at f as the orbit of C under Γ_2. Figure 5 illustrates the twisted cubic C and the cylinder Ψ, together with some of their images under affinities of Γ_2. Figure 6 shows the ruled surface which is generated by applying Γ_2 to the curve C. This surface is a proper subset (only the points of $F \setminus \{f\}$ are missing) of the (ruled) *Cayley surface* with equation $2x_0x_1x_2 - x_1^3 = x_0^2x_3$.

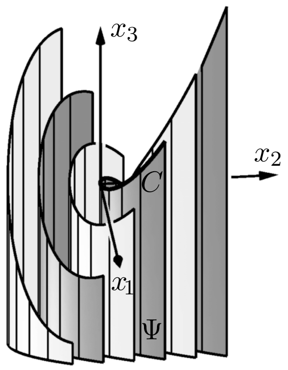

Figure 5

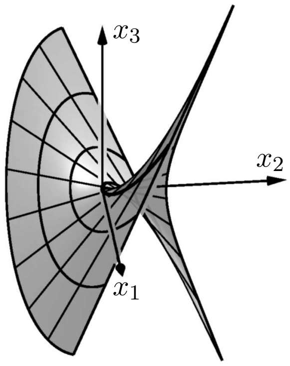

Figure 6

3 The Laguerre Geometry $\Sigma(\mathbb{R}, \mathbb{L})$

Let $\mathbb{R}[X]$ be the polynomial ring over the reals. The factor ring $\mathbb{R}[X]/\langle X^3\rangle =: \mathbb{L}$ is a 3-dimensional real commutative local algebra with an $\mathbb{R}$-basis $1_{\mathbb{L}}, \varepsilon, \varepsilon^2$, where $\varepsilon := X + \langle X^3\rangle$. Its non-invertible elements form the only maximal ideal $N := \mathbb{R}\varepsilon + \mathbb{R}\varepsilon^2$. We consider $\mathbb{R}$ as a subring of $\mathbb{L}$ by identifying $x \in \mathbb{R}$ with $x \cdot 1_{\mathbb{L}} \in \mathbb{L}$. (Our ring $\mathbb{L}$ is the ring $\mathfrak{L}_4$ in [1, p. 306].) Let us recall the definition of the *projective line over* $\mathbb{L}$, in symbols $\mathbb{P}(\mathbb{L})$: We consider the free left $\mathbb{L}$-module $\mathbb{L}^2$. A cyclic submodule $\mathbb{L}(u, v) \subset \mathbb{L}^2$ is a point of $\mathbb{P}(\mathbb{L})$ if, and only if, u or v is a unit in $\mathbb{L}$. Two such pairs (u, v) and (u', v') in $\mathbb{L}^2$ determine the same point precisely when they are proportional by a unit in $\mathbb{L}$. (Cf. [6, p. 785] for a definition of the projective line over an arbitrary ring with a unit element.). We embed the real projective line $\mathbb{P}(\mathbb{R})$ in $\mathbb{P}(\mathbb{L})$ by $\mathbb{R}(x, y) \mapsto \mathbb{L}(x, y)$. The point set of the *chain geometry* $\Sigma(\mathbb{R}, \mathbb{L})$ is the projective line over $\mathbb{L}$, the *chains* are the images of $\mathbb{P}(\mathbb{R}) \subset \mathbb{P}(\mathbb{L})$ under the natural right action of $\mathrm{GL}_2(\mathbb{L})$ on $\mathbb{L}^2$; cf. [6, p. 790]. Since $\mathbb{L}$ is a local ring, our chain geometry is a *Laguerre geometry* [6, p. 793]. If two distinct points of $\mathbb{P}(\mathbb{L})$ can be joined by a chain then they are said to be *distant*. Non-distant points are also said to be *parallel* ($\parallel$). Letting $p = \mathbb{L}(a, b)$ and $q = \mathbb{L}(c, d)$ gives

$$p \parallel q \Leftrightarrow \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N. \tag{7}$$

This parallelism is an equivalence relation. There is a unique chain through any three mutually distant points.

We fix the point $\mathbb{L}(1,0) =: \infty \in \mathbb{P}(\mathbb{L})$. Then the point set of $\mathbb{P}(\mathbb{L})$ can be split into two classes: A *proper point* has the form $\mathbb{L}(z,1)$, and we identify such a point with the element $z \in \mathbb{L}$. The proper points are precisely the points which are distant (non-parallel) to ∞. Every other point of $\mathbb{P}(\mathbb{L})$ has the form $\mathbb{L}(1,z)$ with $z \in N$. Such points are said to be *improper*. Hence we can regard $\mathbb{P}(\mathbb{L})$ as the real affine 3-space on $\mathbb{L}$ together with an extra "improper plane" which is just a copy of the maximal ideal N.

The algebra $\mathbb{L}$ has two distinguished ideals, namely the maximal ideal N and its annihilator $\{z \in \mathbb{L} \mid zN = 0\} = \mathbb{R}\varepsilon^2$. Accordingly, there are three types of lines: A line $\mathbb{R}u + v \subset \mathbb{L}$, where $u \in \mathbb{L} \setminus \{0\}$, $v \in \mathbb{L}$ is called *singular* if $u \in N$, and *regular* otherwise. A singular line of the form $\mathbb{R}\varepsilon^2 + v$ is said to be *vertical*. We say that a plane is *regular* provided that it contains at least one regular line. A *singular* plane is just a non-regular plane. By (7), the singular planes are the classes of proper parallel points.

For each subset $\mathcal{S} \subset \mathbb{P}(\mathbb{L})$ let $\mathcal{S}^\circ$ be its *proper part*, i.e. the set of all its proper points. The following is taken from H.-J. SAMAGA [8, Satz 4]; cf. also [2]: A subset $\mathcal{C}$ of $\mathbb{P}(\mathbb{L})$ is a chain of $\Sigma(\mathbb{R},\mathbb{L})$ precisely when one of the following conditions holds:

$$\mathcal{C} = \{t + (a_{02} + a_{12}t)\varepsilon + (a_{03} + a_{13}t)\varepsilon^2 \mid t \in \mathbb{R}\} \cup \{\infty\}, \tag{8}$$

whence $\mathcal{C}^\circ$ is an affine line;

$$\mathcal{C} = \{t + (a_{02} + a_{12}t)\varepsilon + (a_{03} + a_{13}t + a_{33}t^2)\varepsilon^2 \mid t \in \mathbb{R}, a_{33} \neq 0\} \cup \{\mathbb{L}(1, -a_{33}\varepsilon^2)\}, \tag{9}$$

whence $\mathcal{C}^\circ$ is a parabola;

$$\begin{aligned}\mathcal{C} = \{&t + (a_{02} + a_{12}t + a_{22}t^2)\varepsilon + (a_{03} + a_{13}t + a_{23}t^2 + a_{33}t^3)\varepsilon^2 \mid t \in \mathbb{R}, a_{33} = a_{22}^2 \neq 0\} \\ &\cup \{\mathbb{L}(1, -a_{22}\varepsilon + (-a_{23} + 2a_{12}a_{22})\varepsilon^2)\},\end{aligned} \tag{10}$$

whence $\mathcal{C}^\circ$ is a cubic parabola. In either case the a_{ij}'s are real constants subject to the conditions stated above. Obviously, the lines given by (8) are precisely the regular ones. So, all regular lines are *representatives* for the point ∞. We say that a (cubic) parabola in $\mathbb{L}$ is *admissible* if it is the proper part of a chain. By (9), a parabola is admissible if, and only if, its diameters are vertical lines and its plane is regular. Each admissible parabola is a representative of a unique improper point. We describe admissible parabolas which determine the same improper point:

Theorem 3 *Let $\mathcal{C}^\circ$ and $\widetilde{\mathcal{C}}^\circ$ be admissible parabolas of $\mathbb{L}$. Then the chains $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ have the same improper point if, and only if, the parallel projection of $\widetilde{\mathcal{C}}^\circ$ onto the plane of $\mathcal{C}^\circ$, in the direction of an arbitrary non-vertical singular line, is a translate of $\mathcal{C}^\circ$.*

Proof. Let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ be given according to (9) with coefficients a_{ij} and $\widetilde{a}_{ij}$, respectively. The parallel projection of $\widetilde{\mathcal{C}}^\circ$ onto the plane of $\mathcal{C}^\circ$ is a parabola

$$\{t + (a_{02} + a_{12}t)\varepsilon + (a^*_{03} + a^*_{13}t + a^*_{33}t^2)\varepsilon^2 \mid t \in \mathbb{R}\} \text{ with } a^*_{33} = \widetilde{a}_{33}.$$

An easy calculation shows that the projected parabola is a translate of $\mathcal{C}^\circ$ if, and only if, $\widetilde{a}_{33} = a_{33}$. By (9), this is necessary and sufficient for $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ to have the same improper point. □

Let us consider the *projective closure* $\mathbb{P}_3(\mathbb{R})$ of the affine space on $\mathbb{L}$, where we do not distinguish between $\mathbb{R}(1, x_1, x_2, x_3) \in \mathbb{P}_3(\mathbb{R})$ and $x_1 + x_2\varepsilon + x_3\varepsilon^2 \in \mathbb{L}$. Since we are going to work with *two different extensions* of the affine space on $\mathbb{L}$, we reserve the phrases "at infinity" and "improper" for the projective closure and for the chain-geometric closure, respectively. If $\mathcal{C}$ is a chain of $\Sigma(\mathbb{R}, \mathbb{L})$ then $\mathcal{C}^+ \subset \mathbb{P}_3(\mathbb{R})$ denotes that unique projective line or conic or twisted cubic which contains $\mathcal{C}^\circ$. We denote by f, F, and Φ the point at infinity of the vertical line $\mathbb{R}\varepsilon$, the line at infinity of the singular plane N, and the plane at infinity, respectively.

Let $\mathcal{C}$ be a chain. If $\mathcal{C}^\circ$ is a line then $\mathcal{C}^+ \not\subset \Phi$ is a projective line with a point at infinity not on F and vice versa. If $\mathcal{C}^\circ$ is a parabola then $\mathcal{C}^+ \not\subset \Phi$ is a conic through f touching a line at infinity other than F. As before, all such conics arise from chains. We note that when $\mathcal{C}^\circ$ and $\widetilde{\mathcal{C}}^\circ$ are parabolas in the same plane then the existence of a translation taking $\mathcal{C}^\circ$ to $\widetilde{\mathcal{C}}^\circ$ just means that the projective conics $\mathcal{C}^+$ and $\widetilde{\mathcal{C}}^+$ have contact of second order at the point f. See, e.g., [7]. But, since admissible parabolas in different planes may represent the same improper point, we cannot always describe improper points in terms of conics with second order contact at infinity. Now we turn to the case when $\mathcal{C}^\circ$ is a cubic parabola:

Theorem 4 *The cubic parabola*

$$\{t + t^2\varepsilon + t^3\varepsilon^2 \mid t \in \mathbb{R}\} \tag{11}$$

is admissible. A cubic parabola of $\mathbb{L}$ is admissible if, and only if, its projective extension and the projective extension of (11) have contact of second order at the point $f = \mathbb{R}(0, 0, 0, 1)$.

Proof. By (10), there is a unique chain $\mathcal{D}$, say, such that $\mathcal{D}^\circ$ coincides with the cubic parabola (11). Its projective extension $\mathcal{D}^+$ is given by (1), whence it is a twisted cubic through f, with tangent line F, and osculating plane Φ. Now we apply those collineations of $\mathbb{P}_3(\mathbb{R})$ which are given by regular matrices of type (III.1). So we get all the twisted cubics which have second order contact with $\mathcal{D}^+$ at f and, by (10), these are precisely the projectively extended admissible cubic parabolas. □

Theorem 5 *Let $\mathcal{C}^\circ$ and $\widetilde{\mathcal{C}}^\circ$ be admissible cubic parabolas of $\mathbb{L}$. Then the chains $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ have the same improper point if, and only if, the extended curves $\mathcal{C}^+$ and $\widetilde{\mathcal{C}}^+$ have contact of third order at $f = \mathbb{R}(0, 0, 0, 1)$.*

Proof. (a) First, we consider that chain $\mathcal{D}$ which yields the cubic parabola (11). The improper point of $\mathcal{D}$ is $\mathbb{L}(1, -\varepsilon)$. Now we apply those collineations of $\mathbb{P}_3(\mathbb{R})$ which are given by regular matrices of type (III.1.2). This gives, by Theorem 2, precisely those twisted cubics which have third order contact with $\mathcal{D}^+$ at f and, by (10), we get all the projectively extended cubic parabolas that arise from the chains through $\mathbb{L}(1, -\varepsilon)$. Since contact of third order is a transitive notion, the assertion follows for all chains $\mathcal{C}$ through $\mathbb{L}(1, -\varepsilon)$.

(b) Next, let $\mathcal{C}$ be any chain whose proper part is a cubic parabola, so that its improper point can be written as $\mathbb{L}(1, -a\varepsilon - b\varepsilon^2)$, where $a, b \in \mathbb{R}$ and $a \neq 0$. The matrix

$$\alpha := \begin{pmatrix} a & 0 \\ -\frac{b}{a} & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{L})$$

induces a projectivity of $\mathbb{P}(\mathbb{L})$ taking the improper point $\mathbb{L}(1, -a\varepsilon - b\varepsilon^2)$ to

$$\mathbb{L}\Big(\Big(\frac{1}{a} - \frac{b}{a^2}\,\varepsilon\Big)\Big(a + b\varepsilon + \frac{b^2}{a}\,\varepsilon^2, -a\varepsilon - b\varepsilon^2\Big)\Big) = \mathbb{L}(1, -\varepsilon).$$

The action of α on the proper points is the affine transformation $\mathbb{L} \to \mathbb{L} : z \mapsto za - \frac{b}{a}$ which in turn can be extended to a collineation of $\mathbb{P}_3(\mathbb{R})$. Since contact of any order is preserved under collineations, we can apply the results from (a) in order to complete the proof. □

From the affine point of view, the previous results are not satisfying, because they are formulated in projective terms. However, in Section 2 we have explained how one can "see" contact of higher order at f via an affine transformation taking $\widetilde{\mathcal{C}}^\circ$ to $\mathcal{C}^\circ$. Another basic topic is to characterize chains $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ which *touch* at a common improper point. If $\mathcal{C}^\circ$ is an affine line then this means, by definition, that $\mathcal{C}^\circ$ and $\widetilde{\mathcal{C}}$ are parallel lines. If $\mathcal{C}^\circ$ is a parabola then a characterization as in Theorem 2 can be given, but now the parallel projection of $\widetilde{\mathcal{C}}^\circ$ has to arise from $\mathcal{C}^\circ$ under a translation in the direction of ε^2. (This means contact of third order at f.) Likewise, Theorem 5 can be modified as to describe touching chains, by replacing "third order contact" with "fourth order contact". The proofs are left to the reader. The affine space on $\mathbb{L}$ is closely related with the *flag space* (*two-fold isotropic space*), as the triple (f, F, Φ) can be considered as its *absolute flag*. Cf., among others, the papers [4] by H. Brauner. Due to lack of space we have to refrain from presenting here the interesting connections between these two geometries.

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