Weak Linear Mappings – A Survey

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There are various concepts of *structure preserving mappings* in geometry. It is the aim of the present paper to give a survey on geometrical characterizations of some of those mappings. We discuss the results for projective spaces in some detail and report on generalizations to other spaces.

We shall come across *partially defined mappings*. The notation $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ is used in order to point out that φ is a mapping with *domain* $\operatorname{dom}\varphi \subset \mathcal{P}$ and *image set* $\operatorname{im}\varphi \subset \mathcal{P}'$. The *exceptional set* of φ (with respect to $\mathcal{P}$) is given as $\operatorname{ex}\varphi := \mathcal{P} \setminus \operatorname{dom}\varphi$. The mapping φ is said to be *globally defined* (with respect to $\mathcal{P}$) provided that $\operatorname{ex}\varphi$ is empty. The notation $\varphi : \mathcal{P} \to \mathcal{P}'$ is maintained for globally defined mappings only.

If we are given any subset $\mathcal{M} \subset \mathcal{P}$, then $\{X^\varphi \mid X \in \mathcal{M} \cap \operatorname{dom}\varphi\}$ is a well–defined set. By abuse of notation, it is written as $\mathcal{M}^\varphi$. Hence $\mathcal{M}^\varphi = \emptyset$ exactly for $\mathcal{M} \subset \operatorname{ex}\varphi$.

1 Projective Spaces

1.1 Weak Semilinear Mappings of Vector Spaces

Throughout this paper $\mathbf{V}$, $\mathbf{V}'$ denote right vector spaces over (not necessarily commutative) fields K, K', respectively. We generalize the well–known concept of a semilinear mapping as follows: A *weak semilinear mapping*[1] $f : \mathbf{V} \to \mathbf{V}'$ with respect to a mapping $\zeta : K \to K'$ is additive and ζ–homogeneous, i.e.

$$(\mathbf{a}+\mathbf{b})^f = \mathbf{a}^f + \mathbf{b}^f \text{ and } (\mathbf{a}x)^f = \mathbf{a}^f x^\zeta \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbf{V},\ x \in K.$$

If $f \neq 0$, then ζ is a monomorphism which is uniquely determined by f.

Given such an $f \neq 0$ there exists a right vector space $\mathbf{W}$ over $F := K^\zeta$ and a semilinear bijection $g : \mathbf{V} \to \mathbf{W}$ with respect to ζ (regarded as isomorphism $K \to F$). By the universal property of the tensor product $\mathbf{W} \otimes_F K'$, there is a K'–linear mapping $h : \mathbf{W} \otimes_F K' \to \mathbf{V}'$ such that $\mathbf{a}^f = (\mathbf{a}^g \otimes 1)^h$ for all $\mathbf{a} \in \mathbf{V}$.

The one– and two–dimensional subspaces of $\mathbf{V}$ are the points and lines of the projective space on $\mathbf{V}$ which is denoted by $(\mathcal{P}(\mathbf{V}), \mathcal{L}(\mathbf{V}))$. More generally, we put $\mathcal{P}(\mathbf{M}) := \{\mathbf{a}K \mid \mathbf{a} \in \mathbf{M} \setminus \{0\}\}$ for $\mathbf{M} \subset \mathbf{V}$; $\mathcal{P}(\mathbf{W})$ etc. is defined likewise. Each weak semilinear mapping $f : \mathbf{V} \to \mathbf{V}'$ determines a mapping of points

$$\varphi : \mathcal{P}(\mathbf{V}) \rightarrowtail \mathcal{P}(\mathbf{V}'),\ \mathbf{a}K \mapsto (\mathbf{a}^f)K' \text{ for all } \mathbf{a}K \in \mathcal{P}(\mathbf{V} \setminus \ker f). \tag{1}$$

The exceptional set of φ is the subspace $\mathcal{P}(\ker f)$. It follows that φ has the property

$$(\mathcal{X} \vee \mathcal{Y})^\varphi \subset \operatorname{span}(\mathcal{X}^\varphi) \vee \operatorname{span}(\mathcal{Y}^\varphi) \text{ for all subspaces } \mathcal{X}, \mathcal{Y} \subset \mathcal{P}(\mathbf{V}). \tag{2}$$

[1] Such a mapping should not be mixed up with a *generalized semilinear mapping* [42] which yields a *homomorphism* of projective spaces.

We remark that φ–images of subspaces need not be subspaces.

Suppose now that ζ is bijective, whence f is semilinear. If $\mathcal{X} \subset \mathcal{P}$ is a subspace, then so is $\mathcal{X}^\varphi$. Moreover, (2) improves to

$$(\mathcal{X} \vee \mathcal{Y})^\varphi = \mathcal{X}^\varphi \vee \mathcal{Y}^\varphi \text{ for all subspaces } \mathcal{X}, \mathcal{Y} \subset \mathcal{P}(\mathbf{V}). \tag{3}$$

1.2 Definition and Examples of Linear Mappings

In the sequel $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ denote arbitrary projective spaces. Property (3) is adopted in the following purely geometric definition:

A *linear mapping*[2] of projective spaces is a mapping $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ (defined on some subset of $\mathcal{P}$) satisfying the following conditions:

(L1) $(\{X\} \vee \{Y\})^\varphi = \{X\}^\varphi \vee \{Y\}^\varphi$ for all $X, Y \in \mathcal{P}$, $X \neq Y$.

(L2) If $X^\varphi = Y^\varphi$ for distinct points $X, Y \in \operatorname{dom}\varphi$, then there is at least one exceptional point on the line $\{X\} \vee \{Y\}$.

Condition (L2) is only used to rule out the possibility that *all* points of a line are mapped to the same point. There are three possibilities for the image of a line $l \in \mathcal{L}$:

$\#(l \cap \operatorname{ex}\varphi) \geq 2$: Then $l \subset \operatorname{ex}\varphi$, i.e. we have an *exceptional line.*

$\#(l \cap \operatorname{ex}\varphi) = 1$: Then all points of $l \cap \operatorname{dom}\varphi$ are mapped to the same point.

$\#(l \cap \operatorname{ex}\varphi) = 0$: Then $\varphi \mid l$ is injective and $l^\varphi \in \mathcal{L}'$.

We give some examples of linear mappings:

1. The mapping (1) is linear, if $f : \mathbf{V} \to \mathbf{V}'$ is semilinear.
2. Let $\mathcal{S} \subset \mathcal{P}$ be a subspace. Write $\mathcal{P}/\mathcal{S}$ for the set of all subspaces of the form $\mathcal{S} \vee \{X\}$, where $X \in \mathcal{P} \setminus \mathcal{S}$. Then $\mathcal{P}/\mathcal{S}$ is the set of points of the quotient space of $(\mathcal{P}, \mathcal{L})$ modulo $\mathcal{S}$. The *canonical projection*

$$\psi_\mathcal{S} : \mathcal{P} \rightarrowtail \mathcal{P}/\mathcal{S},\ X \mapsto \mathcal{S} \vee \{X\} \text{ for all } X \in \mathcal{P} \setminus \mathcal{S}$$

   is a surjective linear mapping with $\operatorname{ex}\psi_\mathcal{S} = \mathcal{S}$.
3. With $\mathcal{S}$ as above, the *canonical injection* $\iota_\mathcal{S} : \mathcal{S} \to \mathcal{P},\ X \mapsto X$ is a globally defined linear mapping of the projective space on $\mathcal{S}$ in $\mathcal{P}$.
4. A *projection* is based on two complementary subspaces of $\mathcal{S}, \mathcal{T}$ of $(\mathcal{P}, \mathcal{L})$ as follows:

$$\pi : \mathcal{P} \rightarrowtail \mathcal{T},\ X \mapsto X^\pi \text{ with } \{X^\pi\} := (\mathcal{S} \vee \{X\}) \cap \mathcal{T} \text{ for all } X \in \mathcal{P} \setminus \mathcal{S}.$$

   This π is a surjective linear mapping with $\operatorname{ex}\pi = \mathcal{S}$.
5. Any collineation of projective spaces is a globally defined bijective linear mapping.
6. Suppose that $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ and $\varphi' : \mathcal{P}'' \rightarrowtail \mathcal{P}'''$ are linear mappings with $\operatorname{im}\varphi \subset \mathcal{P}''$. Then $\varphi\varphi' : \mathcal{P} \rightarrowtail \mathcal{P}'''$ is also a linear mapping (possibly an empty mapping).
7. Let $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ be linear. The hyperplanes of $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ are, respectively, the points of the dual projective spaces $(\mathcal{P}^*, \mathcal{L}^*)$ and $(\mathcal{P}'^*, \mathcal{L}'^*)$. The *extended* φ–*preimage* of any hyperplane $\mathcal{H}' \subset \mathcal{P}'$ is given by $\operatorname{ex}\varphi \cup \mathcal{H}'^{\varphi^{-1}}$. Define

$$\varphi^T : \mathcal{P}'^* \rightarrowtail \mathcal{P}^*,\ \mathcal{H}' \mapsto \operatorname{ex}\varphi \cup \mathcal{H}'^{\varphi^{-1}} \text{ for all } \mathcal{H}' \in \mathcal{P}'^* \text{ with } \operatorname{im}\varphi \not\subset \mathcal{H}'$$

   This *transpose mapping* of φ is linear [23], [29].

[2] This name has been used in Descriptive Geometry for more than 100 years. Some authors use other terminologies.

1.3 Brauner's Theorem

Theorem 1 (Brauner [7]) *Let $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ be a mapping of projective spaces.*

1. *If φ satisfies (L1), then the exceptional subset $\operatorname{ex}\varphi \subset \mathcal{P}$ and the image set $\operatorname{im}\varphi \subset \mathcal{P}'$ are subspaces.*
2. *If φ satisfies (L1) and $\#\operatorname{im}\varphi \neq 1$, then φ is linear.*
3. *Each linear mapping φ is decomposable into the canonical projection $\mathcal{P} \rightarrowtail \mathcal{P}/\operatorname{ex}\varphi$, a collineation of this quotient space onto the subspace $\operatorname{im}\varphi$, and the canonical injection of $\operatorname{im}\varphi$ in $\mathcal{P}'$.*

As a matter of fact Brauner did not discuss mappings φ satisfying (L1) and $\#\operatorname{im}\varphi = 1$. It is immediate that here $\operatorname{ex}\varphi$ can be any subspace of $(\mathcal{P}, \mathcal{L})$ other than $\mathcal{P}$.

By the second fundamental theorem of projective geometry we obtain:

Corollary 1 *Each linear mapping $\varphi : \mathcal{P}(\mathbf{V}) \rightarrowtail \mathcal{P}(\mathbf{V}')$ such that $\operatorname{im}\varphi$ contains a triangle is induced by a semilinear mapping $f : \mathbf{V} \to \mathbf{V}'$ which is determined to within a non–zero factor in K'.*

The existence of a triangle in $\operatorname{im}\varphi$ is needed in Corollary 1 to avoid "degenerate" cases:

1. If $\operatorname{im}\varphi$ is a line, then such an f needs not exist, since a collineation of 1–dimensional projective spaces is just a bijection of their point sets. However, when $\#K \in \{2, 3, 4\}$, then φ is nevertheless induced by a semilinear mapping.
2. If $\operatorname{im}\varphi$ is a singleton or empty, then K and K' need not be isomorphic. If K and K' are assumed to be isomorphic, then φ can be induced by a semilinear mapping.

We remark that under the assumptions of Corollary 1 the transpose of the semilinear mapping f induces the transpose of φ.

A geometric characterization of linear mappings of real projective spaces is due to Rehbock [48], [49]. Timmermann [54] characterizes the projections in projective spaces by conditions similar to our (L1). [9, 4.5] and a paper by Faure and Frölicher [21] contain proofs of Brauner's Theorem. A characterization of a linear mapping φ of a projective space in its dual space in terms of *two* mappings is given by Faure and Frölicher [23]. The two mappings are φ and $\varphi^T|\mathcal{P}$, where $\mathcal{P}$ is identified with a subspace of the bidual projective space. When these two mappings coincide, then one obtains a possibly degenerate quasipolarity. See also Lenz [34], [35].

1.4 Definition and Examples of Weak Linear Mappings

Formula (2) motivates the following definition: A *weak linear mapping* of projective spaces is a mapping $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ (defined on some subset of $\mathcal{P}$) satisfying

(WL1) $(\{X\} \vee \{Y\})^\varphi \subset \{X\}^\varphi \vee \{Y\}^\varphi$ for all $X, Y \in \mathcal{P}$, $X \neq Y$.

By (WL1), there are four possibilities for the image of a line $l \in \mathcal{L}$:

$\#(l \cap \operatorname{ex}\varphi) \geq 2$: Then $l \subset \operatorname{ex}\varphi$, i.e. we have an *exceptional line*.

$\#(l \cap \operatorname{ex}\varphi) = 1$: Then all points of $l \cap \operatorname{dom}\varphi$ are mapped to the same point.

$\#(l \cap \operatorname{ex}\varphi) = 0$ and all points of l are mapped to the same point.

$\#(l \cap \operatorname{ex}\varphi) = 0$ and $\varphi \mid l$ is injective, but l^φ is not necessarily a line.

Thus now, in contrast to a linear mapping, we do not rule out the possibility that *all* points of a line are mapped onto the same point.

Each mapping $\varphi : \mathcal{P} \to \mathcal{P}'$, which is globally defined, injective, and collinearity–preserving, is a weak linear mapping. If non–collinearity of points is also preserved, then φ is called an *embedding*. An embedding is called *strong*, if independent points in $(\mathcal{P}, \mathcal{L})$ always go over to independent points in $(\mathcal{P}', \mathcal{L}')$.

We give some examples of weak linear mappings:

1. Formula (1) yields a weak linear mapping.
2. Each linear mapping is also a weak linear mapping.
3. Let L/K be a field extension and let $\mathbf{V}$ be a vector space over K. Then

$$\varphi : \mathcal{P}(\mathbf{V}) \to \mathcal{P}(\mathbf{V} \otimes_K L),\ \mathbf{a}K \mapsto (\mathbf{a} \otimes 1)L$$

   is a strong embedding. This φ is a collineation if, and only if, $K = L$ or $\dim \mathbf{V} \leq 1$.
4. The following embedding is not strong (Brezuleanu and Rădulescu [11]): Let L/K be a field extension such that there exist elements $1, y_0, y_1, y_2 \in L$ which are linearly independent in the *left* vector space L over K. Then define

$$\varphi : \mathcal{P}(K^4) \to \mathcal{P}(L^3),\ (a_0,\ a_1,\ a_2,\ a_3)K \mapsto (a_0 + y_0 a_3,\ a_1 + y_1 a_3,\ a_2 + y_2 a_3)L.$$

5. Each product of weak linear mappings is a weak linear mapping.

1.5 A Fundamental Theorem for Weak Linear Mappings

Theorem 2 (Faure and Frölicher [22], Havlicek [31]) *Each weak linear mapping $\varphi : \mathcal{P}(\mathbf{V}) \rightarrowtail \mathcal{P}(\mathbf{V}')$ such that* $\operatorname{im}\varphi$ *contains a triangle is induced by a weak semilinear mapping $f : \mathbf{V} \to \mathbf{V}'$ which is determined to within a non–zero factor in K'.*

We infer from the decomposition of a weak semilinear mapping in 1.1 the following

Corollary 2 *Each weak linear mapping $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ of desarguesian projective spaces such that* $\operatorname{im}\varphi$ *contains a triangle can be decomposed into a strong embedding $\mathcal{P} \to \mathcal{P}''$ in some projective space $(\mathcal{P}'', \mathcal{L}'')$ and a linear mapping $\mathcal{P}'' \rightarrowtail \mathcal{P}'$.*

Corollary 2 does not remain true if the hypothesis on $\operatorname{im}\varphi$ is dropped: If $\operatorname{im}\varphi$ does not contain a triangle of $(\mathcal{P}', \mathcal{L}')$, then $\operatorname{ex}\varphi$ is still a subspace. For each point $Y \in \operatorname{im}\varphi$ the extended preimage $\operatorname{ex}\varphi \cup \{Y\}^{\varphi^{-1}}$ is a subspace. Thus $\mathcal{P}/\operatorname{ex}\varphi$ is partitioned into subspaces. However, those subspaces are not necessarily isomorphic. Conversely, any subspace $\mathcal{S} \subset \mathcal{P}$ and any partition of $\mathcal{P}/\mathcal{S}$ into subspaces determines a weak linear mapping in some projective line.

Corollary 3 *Let $\varphi : \mathcal{P}(\mathbf{V}) \rightarrowtail \mathcal{P}(\mathbf{V}')$ be given as in Theorem 2. Furthermore, assume that each monomorphism $K \to K'$ is surjective. Then φ is a linear mapping.*

For example, each monomorphism of $\mathbb{R}$ is surjective [1, p. 88], whereas $\mathbb{C}$ admits non–surjective monomorphisms [1, p. 114].

A recent result due to Kreuzer [33] says that each pappian projective space is embeddable in a pappian projective plane. Other results on embeddings and specific examples are due to Benz [1, p. 109], Brezuleanu and Rădulescu [11], Brown [13], Carter and Vogt [16], [17], Havlicek [30], and Limbos [37], [39], [40].

Another special class of weak linear mappings are *semicollineations* of projective spaces, i.e. globally defined bijective mappings which preserve the collinearity, but not necessarily the non–collinearity of points. CECCHERINI [18] has given an example of a semicollineation of a 4–dimensional projective space onto a non–desarguesian projective plane. Cf. also BERNARDI and TORRE [2], and MAROSCIA [43]. It seems to be an open problem, if each semicollineation of desarguesian projective spaces with dimensions ≥ 2 is a collineation.

1.6 Local Characterizations

The definition of a linear mapping φ uses all points of $\mathcal{P}$. The following "local" results characterize a linear mapping only in terms of its domain or in terms of a subset of its domain.

Firstly, we state the following variant of axiom (L1):

(L1') $(\{X\} \vee \{Y\})^\varphi = \{X\}^\varphi \vee \{Y\}^\varphi$ for all $X, Y \in \operatorname{dom}\varphi$, $X \neq Y$.

Theorem 3 (SÖRENSEN [52]) *Let $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ be a mapping of projective spaces satisfying (L1'). Then the following conditions hold true:*

1. *The image set* $\operatorname{im}\varphi \subset \mathcal{P}'$ *is a subspace.*
2. *If* $\operatorname{span}(\operatorname{ex}\varphi) \neq \mathcal{P}$ *and if* $\#\operatorname{im}\varphi \geq 2$*, then the exceptional subset* $\operatorname{ex}\varphi \subset \mathcal{P}$ *is a subspace.*
3. *If* $\operatorname{span}(\operatorname{ex}\varphi) \neq \mathcal{P}$ *and if* $\operatorname{im}\varphi$ *contains a triangle, then* φ *is a linear mapping.*

The subsequent examples illustrate that the assumptions in Theorem 3 are essential:

1. Let $\varepsilon : \mathcal{P}' \to \mathcal{P}$ be a non–surjective strong embedding with $\mathcal{P}' \neq \emptyset$, e.g., the canonical injection of a proper subspace $\mathcal{P}' \subset \mathcal{P}$. Define $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ as inverse mapping of ε with domain $\mathcal{P}'^{\varepsilon}$. This φ satisfies (L1'), but $\operatorname{ex}\varphi \subset \mathcal{P}$ is not a subspace.
2. Let $\#\mathcal{P} > 1$ and $\#\mathcal{P}' = 1$. There exists an $\mathcal{M} \subset \mathcal{P}$ other than a subspace. Define $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ by $\operatorname{ex}\varphi = \mathcal{M}$. Hence $\#\operatorname{im}\varphi = 1$ and (L1') is trivially true.
3. Let $(\mathcal{P}, \mathcal{L})$ be a projective space containing a line t_1 with more than three points. Choose a complement $\mathcal{S}_1$ of t_1 and let $\mathcal{S}$ be a hyperplane containing $\mathcal{S}_1$. Write $\varphi_1 : \mathcal{P} \rightarrowtail t$ for the projection onto t with exceptional subspace $\mathcal{S}_1$ and put $t := t_1 \backslash \mathcal{S}$. The mapping $\varphi : \mathcal{P} \setminus \mathcal{S} \to t,\ X \mapsto X^{\varphi_1}$ satisfies (L1') with $(\mathcal{P}', \mathcal{L}') = (t, \{t\})$, but clearly (L1) is violated. Thus φ is not a linear mapping [52].

Secondly, axioms (L1) and (L2) are modified as follows:

(L1") X, Y, Z collinear implies $X^\varphi, Y^\varphi, Z^\varphi$ collinear, for all $X, Y, Z \in \operatorname{dom}\varphi$.

(L2") If $X^\varphi = Y^\varphi$ for distinct points $X, Y \in \operatorname{dom}\varphi$, then $\varphi \mid ((\{X\} \vee \{Y\}) \cap \operatorname{dom}\varphi)$ is a constant mapping.

Moreover, we need some topological tools: A projective space $(\mathcal{P}, \mathcal{L})$ is said to carry a *linear topology*, if each line $x \in \mathcal{L}$ is endowed with a non–trivial[3] topology T_x such that all perspectivities between intersecting lines are continuous. A subset $\mathcal{O}$ of $\mathcal{P}$ is called *linearly open*, if $\mathcal{O} \cap x$ is open in the topological space (x, T_x) for all lines $x \in \mathcal{L}$ [3]. If $(\mathcal{P}, \mathcal{L})$ is a topological projective space [14, Ch. 23], then the induced topologies on the

[3]We rule out the coarsest and the finest topology.

lines yield a linear topology. Each open set $\mathcal{M} \subset \mathcal{P}$ is also linearly open. However, there are linear topologies that do not arise in this way. An example is given by the *cofinite topology* in a projective space with infinite order: A subset m of a line a is defined to be open, if $a \setminus m$ is finite.

Theorem 4 (FRANK [24]) *Let $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ be projective spaces satisfying the minor axiom of Desargues. Suppose that $\varphi : \mathcal{P} \rightarrowtail \mathcal{P}'$ is a mapping satisfying (L1") and (L2") such that* $\operatorname{im}\varphi$ *contains a triangle. Each of the following conditions is sufficient for the existence of a unique linear mapping $\overline{\varphi} : \mathcal{P} \rightarrowtail \mathcal{P}'$ extending φ:*

(a) $(\mathcal{P}, \mathcal{L})$ admits a linear topology such that $\operatorname{dom}\varphi$ *is linearly open. Moreover, there exists a line $l \in \mathcal{L}$ such that* $(\operatorname{dom}\varphi \cap l)^{\varphi}$ *contains non–empty open set with respect to some linear topology of $(\mathcal{P}', \mathcal{L}')$.*

(b) $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ are spaces of the same finite order N. Moreover, $x \cap \operatorname{dom}\varphi \neq \emptyset$ implies $\#(x \cap \operatorname{dom}\varphi) > \frac{2}{3}N$ for all $x \in \mathcal{L}$.

Theorem 4 generalizes a result of LENZ [36] for real projective spaces. There are also several theorems in the literature giving local characterizations of embeddings of projective planes and spaces. Refer to BOGNÁR and KERTÉSZ [6], BREZULEANU and RĂDULESCU [12], and the references given in [24].

2 Other Spaces

2.1 Partial Linear Spaces and Linear Spaces

The geometric conditions (L1), (L2), (L1') etc. still do make sense in a wider context.

Let $(\mathcal{P}, \mathcal{L})$ be a pair consisting of a *point set* $\mathcal{P}$ and a set $\mathcal{L} \subset 2^{\mathcal{P}}$ whose elements are called *lines*. If any two distinct points are on at most one common line, and if each line contains at least two points, then $(\mathcal{P}, \mathcal{L})$ is called a *partial linear space*. A *linear space* is a partial linear space such that any two distinct points are on a common line.

SÖRENSEN [52] has discussed mappings between linear spaces $(\mathcal{P}, \mathcal{L})$ and $(\mathcal{P}', \mathcal{L}')$ satisfying (L1). Given $\mathcal{M} \subset \mathcal{P}$ the *trace space* $(\mathcal{M}, \mathcal{L}_{\mathcal{M}})$, which is defined by putting

$$\mathcal{L}_{\mathcal{M}} := \{l \cap \mathcal{M} \mid l \in \mathcal{L},\ \#(l \cap \mathcal{M}) \geq 2\},$$

is also a linear space. Thus, if a linear mapping is not globally defined, one can go over to the trace space on its domain. Hence it means no loss of generality, that all mappings in [52] are assumed to be globally defined.

Examples of semicollineations of linear spaces and necessary conditions for a semicollineation of linear spaces to be a collineation are due to KREUZER [32]. There is a widespread literature on embeddings of linear spaces in projective spaces [14, Ch. 6].

2.2 Affine Spaces

The following seems to be part of the folklore: Suppose that a globally defined mapping φ of the affine space on $\mathbf{V}$ (over K with $\#K > 2$) in the affine space on $\mathbf{V}'$ is satisfying (L1). If $\operatorname{im}\varphi$ contains a triangle, then $\varphi - 0^{\varphi}$ is a semilinear mapping $\mathbf{V} \to \mathbf{V}'$.

This result follows from Theorem 4 by going over to the projective closures and, for infinite order, by applying their cofinite topology. See SÖRENSEN [52] for a direct proof.

Globally defined mappings of affine spaces with property (WL1) have been discussed by ZICK [58], [59]. Since such a mapping needs not preserve parallelity of lines, *fractional weak semilinear mappings* have been introduced to obtain an algebraic description.

Even an *embedding* φ of the affine space on $\mathbf{V}$ in the affine space on $\mathbf{V}'$ has in general not the property that $\varphi - 0^\varphi$ is a weak semilinear mapping. One class of examples is given by spaces over GF(2). A more interesting example is the embedding of an affine plane over GF(3) in the complex affine plane which is based on the nine inflections of an elliptic cubic curve, another example is due to BENZ [1, p. 113]. See BICHARA and KORCHMÁROS [4], CARTER and VOGT [16], [17], LIMBOS [37], [38], [39], [40], OSTROM and SHERK [44], RIGBY [50], THAS [53]. SCHAEFFER [51] has given a local characterization of some embeddings which can also be found in [1, 3.2].

2.3 Grassmann Spaces

Write ${}^d\mathcal{P}$ for the set of all d–flats (i.e. d–dimensional subspaces) of an n–dimensional projective space $(\mathcal{P}, \mathcal{L})$ $(1 \leq d \leq n-2)$ and ${}^d\mathcal{L}$ for the set of all pencils of d–flats. Then $({}^d\mathcal{P}, {}^d\mathcal{L})$ is a partial linear space, called *Grassmann space*. Two distinct "points" ${}^dX, {}^dY \in {}^d\mathcal{P}$ are "collinear" if, and only if, ${}^dX \cap {}^dY$ is a $(d-1)$–flat. The Grassmann space $({}^d\mathcal{P}, {}^d\mathcal{L})$ is covered by one–dimensional projective spaces, namely the pencils of d–flats.

A mapping $\varphi : {}^d\mathcal{P} \rightarrowtail \mathcal{P}'$ in the point set of a projective space $(\mathcal{P}', \mathcal{L}')$ is called *linear*, if the restriction of φ to each pencil is a linear mapping of projective spaces [26].

If $(\mathcal{P}, \mathcal{L})$ is pappian, then each linear mapping $\varphi : {}^d\mathcal{P} \rightarrowtail \mathcal{P}'$ determines a mapping $\widehat{\varphi} : G_{n,d} \rightarrowtail \mathcal{P}'$ of the associated Grassmann variety $G_{n,d}$ (cf. BURAU [15]) such that the restriction to each line $l \subset G_{n,d}$ is linear. By HAVLICEK [27], this $\widehat{\varphi}$ extends to a unique linear mapping of the ambient space of $G_{n,d}$. A weaker result is due to WELLS [55].

If $(\mathcal{P}, \mathcal{L})$ is not pappian, then Grassmann varieties are not available due to a non–existence theorem [27], and little seems to be known here.

A *geometric hyperplane* of $({}^d\mathcal{P}, {}^d\mathcal{L})$ is a subset of ${}^d\mathcal{P}$ which intersects each pencil of d–flats in exactly one or in all elements. The geometric hyperplanes are exactly the exceptional sets of linear mappings $\varphi : {}^d\mathcal{P} \rightarrowtail \mathcal{P}'$ with $\#\mathrm{im}\,\varphi = 1$. Hence in the pappian case all geometric hyperplanes arise as hyperplane sections of the associated Grassmann variety [27, 4.5]. All geometric hyperplanes in the non–pappian case have been descibed by HALL and SHULT [25].

It seems that ECKHART [19] and REHBOCK [48], [49] have been the first geometers to discuss linear mappings of the Grassmann space formed by the lines of the real projective 3–space. See BRAUNER [7], [8], [10], HAVLICEK [28], and LÜBBERT [41] for further references and *kinematic line mappings*. ZANELLA [56] has investigated linear mappings of Grassmann spaces which are globally defined and injective. He has given sufficient conditions for the image of such a mapping to be projectively equivalent to the corresponding Grassmann variety and rather sophisticated examples where this is not the case. Cf. also BICHARA and ZANELLA [5].

2.4 Product Spaces

Another partial linear space is the *product space* of two projective spaces. If both spaces have isomorphic commutative ground fields, then their product space can be represented as a *Segre variety*. Linear mappings of a product space in a projective space may be defined as for Grassmann spaces. A thorough discussion of globally defined injective linear mappings can be found in a paper by ZANELLA [57]. Unfortunately, we cannot give the details here due to the lack of space. Let us just remark that the situation is more complicated than for Grassmann spaces. Cf. also BICHARA and ZANELLA [5].

2.5 Lattice Geometries

A significant generalization of linear mappings to *projective* and *affine lattice geometries* has been given by PFEIFFER [45], [46], and PFEIFFER and SCHMIDT [47]. See furthermore FAIGLE [20].

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