Pencilled regular parallelisms

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Basic notions

- In $\mathrm{PG}(n, \mathbb{K})$, the point set is written as $\mathcal{P}_n$ and the line set is denoted by $\mathcal{L}_n$.
- A *spread* of $\mathrm{PG}(3, \mathbb{K})$ is a partition of $\mathcal{P}_3$ by (disjoint) lines.
- A *parallelism* on $\mathrm{PG}(3, \mathbb{K})$ is a partition of $\mathcal{L}_3$ by (disjoint) spreads.
- The spreads of a parallelism are called *parallel classes*.
- Parallelisms are known as *packings* when $\mathbb{K}$ is a finite field.

See, among others, Hirschfeld [13], Johnson [14], [15], Karzel and Kroll [16], and Knarr [17].

Additional properties

In PG$(3, \mathbb{K})$, we shall only be concerned with spreads and parallelisms that satisfy some additional properties.

- A spread is are called *regular* if it is closed under reguli.
- A parallelism is called *regular* if all its parallel classes are regular spreads.

Regular spreads of $\mathrm{PG}(3, \mathbb{K})$ and external lines to H_5

Let $\mathcal{C}$ be a regular spread of $\mathrm{PG}(3, \mathbb{K})$.

- The **Klein correspondence** λ sends $\mathcal{C}$ to an elliptic subquadric $\lambda(\mathcal{C})$ of the Klein quadric H_5.
- The **span** of $\lambda(\mathcal{C})$ is a solid of $\mathrm{PG}(5, \mathbb{K})$.
- The **polarity** π_5 of the Klein quadric sends the solid spanned by $\lambda(\mathcal{C})$ to

$$\pi_5\big(\mathrm{span}\,\lambda(\mathcal{C})\big),$$

which is an external line to the Klein quadric.

The bijection γ

Theorem 1.

Let $\boldsymbol{C}$ *denote the set of all regular spreads of* $\mathrm{PG}(3,\mathbb{K})$ *and write* $\mathcal{Z}$ *for the set of all lines of* $\mathrm{PG}(5,\mathbb{K})$ *that are external to the Klein quadric* H_5*. Then the mapping*

$$\gamma\colon \boldsymbol{C} \to \mathcal{Z} : \mathcal{C} \mapsto \pi_5\big(\operatorname{span}\lambda(\mathcal{C})\big)$$

is bijective.

Hfd line sets

Let $\boldsymbol{P}$ be a regular parallelism on $\mathrm{PG}(3, \mathbb{K})$. Below we follow Betten and Riesinger [3].

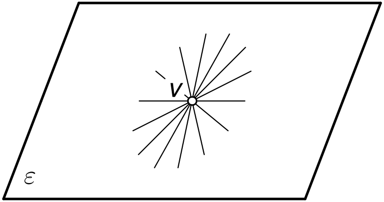
- The λ-image of $\boldsymbol{P}$ is a *hyperflock* of the Klein quadric H_5, that is, a partition of H_5 by (disjoint) elliptic subquadrics.
- The γ-image of $\boldsymbol{P}$ is a *hyperflock determining line set* with respect to the Klein quadric, that is, a set $\mathcal{H}$ with the following properties:

1. $\mathcal{H} \subset \mathcal{L}_5$ consists of lines that are external to the Klein quadric.
2. Each tangent hyperplane of the Klein quadric contains exactly one line of $\mathcal{H}$.

Such a set $\mathcal{H}$ will shortly be called an *hfd line set*.

Pencilled regular parallelisms (H. and Riesinger [12])

Notation.
For any incident point-plane pair (v, ε) we denote by $\mathcal{L}[v, \varepsilon]$ the pencil of lines with vertex v and plane ε.



Definition 1.

An hfd line set $\mathcal{H}$ is said to be *pencilled* if each element of $\mathcal{H}$ belongs to at least one pencil of lines contained in $\mathcal{H}$.

Definition 2.

A regular parallelism $\boldsymbol{P}$ on $\mathrm{PG}(3, \mathbb{K})$ is called *pencilled* if the hfd line set $\gamma(\boldsymbol{P})$ is pencilled.

Construction of pencilled hfd line sets

Theorem 2.

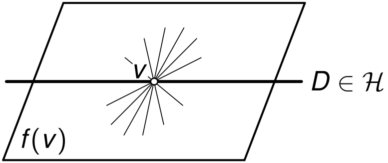
In PG$(5, \mathbb{K})$, *let* D *be a line such that*

$$\mathcal{E}_D := \left\{\varepsilon \subset \mathcal{P}_5 \mid D \subset \varepsilon \text{ and } \varepsilon \text{ is an external plane to } H_5\right\}$$

is non-empty. Then, upon choosing any mapping $f\colon D \to \mathcal{E}_D$, *the union*

$$\bigcup_{v \in D} \mathcal{L}[v, f(v)] =: \mathcal{H}$$

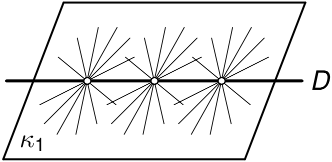
is a pencilled hfd line set.



Example

Suppose that the mapping $f\colon D \to \mathcal{E}_D$ in Theorem 2 is constant.

- The image of f contains a single plane, say κ_1.
- $\mathcal{H}$ is the plane of lines in κ_1.



- $\gamma^{-1}(\mathcal{H})$ is a *Clifford parallelism*.
  These parallelisms are commonly defined in various ways; see Betten and Riesinger [4], Blunck, Pianta and Pasotti [6], Karzel and Kroll [16], or H. [8], [9], [10], [11].
  Cf. also Blunck, Knarr, Stroppel and Stroppel [5].

Example

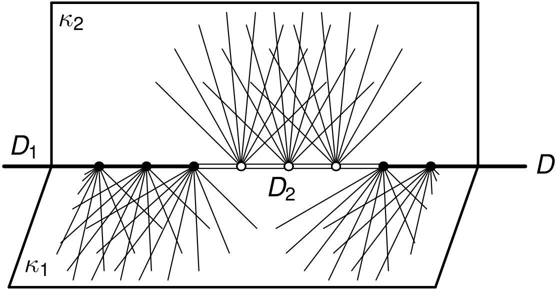
Let the image of the mapping f in Theorem 2 consist of two distinct planes κ_1 and κ_2.

- The mapping f decomposes the line D into two non-empty subsets D_1 and D_2, namely the pre-images of κ_1 and κ_2, respectively.
- The corresponding hfd line set can be written in the form

$$\Big(\bigcup_{v\in D_1}\mathcal{L}[v,\kappa_1]\Big)\cup\Big(\bigcup_{v\in D_2}\mathcal{L}[v,\kappa_2]\Big).$$

Example (cont.)

- Over the real numbers, f can be chosen in such a way that D_1 is a connected component of D with respect to the standard topology in $\mathrm{PG}(5, \mathbb{R})$. Then D_2 is also connected.



Main theorem on pencilled hfd line sets

Theorem 3.

In $\mathrm{PG}(5, \mathbb{K})$*, any hfd line set admits at least one construction as in Theorem 2.*

Existence of pencilled regular parallelisms

Theorem 4.

Given any field $\mathbb{K}$ the following assertions are equivalent.

1. *In* $\mathrm{PG}(3,\mathbb{K})$ *there exists a Clifford parallelism.*
2. *There exists an algebra* $\mathbb{H}$ *over the field* $\mathbb{K}$ *such that one of the following conditions,* (A) *or* (B), *is satisfied:*

   (A) $\mathbb{H}$ *is a quaternion skew field with centre* $\mathbb{K}$.

   (B) $\mathbb{H}$ *is an extension field of* $\mathbb{K}$ *with degree* $[\mathbb{H}:\mathbb{K}]=4$ *and such that* $a^2\in\mathbb{K}$ *for all* $a\in\mathbb{H}$.
3. *In* $\mathrm{PG}(3,\mathbb{K})$ *there exists a pencilled regular parallelism that is not Clifford.*

Conclusion

- Hirschfeld [13, p. 69], who follows Conwell [7], uses hfd line sets to construct regular parallelisms of $\mathrm{PG}(3,2)$.
  (Hirschfeld's terminology is different from ours.)
  These parallelisms give rise to solutions of Kirkman's Fifteen Schoolgirls problem (1850).
- Further examples of hfd line sets (pencilled or not) in $\mathrm{PG}(5,\mathbb{R})$ can be found in Betten and Riesinger [1], [2, Ex. 16 and 22], [3], and Löwen [18]. Many of these hfd line sets satisfy additional topological conditions.
- If $\mathbb{K}$ has characteristic two then hfd line sets feature several additional properties [12].

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