Preserver Problems in Geometry

Hans Havlicek

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Introduction
We consider a $k$-dimensional left vector space $V$ over a (not necessarily commutative) field $F$, and denote by

$$G_m(V)$$

the Grassmannian of all $m$-subspaces of the vector space $V$.

Thereby is always assumed that $k$ and $m$ are integers satisfying

$$1 \leq m \leq k - 1.$$
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Since proper skew fields are included, we cannot use tools from exterior algebra.
Standard Transformations

The following mappings on vectors determine the *standard transformations* of the Grassmannian $G_m(V)$. 
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- **Semilinear bijections** $f : V \rightarrow V$:
  
  $X \mapsto X^f := \{ v^f | v \in X \}$;

  there is a unique *automorphism* of $K$ accompanying $f$. 
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  $$X \mapsto X^f := \{v^f | v \in X\};$$

  there is a unique automorphism of $K$ accompanying $f$.

- **For $k = 2m$ only**: Semilinear bijections $f : V \rightarrow V^*$, where $V^*$ denotes the dual of $V$:

  $$X \mapsto \text{annihilator of } X^f.$$  

  Any such $f$ is accompanied by a unique antiautomorphism of $K$. (There are skew fields admitting no antiautomorphism.)
Problem

Characterise the standard transformations of Grassmannians from the previous slide by as few geometric invariants as possible.
The Grassmann Graph

- Subspaces $X_1, X_2 \in \mathcal{G}_m(V)$ are called adjacent (in symbols: $X_1 \sim X_2$) if

\[ \dim(X_1 \cap X_2) = m - 1. \]
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- We consider $\mathcal{G}_m(V)$ as the set of vertices of an *undirected graph*, called the *Grassmann graph*. Its edges are the (unordered) pairs of adjacent $m$-subspaces.
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The automorphisms of the Grassmann graph are precisely those bijections of $\mathcal{G}_m(V)$ that preserve adjacency in both directions.
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- The automorphisms of the Grassmann graph are precisely those bijections of \( G_m(V) \) that preserve adjacency in both directions.

- We shall often assume \( 2 \leq m \leq k - 2 \) in order to avoid a complete graph.
Theorem (W. L. Chow (1949) [7])

Let \( 2 \leq m \leq k - 2 \).

A bijective mapping

\[
\varphi : \mathcal{G}_m(V) \rightarrow \mathcal{G}_m(V) : X \mapsto X^\varphi
\]

preserves adjacency in both directions, i.e.,

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X_1 \sim X_2 \Leftrightarrow X_1^\varphi \sim X_2^\varphi \quad \text{for all } X_1, X_2 \in \mathcal{G}_m(V),
\]

if, and only if, \( \varphi \) is a standard transformation.
The Matrix Approach
Each element of the Grassmannian $G_m(F^k)$ can be viewed as the left row space of a matrix $A|B$ with left row rank $m$, where $A \in F^{m \times (k-m)}$, $B \in F^{m \times m}$, and vice versa. We let $n := k - m$. 
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- Let $\text{rk}(A|B) = m$. Then $A|B$ and $A'|B'$ have the same left row space, if and only if, there is a $T \in \text{GL}_m(F)$ with

$$A' = TA \quad \text{and} \quad B' = TB.$$
Projective Matrix Spaces

Each element of the Grassmannian $G_m(F^k)$ can be viewed as the left row space of a matrix $A|B$ with left row rank $m$, where $A \in F^{m \times (k-m)}$, $B \in F^{m \times m}$, and vice versa. We let $n := k - m$.

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- One may consider a matrix pair
  $$(A, B) \in F^{m \times n} \times F^{m \times m} \quad \text{with} \quad \text{rk}(A|B) = m$$
  as left homogeneous coordinates of an element of $G_m(F^k)$.  

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- $G_m(F^k)$ is also called the point set of the \textit{projective space of $m \times n$ matrices} over $F$. 
An Embedding

We have an injective mapping:

\[ F^{m \times n} \rightarrow F^{m \times k} \rightarrow G_m(F^k) \]

\[ A \mapsto A | I_m \mapsto \text{left rowspace of } A | I_m \]

Here \( I_m \) denotes the \( m \times m \) identity matrix over \( F \).
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- Matrices \( A_1, A_2 \in F^{m \times n} \) are called \textit{adjacent} (in symbols: \( A_1 \sim A_2 \)) if \( \text{rk}(A_1 - A_2) = 1 \).
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- Matrices from \( F^{m \times n} \) are adjacent precisely when their images in \( G_m(F^k) \) are adjacent.
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- **For arbitrary** $m, n$:

  $$A \mapsto P \cdot A^\sigma \cdot Q + R,$$

  where $P \in \text{GL}_m(F)$, $Q \in \text{GL}_n(F)$, $R \in F^{m \times n}$, and $\sigma$ is an automorphism of $F$. 
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  where \( P \in \text{GL}_m(F), \ Q \in \text{GL}_n(F), \ R \in F^{m \times n}, \) and \( \sigma \) is an automorphism of \( F \).

- **For** \( m = n \) **only**:

  \[ A \mapsto P \cdot (A^\sigma)^T \cdot Q + R, \]

  where \( P, Q, R \) are as above, \( \sigma \) is an antiautomorphisms of \( F \), and \( T \) denotes transposition.
Theorem (L. K. Hua (1951) [10])

Let \( m, n \geq 2 \).

A bijective mapping \( \varphi : F^{m \times n} \rightarrow F^{m \times n} : A \mapsto A^\varphi \) preserves adjacency in both directions, i.e.,

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A_1 \sim A_2 \iff A_1^\varphi \sim A_2^\varphi \quad \text{for all } A_1, A_2 \in F^{m \times n},
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For \( \#F = 2 \) the result was established by Z.-X. Wan and Y.-X. Wang (1962, in Chinese); cf. [27].
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A link between the theorems of Chow and Hua is provided by the theory of spine spaces; see K. Prażmowski and M. Żynel [24].
Other Matrix Spaces

and Related Topics
Transformations on Symmetric Matrices

Similar results hold (up to certain exceptions) for bijections that preserve adjacency in both directions for the following spaces:

For any commutative field $F$:
- The space of $m \times m$ symmetric matrices over $F$. 
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- The space of $m \times m$ symmetric matrices over $F$.

- The space of maximal totally isotropic subspaces of $F^{2m}$ w.r.t. a symplectic form.

This is also called the *projective space of* $m \times m$ *symmetric matrices* over $F$. 
Transformations on $\sigma$-Hermitian Matrices

Similar results hold (up to certain exceptions) for bijections that preserve adjacency in both directions for the following spaces:

For any field $F$ that admits an antiautomorphism $\sigma$ of order two:

- The space of $m \times m \sigma$-Hermitian matrices over $F$.

- The space of maximal totally isotropic subspaces of $F^{2m}$ w.r.t. a particular skew $\sigma$-Hermitian sesquilinear form.

This is also called the *projective space of $m \times m \sigma$-Hermitian matrices* over $F$. 
Transformations on Alternating Matrices

Similar results hold (up to certain exceptions) for bijections that preserve adjacency in both directions for the following spaces:

For any commutative field $F$:

- The space of $m \times m$ alternating matrices over $F$.
  Adjacency is not inherited from $F^{n \times n}$.

- The space of maximal totally singular subspaces of $F^{2n}$ w.r.t.
  a particular quadratic form.
  This is also called the projective space of $m \times m$ alternating
  matrices over $F$. 
Monographs and Surveys

- W. Benz: *Geometrische Transformationen* (1992) [1].
- W. Benz: *Real Geometries* (1994) [2].
- J. Lester: Distance preserving transformations (1995) [19].
- P. Šemrl: Maps on matrix and operator algebras (2006) [26].

Applications: light cone preservers, Jordan homomorphisms, ...
Chow’s Theorem
Key Questions

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2. Is it possible to further weaken the assumptions?

3. Why adjacency, why not ...?

4. Is there a unified theory?
Grassmannians Revisited

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- 0-flats are called *points*. 
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- 0-flats are called **points**.
- 1-flats are called **lines**.
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- \( \ldots \)
- \((k - 2)\)-flats are called *hyperplanes*.
Techniques: Maximal Cliques

For \(2 \leq m \leq k - 2\) the maximal cliques of the Grassmann graph \((G_m(V), \sim)\) fall into two classes.

- A star is the set of all \((m - 1)\)-flats through a fixed \((m - 2)\)-flat, called the centre of the star.
- A top is the set of all \((m - 1)\)-flats within a fixed \(m\)-flat, called the carrier of the top.
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$k = 4, m = 2$
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- A *top* is the set of all $(m - 1)$-flats within a fixed $m$-flat, called the *carrier* of the top.
Techniques: Intersection of Maximal Cliques

- The intersection of two distinct stars (tops) is either empty or it contains a single \((m - 1)\)-flat.

- The intersection of a star and a top is either empty or it contains at least three \((m - 1)\)-flats.

The second case characterises stars (tops) with adjacent centres (carriers).
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The second case yields a pencil of \((m - 1)\)-flats.
Techniques: Collineations

**Fundamental Theorem of Projective Geometry**

All collineations between the point sets of projective spaces on vector spaces $V, V'$ of dimension $\geq 3$ stem from semilinear bijections $V \to V'$, and vice versa.
Proof of Chow’s Theorem

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Proof of Chow’s Theorem

The proof of Chow’s theorem is essentially based on:

- the intersection properties of maximal cliques,

- an recursion argument,

- the fundamental theorem of projective geometry.
Theorem (R. Westwick (1974) [28], W. I. Huang (1998) [13])

Let \( 2 \leq m \leq k - 2 \).

A bijective mapping

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\varphi : \mathcal{G}_m(V) \rightarrow \mathcal{G}_m(V) : X \mapsto X^\varphi
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preserves adjacency, i. e.,

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X_1 \sim X_2 \Rightarrow X_1^\varphi \sim X_2^\varphi \quad \text{for all} \quad X_1, X_2 \in \mathcal{G}_m(V),
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if, and only if, \( \varphi \) is a standard transformation.
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For \( m = 2 \) (Grassmannians of lines) see also H. Brauner [6] in combination with H. H. [8].
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A (rather intricate) example of an adjacency preserving bijection $G_2(F^4) \rightarrow G_2(F'^3)$ is due to A. Kreuzer [18].
Westwick’s proof runs along the lines of Chow. Huang’s reasoning is quite different. Her proof is based on a detailed study of maximal distances between a single element and certain subsets of the Grassmannian $G_m(V)$. 
Techniques: Distances

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Its basic properties are:

- $\text{dist}(X, Y) = s \iff \dim(X \cap Y) = m - s$. 
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- $\text{dist}(X, Y) = s \iff \dim(X \cap Y) = m - s$.

- The diameter of the Grassmann graph $G_m(V)$ equals

$$\text{diam} \ G_m(V) = \min\{m, k - m\}.$$
Theorem (M.-H. Lim (2010) [20])

Let \( 2 \leq m \leq k - 2 \) and chose an integer \( s \) such that

\[
1 \leq s < \text{diam}\, G_m(V).
\]

A surjective mapping

\[
\varphi : G_m(V) \rightarrow G_m(V) : X \mapsto X^\varphi
\]

satisfies

\[
\text{dist}(X_1, X_2) \leq s \iff \text{dist}(X_1^{\varphi}, X_2^{\varphi}) \leq s \quad \text{for all} \quad X_1, X_2 \in G_m(V),
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if, and only if, \( \varphi \) is a standard transformation.
Techniques: Balls of Radius $s$

For each subset $\mathcal{T} \subset G_m(V)$ let

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Then for all $X_1, X_2 \in G_m(V)$ with $1 \leq \text{dist}(X_1, X_2) \leq s$ the following characterisations hold:

- $\text{dist}(X_1, X_2) \neq 1 \iff (\{X_1, X_2\}^{[s]})^{[s]} = \{X_1, X_2\}$.
- $\text{dist}(X_1, X_2) = 1 \iff (\{X_1, X_2\}^{[s]})^{[s]} \text{ has at least three elements. (It is a pencil).}$
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\]

Then for all \( X_1, X_2 \in \mathcal{G}_m(V) \) with \( 1 \leq \text{dist}(X_1, X_2) \leq s \) the following characterisations hold:

- \( \text{dist}(X_1, X_2) \neq 1 \iff (\{X_1, X_2\}^{[s]}{[s]} = \{X_1, X_2\} \).  
- \( \text{dist}(X_1, X_2) = 1 \iff (\{X_1, X_2\}^{[s]}{[s]} \text{ has at least three elements. (It is a pencil)}\).
Techniques: Balls of Radius $s$

For each subset $\mathcal{T} \subset G_m(V)$ let

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Then for all $X_1, X_2 \in G_m(V)$ with $1 \leq \text{dist}(X_1, X_2) \leq s$ the following characterisations hold:

- $\text{dist}(X_1, X_2) \neq 1 \iff (\{X_1, X_2\}[^s])[^s] = \{X_1, X_2\}$.
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A more general example:

$\text{dist}(X_1, X_2) = 2, \ s = 1$

$k = 4, \ m = 2$
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\[ k = 4, \ m = 2 \] 

\[ \text{dist}(X_1, X_2) = 1, \ s = 1 \]
Corollary (Lim’s theorem for $s = d - 1$)

Let $2 \leq m \leq k - 2$ and define $d := \text{diam} \mathcal{G}_m(V)$. A surjective mapping 
\[
\varphi : \mathcal{G}_m(V) \rightarrow \mathcal{G}_m(V) : X \mapsto X^\varphi
\]
satisfies 
\[
\text{dist}(X_1, X_2) = d \iff \text{dist}(X_1^\varphi, X_2^\varphi) = d \quad \text{for all} \quad X_1, X_2 \in \mathcal{G}_m(V),
\]
if, and only if, $\varphi$ is a standard transformation.
Let $2 \leq m \leq k - 2$ and define $d := \text{diam } G_m(V)$. A surjective mapping
\[ \varphi : G_m(V) \to G_m(V) : X \mapsto X^\varphi \]
satisfies
\[ \text{dist}(X_1, X_2) = d \Leftrightarrow \text{dist}(X_1^\varphi, X_2^\varphi) = d \quad \text{for all } X_1, X_2 \in G_m(V), \]
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Lim’s result generalises previous work on diameter preservers by A. Blunck, W. I. Huang, M. Pankov, and H. H. [3], [15], [9].
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Lim’s result generalises previous work on diameter preservers by A. Blunck, W. I. Huang, M. Pankov, and H. H. [3], [15], [9].

It overlaps with a characterisation of (not necessarily surjective) distance preserving mappings due to J. Kosiorek, A. Matraś, and M. Pankov [17], [22].
Final Remarks

- W. I. Huang [14] generalised Lim’s result to wide class of graphs satisfying certain axioms.
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- For **Grassmannians over rings** refer to L. P. Huang [11], [12].
Serdecznie dziękuję za zaproszenie i za Państwa uwagę!
The bibliography focusses on preserver problems for Grassmannians, and includes only a few items of related work.
References (cont.)


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