Designs:

t-(v, k, λ) - design ... (P, B)

blocks

blocks per t points

points

points per block

number of points

Examples:

Projective planes : 2-$(n^2+n+1,\ n+1,\ 1)$

Affine planes : 2-$(n^2, n, 1)$

Witt's 5-$(12, 6, 1)$ design W_{12}

12 points, 132 blocks

Choose 3 points $\longrightarrow$ 9 points remaining

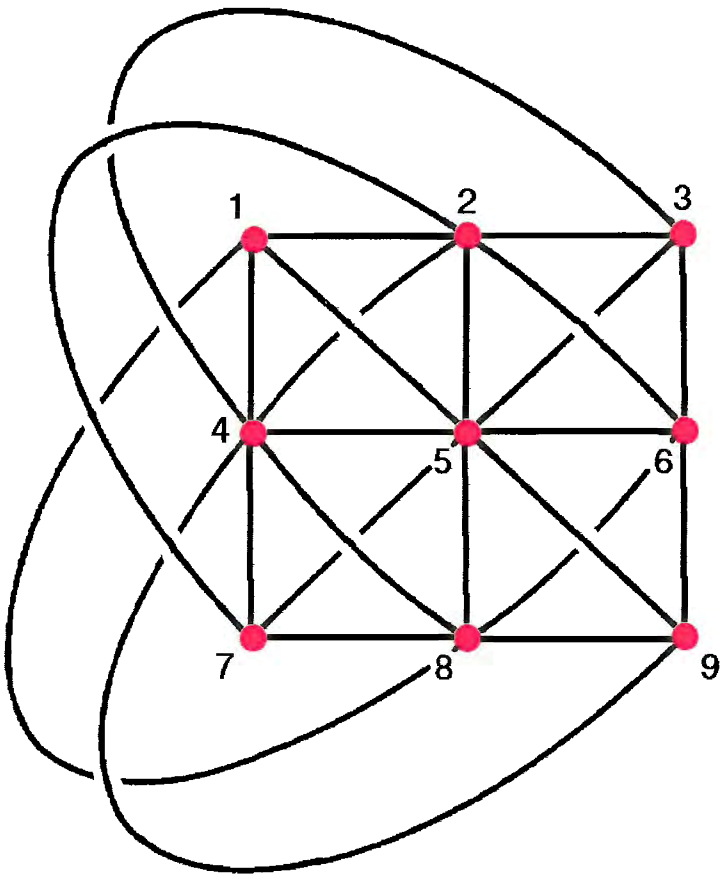
... blocks through them $\longrightarrow$ 12 blocks $\longrightarrow$ 12 lines

$\underbrace{\hphantom{12 \text{ blocks} \longrightarrow 12 \text{ lines}}}$

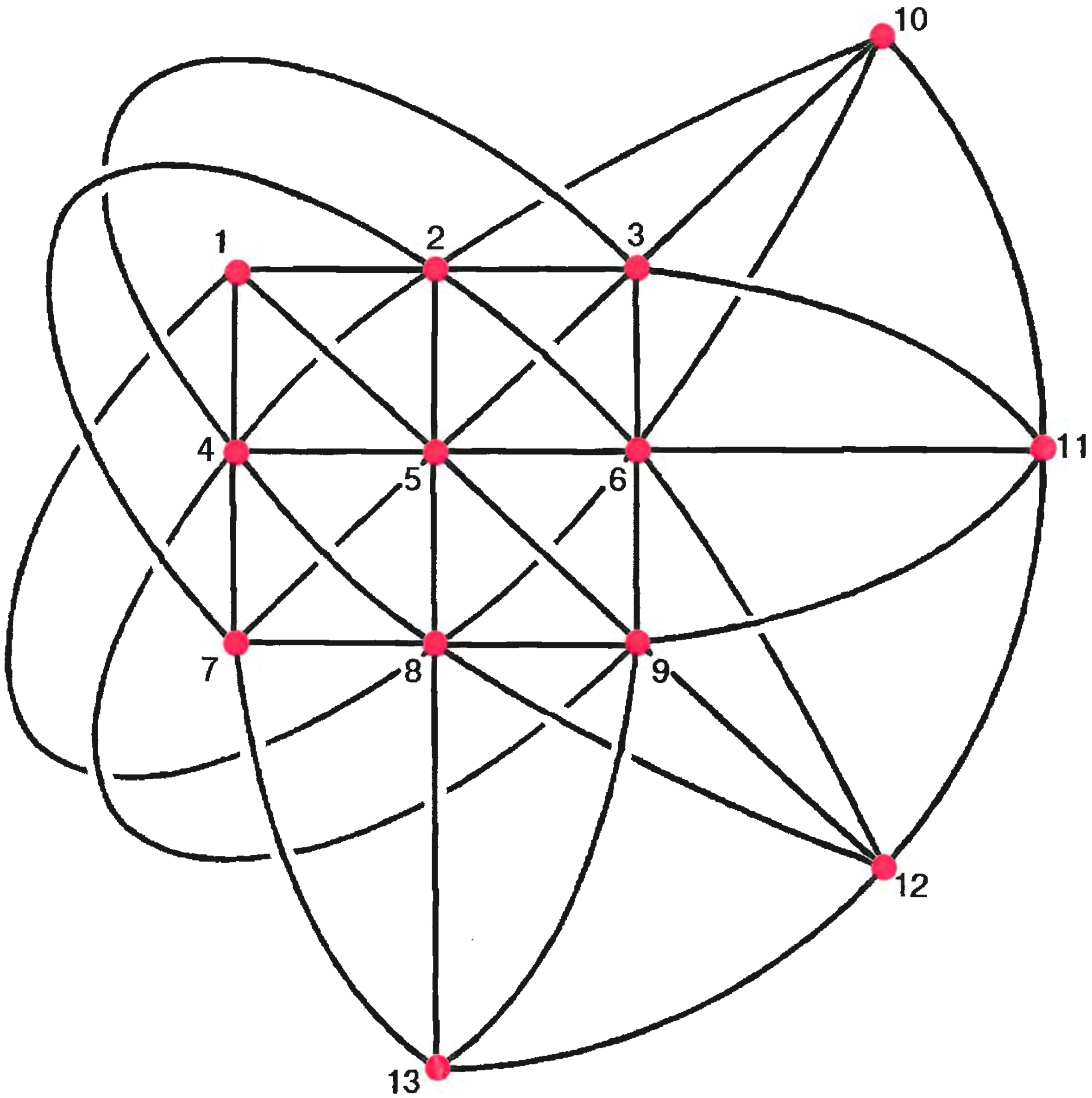
affine plane $AG(2,3)$

$\Big\downarrow$ 3 fold extension

W_{12}



The affine plane AG(2,3)



The projective plane PG(2,3)

Point model of W_{12} in $PG(5,3)$

$\mathcal{K}$ ... 12 points

5 points in $\mathcal{K}$ $\Rightarrow$ $\exists^*$ hyperplane $\mathcal{H}$

$$\#(\mathcal{H} \cap \mathcal{K}) = 6$$

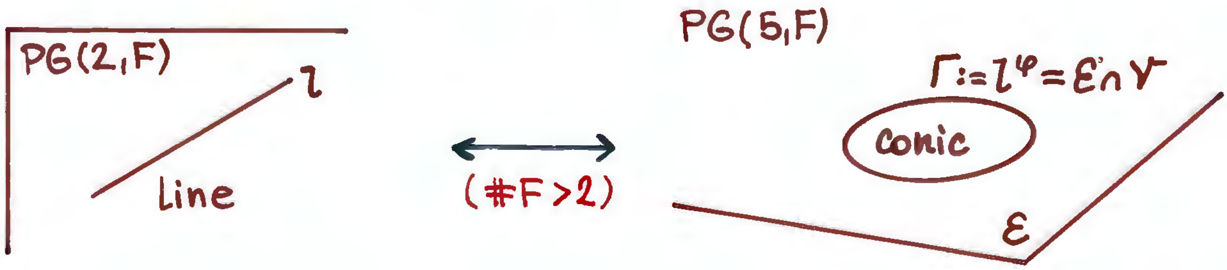
H.S.M. Coxeter, G. Pellegrino, J.A. Todd

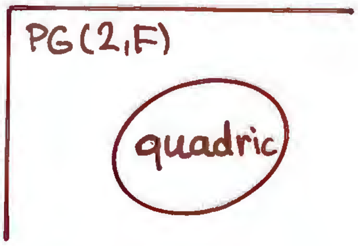
Veronese surface:

Veronese mapping:

$$\underbrace{F(x_0, x_1, x_2)}_{PG(2,F)} \xmapsto{\varphi} \underbrace{F(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)}_{PG(5,F)}$$

$\mathrm{im}\,\varphi =: \mathcal{V}$ ... Veronese surface





$\longleftrightarrow$

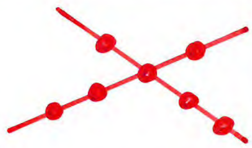
PG(5,F)

hyperplane section of $\mathcal{V}$

$$\sum_{i\le j} a_{ij} x_i x_j = 0 \qquad \sum_{i\le j} a_{ij} y_{ij} = 0$$

Zanella – H.

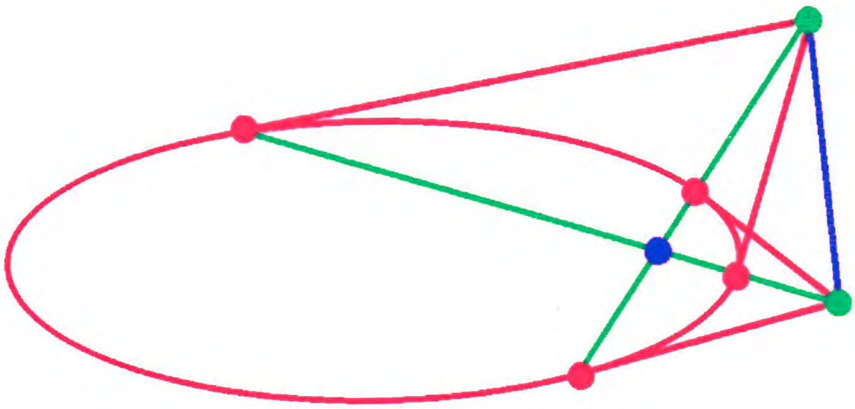
Quadrics in PG(2,3):

Equation	Name	Picture	# points
$x_0^2 + x_1^2 + x_2^2 = 0$	conic		4
$x_0^2 + x_1^2 = 0$	one point		1
$x_0^2 - x_1^2 = 0$	cross of lines		7
$x_0^2 = 0$	repeated line		4

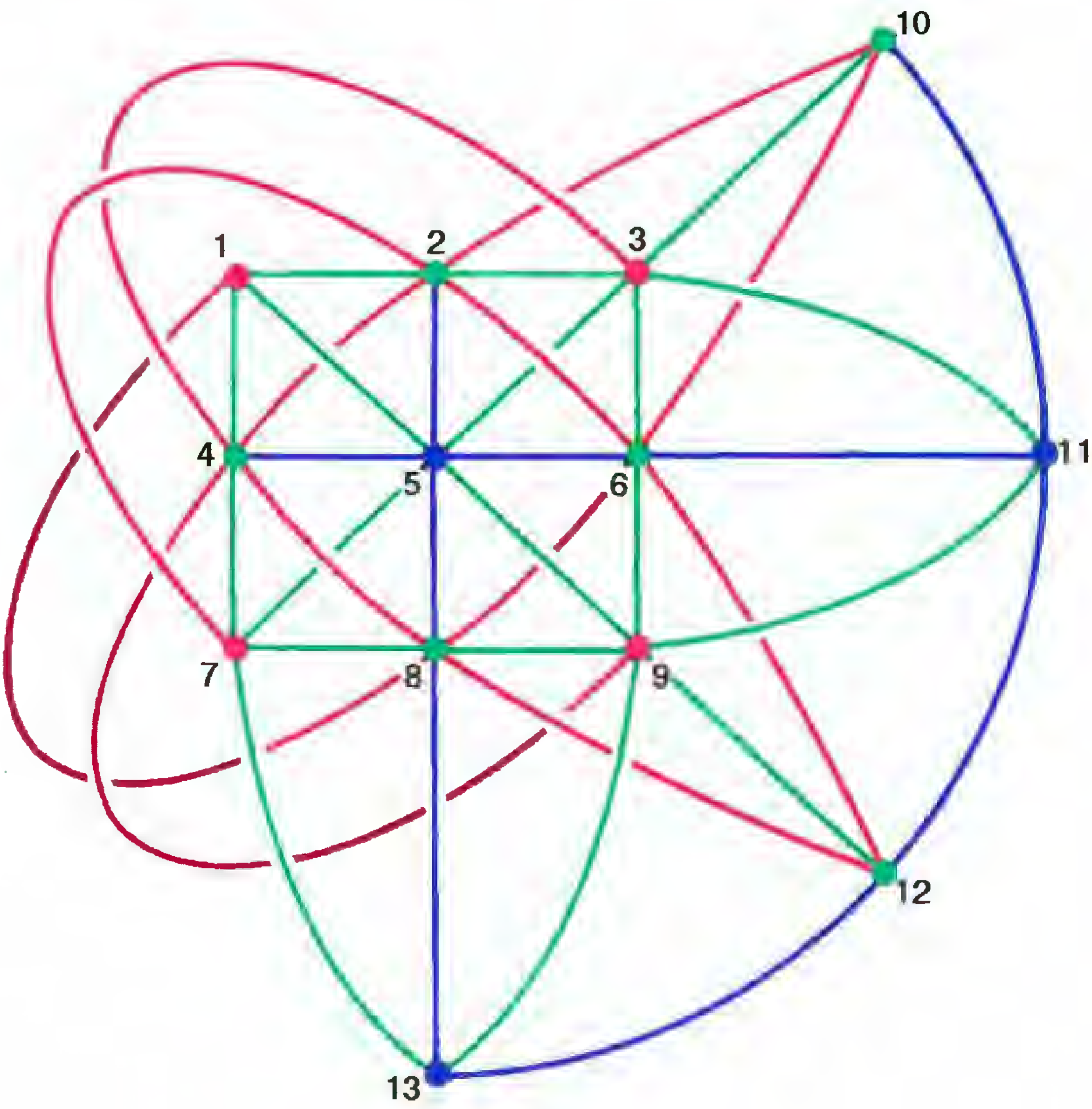
$\mathcal{H}$.. hyperplane of PG(5,3)

$$\underline{\underline{c := \#(\mathcal{H} \cap \mathcal{V}) \in \{1,4,7\}}}$$

Thas-Hirschfeld



Conic in the real projective plane:
Red: Points (of the conic) and tangents.
Blue: An internal point and an exterior line.
Green: External points and bisecant lines.



Conic in PG(2,3):
Red: 4 points (of the conic) and 4 tangents.
Blue: 3 internal points and 3 exterior lines.
Green: 6 external points and 6 bisecant lines.

Replacement :

l_∞ ... a line in $PG(2,3)$

$\Gamma_\infty := l_\infty^\varphi$ ... a conic ($\subset \mathcal{V}$)

Δ_∞ ... diagonal triangle of the quadrangle Γ_∞

ε_∞ ... plane of Γ_∞

$$\mathcal{K} := (\mathcal{V} \setminus \Gamma_\infty) \cup \Delta_\infty$$

$\# \mathcal{K} = 12$

φ

	1,0,0	1,0,0,0,0,0	
	1,0,1	1,0,1,0,0,1	
	1,0,2	1,0,2,0,0,1	
	1,1,0	1,1,0,1,0,0	
$AG(2,3)$	1,1,1	1,1,1,1,1,1	$\Upsilon \setminus \Gamma_\infty$
	1,1,2	1,1,2,1,2,1	
	1,2,0	1,2,0,1,0,0	
	1,2,1	1,2,1,1,2,1	
	1,2,2	1,2,2,1,1,1	
		0,0,0,1,0,1	
		0,0,0,2,1,1	Δ_∞
		0,0,0,2,2,1	

Theorem: $\underline{\underline{d := \#(\mathcal{H} \cap \mathcal{K}) \in \{0,3,6\}}}$ for all hyperplanes $\mathcal{H}$ of $PG(5,3)$.

Proof.: $\mathcal{H} \cap \mathcal{V} \xrightarrow{\varphi^{-1}} Q$ .. quadric

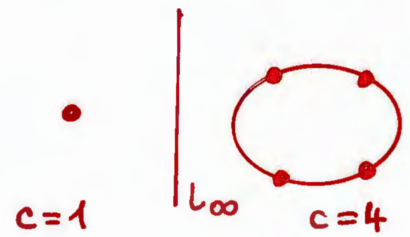
$c := \#(\mathcal{H} \cap \mathcal{V}) \in \{1,4,7\}$.

1. $\underline{\mathcal{E}_\infty \subset \mathcal{H}} \Rightarrow \underline{d} = c-4+3 = \underline{c-1} \in \{0,\underline{3,6}\}$

2. $\underline{\mathcal{E}_\infty \cap \mathcal{H} \text{ external line of } \Gamma_\infty}$

   $\Rightarrow \underline{d} = c-0+2 = \underline{c+2}$

   Q has no points at infinity



$\Rightarrow d \in \{\underline{3,6}\}$

3. $\underline{\mathcal{E}_\infty \cap \mathcal{H} \text{ is a tangent of } \Gamma_\infty} \Rightarrow \underline{d} = c-1+0 = \underline{c-1} \in \{\underline{0,3,6}\}$

4. $\underline{\mathcal{E}_\infty \cap \mathcal{H} \text{ is a bisecant of } \Gamma_\infty} \Rightarrow \underline{d} = c-2+1 = \underline{c-1} \in \{0,\underline{3,6}\}$

Characterizations of $\mathcal{K}$

(1) $\# \mathcal{K} = 12$

(2) $d \in \{0, 3, 6\}$

---

(3) $\# \mathcal{K} \geq 7$

(4) $\# (\mathcal{H} \cap \mathcal{K}) \geq 5 \Rightarrow \# (\mathcal{H} \cap \mathcal{K}) = 6$

(5) Any 5-subset of $\mathcal{K}$ is independent.

---

(1) ...

(5) ...

H.S.M. Coxeter, G. Pellegrino, J.A. Todd, . H.H.

$(1) \wedge (2) \Leftrightarrow (3) \wedge (4) \wedge (5) \Leftrightarrow (1) \wedge (5)$

$\exists^*$ unique $\mathcal{K}$ up to collineations

Model of W_{12} in $PG(5,3)$

$P := \mathcal{K}$

$B := \{ \mathcal{H} \cap \mathcal{K} \mid \#(\mathcal{H} \cap \mathcal{K}) \geq 4 \,,\ \mathcal{H} \text{ a hyperplane of } PG(5,3) \}$

$$\Downarrow$$

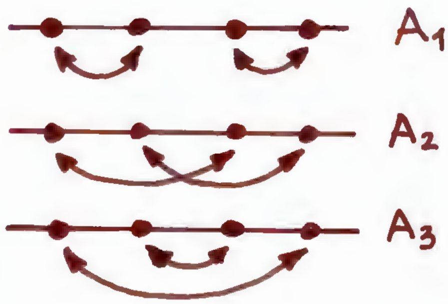
$$\#(\mathcal{H} \cap \mathcal{K}) = 6$$

Model of W_{12} in $PG(5,3) = (\mathcal{P}, \mathcal{L})$

3 elliptic involutions on l_∞

$$\downarrow$$

3 „new" points A_1, A_2, A_3



$P := (\mathcal{P} \setminus l_\infty) \cup \{A_1, A_2, A_3\}$

B ...

- affine line + all elliptic involutions (3)
- ellipse + those elliptic involutions that are not the involution of conjugate points on l_∞ (2)
- union of two disjoint affine lines (0)
- union of two non-parallel affine lines + that involution which interchanges the points at infinity. (1)

$$\underbrace{\text{affine quadric}}_{\text{Veronese}} + \underbrace{\text{some „new points"}}_{\text{replacement}}$$