Regular parallelisms in terms of the Klein quadric

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Basic notions

- $\mathrm{PG}(n, \mathbb{K})$ denotes the n-dimensional projective space over an arbitrary (commutative) field $\mathbb{K}$.
- In $\mathrm{PG}(n, \mathbb{K})$, the point set is written as $\mathcal{P}_n$ and the line set is denoted by $\mathcal{L}_n$.
- A *spread* of $\mathrm{PG}(3, \mathbb{K})$ is a partition of $\mathcal{P}_3$ by (disjoint) lines.
- A *parallelism* on $\mathrm{PG}(3, \mathbb{K})$ is a partition of $\mathcal{L}_3$ by (disjoint) spreads.
- The spreads of a parallelism are called *parallel classes*.
- Parallelisms are known as *packings* when $\mathbb{K}$ is a finite field.

See, among others, Hirschfeld [12], Johnson [13], [14], Karzel and Kroll [15], and Knarr [16].

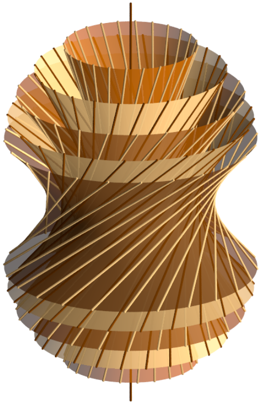
Additional properties

In $\mathrm{PG}(3, \mathbb{K})$, we shall only be concerned with spreads and parallelisms that satisfy some additional properties.

- A spread is are called *regular* if it is closed under reguli.
- Regular spreads are precisely the elliptic linear congruences of lines.
- A parallelism is called *regular* if all its parallel classes are regular spreads.

Example

We consider the projective extension of the 3-dimensional Euclidean space $\mathbb{R}^3$.



The picture illustrates a regular spread $\mathcal{C}$ that consists of the z-axis, the line at infinity of the plane $z = 0$, and reguli lying on hyperboloids of revolution with equations

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - z^2 = 1, \text{ where } a \text{ varies in } \mathbb{R}^+.$$

A regular parallelism $\boldsymbol{P}$ is obtained by applying all rotations about the origin to the regular spread $\mathcal{C}$.

The Klein correspondence λ

- The Klein correspondence is a bijective map

$$\lambda\colon \mathcal{L}_3 \to H_5$$

that maps each line of $\mathrm{PG}(3,\mathbb{K})$ to a point of the Klein quadric H_5 in $\mathrm{PG}(5,\mathbb{K})$.

- In terms of coordinates, the map λ assigns to each line its Plücker coordinates.
- The polarity of $\mathrm{PG}(5,\mathbb{K})$ associated with the Klein quadric H_5 is denoted by π_5.

Regular spreads of PG$(3, \mathbb{K})$ ...

Let $\mathcal{C}$ be a regular spread of PG$(3, \mathbb{K})$.

- The **Klein image** $\lambda(\mathcal{C})$ is an elliptic subquadric of H_5, that is, an elliptic quadric in a solid (three-dimensional subspace) of PG$(5, \mathbb{K})$.
- The **span** of $\lambda(\mathcal{C})$ is a solid of PG$(5, \mathbb{K})$.
- The **polarity** π_5 sends the solid spanned by $\lambda(\mathcal{C})$ to

$$\pi_5\big(\text{span}\,\lambda(\mathcal{C})\big),$$

which is an external line to the Klein quadric or, in other words, a 0-secant of H_5.

. . . and external lines to H_5

Let $D \in \mathcal{L}_5$ be an external line to H_5.

- The **polarity** π_5 sends D to $\pi_5(D)$, which is a solid of $\mathrm{PG}(5, \mathbb{K})$.
- The **intersection** $\pi_5(D) \cap H_5$ is an elliptic subquadric of H_5.
- The **inverse Klein image**

$$\lambda^{-1}\left(\pi_5(D) \cap H_5\right)$$

is a regular spread of $\mathrm{PG}(3, \mathbb{K})$.

The bijection γ

The results from the previous slides can be summarised as follows.

Theorem 1.

Let $\boldsymbol{C}$ *denote the set of all regular spreads of* $\mathrm{PG}(3,\mathbb{K})$ *and write* $\mathcal{Z}$ *for the set of all lines of* $\mathrm{PG}(5,\mathbb{K})$ *that are external to the Klein quadric* H_5*. Then the mapping*

$$\gamma\colon \boldsymbol{C} \to \mathcal{Z} : \mathcal{C} \mapsto \pi_5\big(\operatorname{span}\lambda(\mathcal{C})\big)$$

is bijective.

Hfd line sets

Let ***P*** be a regular parallelism on $\mathrm{PG}(3, \mathbb{K})$. Below we follow Betten and Riesinger [3].

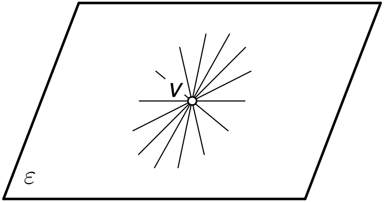
- The λ-image of ***P*** is a *hyperflock* of the Klein quadric H_5, that is, a partition of H_5 by (disjoint) elliptic subquadrics.
- The γ-image of ***P*** is a *hyperflock determining line set* with respect to the Klein quadric, that is, a set $\mathcal{H}$ with the following properties:

1. $\mathcal{H} \subset \mathcal{L}_5$ consists of lines that are external to the Klein quadric.
2. Each tangent hyperplane of the Klein quadric contains exactly one line of $\mathcal{H}$.

Such a set $\mathcal{H}$ will shortly be called an *hfd line set*.

Pencilled regular parallelisms (H. and Riesinger [11])

Notation.
For any incident point-plane pair (v, ε) we denote by $\mathcal{L}[v, \varepsilon]$ the pencil of lines with vertex v and plane ε.



Definition 1.

An hfd line set $\mathcal{H}$ is said to be *pencilled* if each element of $\mathcal{H}$ belongs to at least one pencil of lines contained in $\mathcal{H}$.

Definition 2.

A regular parallelism $\boldsymbol{P}$ on $\mathrm{PG}(3, \mathbb{K})$ is called *pencilled* if the hfd line set $\gamma(\boldsymbol{P})$ is pencilled.

Construction of pencilled hfd line sets

Theorem 2.

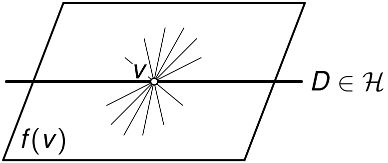
In PG$(5, \mathbb{K})$, *let D be a line such that*

$$\mathcal{E}_D := \{\varepsilon \subset \mathcal{P}_5 \mid D \subset \varepsilon \text{ and } \varepsilon \text{ is an external plane to } H_5\}$$

is non-empty. Then, upon choosing any mapping $f\colon D \to \mathcal{E}_D$, the union

$$\bigcup_{v \in D} \mathcal{L}[v, f(v)] =: \mathcal{H}$$

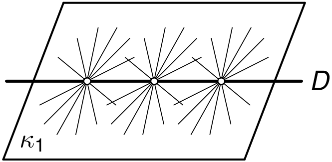
is a pencilled hfd line set.



Example

Suppose that the mapping $f\colon D \to \mathcal{E}_D$ in Theorem 2 is constant.

- The image of f contains a single plane, say κ_1.
- $\mathcal{H} = \bigcup_{v\in D} \mathcal{L}[v, \kappa_1]$ is the plane of lines in κ_1.



- Any point of κ_1 is the vertex of a unique pencil in $\mathcal{H}$.
- $\gamma^{-1}(\mathcal{H})$ is called a *Clifford parallelism*.
  These parallelisms are commonly defined in various ways; see Betten and Riesinger [4], Blunck, Pianta and Pasotti [5], Karzel and Kroll [15], or H. [7], [8], [9], [10].

Example

Let the image of the mapping f in Theorem 2 consist of two distinct planes κ_1 and κ_2.

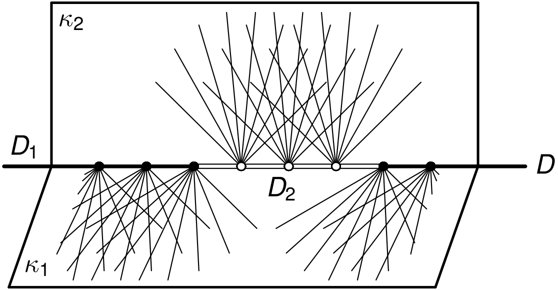
- The mapping f decomposes the line D into two non-empty subsets D_1 and D_2, namely the pre-images of κ_1 and κ_2, respectively.
- The corresponding hfd line set can be written in the form

$$\Big(\bigcup_{v\in D_1} \mathcal{L}[v,\kappa_1]\Big) \cup \Big(\bigcup_{v\in D_2} \mathcal{L}[v,\kappa_2]\Big).$$

- The set D_1 is not subject to any restriction. It may be finite or infinite.

Example (cont.)

- Over the real numbers, f can be chosen in such a way that D_1 is a connected component of D with respect to the standard topology in $\mathrm{PG}(5, \mathbb{R})$. Then D_2 is also connected.



Proof of Theorem 2 (sketched)

Our proof mainly relies on:

Lemma 1.

Let S be a subspace of $\mathrm{PG}(5, \mathbb{K})$. *There exists a tangent hyperplane of the Klein quadric H_5 containing S if, and only if, there exists a subspace M of* $\mathrm{PG}(5, \mathbb{K})$ *satisfying*

$$M \subset S \cap H_5 \quad \textit{and} \quad \dim M \geq \dim S - 2.$$

Corollary 1.

An external plane to the Klein quadric is not contained in any of its tangent hyperplanes.

Towards the Main Theorem

The next lemmas are subject to the following assumptions:

- In PG$(5, \mathbb{K})$, let $\mathcal{H}$ be a pencilled hfd line set.
- We denote by $\mathcal{V}$ the set of all vertices of the pencils in $\mathcal{H}$.
- We denote by $\mathcal{K}$ the set of all planes of the pencils in $\mathcal{H}$.

Towards the Main Theorem (cont.)

Lemma 2.

The sets $\mathcal{K}$ and $\mathcal{V}$ satisfy the following:

1. $|\mathcal{K}| \geq 1$.
2. $|\mathcal{V}| \geq 2$.

Towards the Main Theorem (cont.)

Lemma 3.

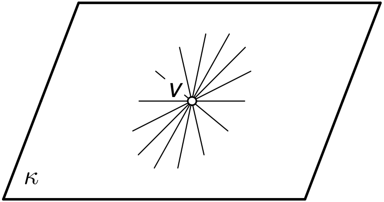
If $G_1, G_2 \in \mathcal{H}$ are distinct coplanar lines, then the plane $G_1 \vee G_2$ is external to the Klein quadric H_5.

Towards the Main Theorem (cont.)

Lemma 4.

Let $\mathcal{L}[v,\kappa] \subset \mathcal{H}$ *be a pencil of lines. Then*

$$\mathcal{L}[v,\kappa] = \{X \in \mathcal{H} \mid v \in X\}.$$

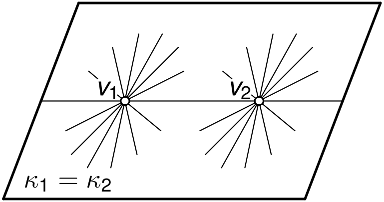


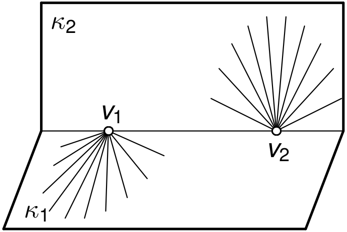
Towards the Main Theorem (cont.)

Lemma 5.

Let $\mathcal{L}[v_1, \kappa_1]$ and $\mathcal{L}[v_2, \kappa_2]$ be distinct pencils of lines that are contained in $\mathcal{H}$. Then the following hold:

1. $v_1 \neq v_2$.
2. $v_1 \vee v_2 \subset \kappa_1 \cap \kappa_2$.
3. $v_1 \vee v_2 \in \mathcal{H}$.





Main theorem on pencilled hfd line sets

Theorem 3.

In PG$(5, \mathbb{K})$*, let* $\mathcal{H}$ *be a pencilled hfd line set. Denote by* $\mathcal{V}$ *the set of all vertices and by* $\mathcal{K}$ *the set of all planes of the pencils in* $\mathcal{H}$*. Then the following hold.*

1. *All planes of* $\mathcal{K}$ *are external to the Klein quadric* H_5*.*
2. *There exists a surjective mapping* $h\colon \mathcal{V} \to \mathcal{K}$ *that assigns to each* $v \in \mathcal{V}$ *a plane* $h(v) \in \mathcal{K}$ *that is incident with* v *and such that*
$$\mathcal{L}[v, h(v)] = \{X \in \mathcal{H} \mid v \in X\}.$$

(To be continued on the next slide.)

Main theorem on pencilled hfd line sets

Theorem 3. (cont.)

In PG$(5, \mathbb{K})$, *let* $\mathcal{H}$ *be a pencilled hfd line set. Denote by* $\mathcal{V}$ *the set of all vertices and by* $\mathcal{K}$ *the set of all planes of the pencils in* $\mathcal{H}$. *Then the following hold.*

1. *All planes of* $\mathcal{K}$ *are external to the Klein quadric* H_5.
2. *There exists a surjective mapping* $h \colon \mathcal{V} \to \mathcal{K}$ ...
3. *If* $\mathcal{V}$ *is a set of non-collinear points, then* $\mathcal{V}$ *is a plane,* $\mathcal{K} = \{\mathcal{V}\}$, *and* $\mathcal{H}$ *is the set of lines in the plane* $\mathcal{V}$.
4. *If* $\mathcal{V}$ *is a set of collinear points, then* $\mathcal{V}$ *is a line,* $\mathcal{V} \in \mathcal{H}$, *and* $|\mathcal{K}| \geq 2$.
5. $\mathcal{V} = \bigcap_{\kappa \in \mathcal{K}} \kappa$.

Main theorem on pencilled hfd line sets (cont.)

Corollary 2.

In $\mathrm{PG}(5, \mathbb{K})$*, any hfd line set admits a construction as in Theorem 2.*

Existence of pencilled regular parallelisms

Theorem 4.

Given any field $\mathbb{K}$ the following assertions are equivalent.

1. *In* $\mathrm{PG}(3, \mathbb{K})$ *there exists a Clifford parallelism.*
2. *There exists an algebra $\mathbb{H}$ over the field $\mathbb{K}$ such that one of the following conditions,* (A) *or* (B)*, is satisfied:*
   - (A) *$\mathbb{H}$ is a quaternion skew field with centre $\mathbb{K}$.*
   - (B) *$\mathbb{H}$ is an extension field of $\mathbb{K}$ with degree $[\mathbb{H} : \mathbb{K}] = 4$ and such that $a^2 \in \mathbb{K}$ for all $a \in \mathbb{H}$.*
3. *In* $\mathrm{PG}(3, \mathbb{K})$ *there exists a pencilled regular parallelism that is not Clifford.*

Conclusion

- Hirschfeld [12, p. 69], who follows Conwell [6], uses hfd line sets to construct regular parallelisms of $\mathrm{PG}(3, 2)$. (Hirschfeld's terminology is different from ours.)
  These parallelisms give rise to solutions of Kirkman's Fifteen Schoolgirls problem (1850).
- Further examples of hfd line sets (pencilled or not) in $\mathrm{PG}(5, \mathbb{R})$ can be found in Betten and Riesinger [1], [2, Ex. 16 and 22], [3], and Löwen [17]. Many of these hfd line sets satisfy additional topological conditions.
- The transfer of properties of a pencilled hfd line set back to $\mathrm{PG}(3, \mathbb{K})$ is a straightforward task, but lengthy [11].
- If $\mathbb{K}$ has characteristic two then hfd line sets feature several additional properties [11].

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