# Projective metric geometry and Clifford algebras

Hans Havlicek

TECHNISCHE UNIVERSITÄT WIEN

Forschungsgruppe Differentialgeometrie und Geometrische Strukturen
Institut für Diskrete Mathematik und Geometrie

Università degli Studi di Brescia
April 7th, 2022

# Metric vector spaces

In this talk we present some results from our recent article [5], which contains a comprehensive bibliography.

- A *metric vector space* is a pair $(\boldsymbol{V}, Q)$ such that $\boldsymbol{V}$ is a vector space over a (commutative) field $F$ and $Q\colon \boldsymbol{V} \to F$ is a quadratic form.
- In what follows, we assume $\boldsymbol{V}$ being of finite dimension $n+1 \geq 0$.
- A non-zero vector $\boldsymbol{a} \in \boldsymbol{V}$ is called *regular* if $Q(\boldsymbol{a}) \neq 0$ and *singular* otherwise.

# Metric vector spaces (cont.)

- The *polar form* of $Q$ is the symmetric bilinear form

$$B\colon \boldsymbol{V} \times \boldsymbol{V} \to F\colon (\boldsymbol{x}, \boldsymbol{y}) \mapsto Q(\boldsymbol{x} + \boldsymbol{y}) - Q(\boldsymbol{x}) - Q(\boldsymbol{y}).$$

- Then, $B(\boldsymbol{x}, \boldsymbol{x}) = 2Q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{V}$.
- *Orthogonality* w.r.t. $B$ is denoted by $\perp$; that is, for all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$, $\boldsymbol{x} \perp \boldsymbol{y}$ means $B(\boldsymbol{x}, \boldsymbol{y}) = 0$.
- Any subset $\boldsymbol{S} \subseteq \boldsymbol{V}$ determines the subspace

$$\boldsymbol{S}^{\perp} := \{\boldsymbol{x} \in \boldsymbol{V} \mid \boldsymbol{x} \perp \boldsymbol{y} \text{ for all } \boldsymbol{y} \in \boldsymbol{S}\} \leq \boldsymbol{V}.$$

- In particular, $\boldsymbol{V}^{\perp}$ is called the *radical* of $B$.

# The weak orthogonal group of $(\boldsymbol{V}, Q)$

- A linear bijection $\psi\colon \boldsymbol{V} \to \boldsymbol{V}$ is called an *isometry* if $Q = Q \circ \psi$.
- All isometries of $(\boldsymbol{V}, Q)$ constitute the *orthogonal group* $\mathrm{O}(\boldsymbol{V}, Q)$.
- The *weak orthogonal group* $\mathrm{O}'(\boldsymbol{V}, Q)$ consists of all isometries of $(\boldsymbol{V}, Q)$ that fix the radical $\boldsymbol{V}^\perp$ elementwise (E. Ellers [2]).
- Given a regular vector $\boldsymbol{r} \in \boldsymbol{V}$ the mapping

$$\xi_{\boldsymbol{r}}\colon \boldsymbol{V} \to \boldsymbol{V}\colon \boldsymbol{x} \mapsto \boldsymbol{x} - B(\boldsymbol{r}, \boldsymbol{x})Q(\boldsymbol{r})^{-1}\boldsymbol{r}.$$

is the *reflection* of $(\boldsymbol{V}, Q)$ in the direction of $\boldsymbol{r}$. We have $\xi_{\boldsymbol{r}} \in \mathrm{O}'(\boldsymbol{V}, Q)$.

# A version of a theorem by E. Cartan and J. Dieudonné

**Theorem** (M. Götzky [3], [4] and M. Kneser [10])

Each isometry $\varphi \in \mathrm{O}'(\boldsymbol{V}, Q)$ is a product of reflections, except when $F$ and $(\boldsymbol{V}, Q)$ satisfy one of the subsequent conditions (1) or (2) for some basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of $\boldsymbol{V}$ and all $\boldsymbol{x} = \sum_{j=0}^{n} x_j \boldsymbol{e}_j$ with $x_j \in F$:

$$|F| = 2,\ \dim \boldsymbol{V} > 2 \ \text{ and } \ Q(\boldsymbol{x}) = x_0 x_1; \tag{1}$$

$$|F| = 2,\ \dim \boldsymbol{V} \geq 4 \ \text{ and } \ Q(\boldsymbol{x}) = x_0 x_1 + x_2 x_3. \tag{2}$$

# Clifford algebras

**Convention**

All our algebras are tacitly assumed to be associative and equipped with a non-zero unit element.

- Let $(\boldsymbol{V}, Q)$ be a metric vector space over $F$ and let $\boldsymbol{A}$ be an $F$-algebra. Then a mapping $\gamma\colon \boldsymbol{V} \to \boldsymbol{A}$ is said to be *Clifford* if it is linear and, for all $\boldsymbol{x} \in \boldsymbol{V}$, we have $\gamma(\boldsymbol{x})^2 = Q(\boldsymbol{x}) \cdot 1_{\boldsymbol{A}}$.
- A *Clifford algebra* $\boldsymbol{C}$ for $(\boldsymbol{V}, Q)$ is an $F$-algebra together with a *universal* Clifford mapping $\iota\colon \boldsymbol{V} \to \boldsymbol{C}$, that is: Given any $F$-algebra $\boldsymbol{A}$ and any Clifford map $\gamma\colon \boldsymbol{V} \to \boldsymbol{A}$ there exists a unique algebra homomorphism $\mu\colon \boldsymbol{C} \to \boldsymbol{A}$ such that
$$\gamma = \mu \circ \iota.$$

# The Clifford algebra of $(\boldsymbol{V}, Q)$

- For each metric vector space $(\boldsymbol{V}, Q)$ there exists (up to an algebra isomorphism) a unique Clifford algebra, say $\mathrm{Cl}(\boldsymbol{V}, Q)$.
- Any universal Clifford mapping is injective, which allows us to consider $\boldsymbol{V}$ as being a subspace of $\mathrm{Cl}(\boldsymbol{V}, Q)$.
- From now on, we identify $1 \in F$ with the unit element of $\mathrm{Cl}(\boldsymbol{V}, Q)$. Thus $F \leq \mathrm{Cl}(\boldsymbol{V}, Q)$.

# The Clifford algebra of $(\boldsymbol{V}, Q)$ (cont.)

- If $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is a basis of $\boldsymbol{V}$, then we obtain a basis of $\mathrm{Cl}(\boldsymbol{V}, Q)$ as

$$\{\boldsymbol{e}_{j_1}\boldsymbol{e}_{j_2}\cdots\boldsymbol{e}_{j_k} \mid 0 \leq j_1 < j_2 < \cdots < j_k \leq n\}, \tag{3}$$

  where an empty product is understood to be $1 \in F$. So, $\dim \mathrm{Cl}(\boldsymbol{V}, Q) = 2^{n+1}$.

- For all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$, we have

$$\boldsymbol{x}^2 = Q(\boldsymbol{x}),$$
$$\boldsymbol{x}\boldsymbol{y} + \boldsymbol{y}\boldsymbol{x} = B(\boldsymbol{x}, \boldsymbol{y}).$$

  By virtue of these formulas one may write up a multiplication table for $\mathrm{Cl}(\boldsymbol{V}, Q)$ in terms of the basis (3).

# The Clifford algebra of $(\boldsymbol{V}, Q)$ (cont.)

- The Clifford algebra $\mathrm{Cl}(\boldsymbol{V}, Q)$ is $\mathbb{Z}/(2\mathbb{Z})$-graded and so it is the direct sum of the *even part* $\mathrm{Cl}_0(\boldsymbol{V}, Q)$, which is a subalgebra of $\mathrm{Cl}(\boldsymbol{V}, Q)$, and the *odd part* $\mathrm{Cl}_1(\boldsymbol{V}, Q)$.
- In particular, $F \leq \mathrm{Cl}_0(\boldsymbol{V}, Q)$ and $\boldsymbol{V} \leq \mathrm{Cl}_1(\boldsymbol{V}, Q)$.
- Given any subset $\boldsymbol{S} \subseteq \mathrm{Cl}(\boldsymbol{V}, Q)$ we let

$$\boldsymbol{S}_0 := \boldsymbol{S} \cap \mathrm{Cl}_0(\boldsymbol{V}, Q) \quad \text{and} \quad \boldsymbol{S}_1 := \boldsymbol{S} \cap \mathrm{Cl}_1(\boldsymbol{V}, Q).$$

  Furthermore, we denote by $\boldsymbol{S}^\times$ the set of all invertible elements (w.r.t. multiplication) in $\boldsymbol{S}$.

# The Clifford algebra of $(\boldsymbol{V}, Q)$ (cont.)

- The *main involution* $\sigma\colon \mathrm{Cl}(\boldsymbol{V}, Q) \to \mathrm{Cl}(\boldsymbol{V}, Q)$ is the only algebra automorphism of $\mathrm{Cl}(\boldsymbol{V}, Q)$ such that $\boldsymbol{x} \mapsto -\boldsymbol{x}$ for all $\boldsymbol{x} \in \boldsymbol{V}$.
- Under the main involution $\sigma$ all elements of $\mathrm{Cl}_0(\boldsymbol{V}, Q)$ remain fixed, any $\boldsymbol{h} \in \mathrm{Cl}_1(\boldsymbol{V}, Q)$ goes over to $-\boldsymbol{h} \in \mathrm{Cl}_1(\boldsymbol{V}, Q)$.

# Reflections in terms of $\mathrm{Cl}(\boldsymbol{V}, Q)$

Let $\xi_{\boldsymbol{r}}$ be the reflection in the direction of a regular vector $\boldsymbol{r} \in \boldsymbol{V}$. Then, for all $\boldsymbol{x} \in \boldsymbol{V}$,

$$\begin{aligned}
\xi_{\boldsymbol{r}}(\boldsymbol{x}) &= \boldsymbol{x} - B(\boldsymbol{r}, \boldsymbol{x}) \cdot Q(\boldsymbol{r})^{-1} \cdot \boldsymbol{r} \\
&= \boldsymbol{x} - (\boldsymbol{rx} + \boldsymbol{xr}) \cdot \boldsymbol{r}^{-2} \cdot \boldsymbol{r} \\
&= \boldsymbol{x} - \boldsymbol{rxr}^{-1} - \boldsymbol{x} \\
&= -\boldsymbol{rxr}^{-1} \\
&= \boldsymbol{rx}(-\boldsymbol{r})^{-1} \\
&= \boldsymbol{rx}\sigma(\boldsymbol{r})^{-1},
\end{aligned}$$

where $\sigma$ denotes the main involution.

# The Lipschitz monoid of $(\boldsymbol{V}, Q)$

Below we present results by J. Helmstetter, as summarised in his survey [7].

**Definition** (J. Helmstetter [7, Def. 2.1])

The *Lipschitz monoid* $\mathrm{Lip}(\boldsymbol{V}, Q)$ is the multiplicative monoid in $\mathrm{Cl}(\boldsymbol{V}, Q)$ generated by the union of $F$, $\boldsymbol{V}$ and the set

$$\{1 + \boldsymbol{st} \mid \boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{V},\ Q(\boldsymbol{s}) = Q(\boldsymbol{t}) = B(\boldsymbol{s}, \boldsymbol{t}) = 0\}. \tag{4}$$

- From this definition, $\mathrm{Lip}(\boldsymbol{V}, Q) = \mathrm{Lip}_0(\boldsymbol{V}, Q) \cup \mathrm{Lip}_1(\boldsymbol{V}, Q)$.
- The Lipschitz monoid $\mathrm{Lip}(\boldsymbol{V}, Q)$ is already generated by $\boldsymbol{V}$ except when one of the following applies: $Q(\boldsymbol{V}) = \{0\}$; $F$ and $(\boldsymbol{V}, Q)$ satisfy (1); $F$ and $(\boldsymbol{V}, Q)$ satisfy (2).

# The Lipschitz group of $(\boldsymbol{V}, Q)$

- All invertible elements of $\mathrm{Lip}(\boldsymbol{V}, Q)$ constitute a group, the so-called *Lipschitz group* $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$.
- The group $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$ is generated by the set comprising all non-zero scalars of $F$, all regular vectors of $\boldsymbol{V}$ and all elements $1 + \boldsymbol{st}$ as in (4). Indeed,

$$(1+\boldsymbol{st})(1+\boldsymbol{ts}) = 1 + \underbrace{\boldsymbol{st}+\boldsymbol{ts}}_{=B(\boldsymbol{s},\boldsymbol{t})=0} + \boldsymbol{s}(\boldsymbol{tt})\boldsymbol{s} = 1 + \boldsymbol{s}\underbrace{Q(\boldsymbol{t})}_{=0}\boldsymbol{s} = 1$$

shows that $1 + \boldsymbol{st}$ is invertible.

# The twisted adjoint representation of $\mathrm{Lip}^\times(\boldsymbol{V}, Q)$

**Theorem** (J. Helmstetter [7, Thm. 3.2])

The mapping

$$\xi\colon \ \mathrm{Lip}^\times(\boldsymbol{V}, Q) \to \mathrm{O}'(\boldsymbol{V}, Q)\colon \boldsymbol{p} \mapsto \left(\xi_{\boldsymbol{p}}\colon \boldsymbol{x} \mapsto \boldsymbol{p}\boldsymbol{x}\sigma(\boldsymbol{p})^{-1}\right) \tag{5}$$

is a surjective homomorphism of groups.

This $\xi$ is known as the *twisted adjoint representation* of $\mathrm{Lip}^\times(\boldsymbol{V}, Q)$; the attribute “twisted” refers to the main involution $\sigma$ appearing in (5); see M. F. Atiyah, R. Bott and A. Shapiro [1].

# The kernel of the twisted adjoint representation $\xi$

- The subalgebra of $\mathrm{Cl}(\boldsymbol{V}, Q)$ generated by the radical $\boldsymbol{V}^\perp$ (together with 1) may be viewed as the Clifford algebra $\mathrm{Cl}(\boldsymbol{V}^\perp, Q|\boldsymbol{V}^\perp)$.
- The kernel of the twisted adjoint representation $\xi$ satisfies

$$F^\times \leq \ker \xi = \mathrm{Lip}^\times(\boldsymbol{V}, Q) \cap \mathrm{Cl}(\boldsymbol{V}^\perp, Q|\boldsymbol{V}^\perp);$$

see J. Helmstetter [6, (22) Cor.], [8, (5.8.7) Lemma], R. Jurk [9, (2.2) Satz].

# Projective spaces

- By the *projective space* $\mathbb{P}(\boldsymbol{V})$ we mean the set of all subspaces of $\boldsymbol{V}$ with *incidence* being symmetrised inclusion.
- The *dimension* of $\mathbb{P}(\boldsymbol{V})$ is one less than the dimension of $\boldsymbol{V}$.
- We adopt the usual geometric terms: *points*, *lines* and *planes* are the subspaces of $\boldsymbol{V}$ with (vector) dimension one, two, and three, respectively.
- The general linear group $\mathrm{GL}(\boldsymbol{V})$ acts in a canonical way on $\mathbb{P}(\boldsymbol{V})$: any $\varkappa \in \mathrm{GL}(\boldsymbol{V})$ determines a *projective collineation* on $\mathbb{P}(\boldsymbol{V})$, which is given by $\boldsymbol{X} \mapsto \varkappa(\boldsymbol{X})$ for all $\boldsymbol{X} \in \mathbb{P}(\boldsymbol{V})$. This action of $\mathrm{GL}(\boldsymbol{V})$ has the kernel $F^{\times}\,\mathrm{id}_{\boldsymbol{V}}$.

# Projective metric spaces

If $(\boldsymbol{V}, Q)$ is a metric vector space, then $Q$ can been used to furnish the projective space with “additional structure”, thus making it into a *projective metric space* $\mathbb{P}(\boldsymbol{V}, Q)$. For example:

- A point $F\boldsymbol{p}$ of $\mathbb{P}(\boldsymbol{V}, Q)$ is said to be *regular* (*singular*) if $\boldsymbol{p}$ is a regular (singular) vector.
- All singular points constitute the *absolute quadric* of $\mathbb{P}(\boldsymbol{V}, Q)$.
- The *Q-distance* of two regular points $F\boldsymbol{p}$, $F\boldsymbol{q}$ is given as

$$\mathrm{dist}_Q(F\boldsymbol{p}, F\boldsymbol{q}) = \frac{B(\boldsymbol{p}, \boldsymbol{q})^2}{Q(\boldsymbol{p})Q(\boldsymbol{q})}.$$

See E. M. Schröder [11] for further details.

# The projective weak orthogonal group

- Any isometry $\varphi \in \mathrm{O}'(\boldsymbol{V}, Q)$ determines a projective collineation of $\mathbb{P}(\boldsymbol{V}, Q)$.
- This action of the weak orthogonal group $\mathrm{O}'(\boldsymbol{V}, Q)$ on $\mathbb{P}(\boldsymbol{V}, Q)$ has the kernel

$$\mathrm{I}'(\boldsymbol{V}, Q) := \mathrm{O}'(\boldsymbol{V}, Q) \cap \{\mathrm{id}_{\boldsymbol{V}}, -\mathrm{id}_{\boldsymbol{V}}\}.$$

- The quotient $\mathrm{O}'(\boldsymbol{V}, Q) / \mathrm{I}'(\boldsymbol{V}, Q) =: \mathrm{PO}'(\boldsymbol{V}, Q)$ is the *projective weak orthogonal group*.
- One easily verifies

$$|\mathrm{I}'(\boldsymbol{V}, Q)| = 1 \;\Leftrightarrow\; \Big(\boldsymbol{V} = \{0\} \text{ or } \boldsymbol{V}^{\perp} \neq \{0\} \text{ or } \operatorname{Char} F = 2\Big).$$

# The point set $\mathcal{M}(\boldsymbol{V}, Q)$

- Given any set of points, say $\mathcal{S}$, in $\mathbb{P}\big(\mathrm{Cl}(\boldsymbol{V}, Q)\big)$ we denote by $\mathcal{S}_0$ (resp. $\mathcal{S}_1$) the subset of $\mathcal{S}$ comprising all points that are contained in $\mathrm{Cl}_0(\boldsymbol{V}, Q)$ (resp. $\mathrm{Cl}_1(\boldsymbol{V}, Q)$).
- The Lipschitz monoid $\mathrm{Lip}(\boldsymbol{V}, Q)$ gives rise to the point set

$$\mathcal{M}(\boldsymbol{V}, Q) := \big\{F\boldsymbol{p} \mid 0 \neq \boldsymbol{p} \in \mathrm{Lip}(\boldsymbol{V}, Q)\big\}.$$

- $\mathcal{M}(\boldsymbol{V}, Q)$ is the disjoint union of $\mathcal{M}_0(\boldsymbol{V}, Q)$ and $\mathcal{M}_1(\boldsymbol{V}, Q)$.
- The sets $\mathcal{M}_0(\boldsymbol{V}, Q)$ and $\mathcal{M}_1(\boldsymbol{V}, Q)$ are algebraic varieties of the projective spaces on $\mathrm{Cl}_0(\boldsymbol{V}, Q)$ and $\mathrm{Cl}_1(\boldsymbol{V}, Q)$, respectively (J. Helmstetter [7, p. 673]).
- If $\dim \boldsymbol{V} \leq 3$, then $\mathcal{M}_0(\boldsymbol{V}, Q)$ resp. $\mathcal{M}_1(\boldsymbol{V}, Q)$ comprises all points of $\mathbb{P}\big(\mathrm{Cl}_0(\boldsymbol{V}, Q)\big)$ resp. $\mathbb{P}\big(\mathrm{Cl}_1(\boldsymbol{V}, Q)\big)$ (J. Helmstetter [6, (31) Lemma]).

# The group $\mathcal{G}(\boldsymbol{V}, Q)$

- Next, we take the Lipschitz group $\mathrm{Lip}^\times(\boldsymbol{V}, Q)$ and introduce the point set

$$\mathcal{G}(\boldsymbol{V}, Q) := \left\{ F\boldsymbol{p} \mid \boldsymbol{p} \in \mathrm{Lip}^\times(\boldsymbol{V}, Q) \right\},$$

which can be made into (multiplicative) group in the following way:

$$(F\boldsymbol{p})(F\boldsymbol{q}) := F(\boldsymbol{pq}) \;\text{ for all }\; F\boldsymbol{p}, F\boldsymbol{q} \in \mathcal{G}(\boldsymbol{V}, Q).$$

- Then $\mathcal{G}(\boldsymbol{V}, Q) \cong \mathrm{Lip}^\times(\boldsymbol{V}, Q)/F^\times$.

# Action of $\mathcal{G}(\boldsymbol{V}, Q)$ on $\mathbb{P}(\boldsymbol{V}, Q)$

- The group $\mathcal{G}(\boldsymbol{V}, Q)$ acts on the projective space $\mathbb{P}(\boldsymbol{V}, Q)$ as follows: for all $F\boldsymbol{p} \in \mathcal{G}(\boldsymbol{V}, Q)$ and all $\boldsymbol{X} \in \mathbb{P}(\boldsymbol{V}, Q)$, we have

$$F\boldsymbol{p} \mapsto \big(\boldsymbol{X} \mapsto \xi_{\boldsymbol{p}}(\boldsymbol{X}) = \boldsymbol{p}\boldsymbol{X}\sigma(\boldsymbol{p})^{-1}\big). \tag{6}$$

- This action of $\mathcal{G}(\boldsymbol{V}, Q)$ on $\mathbb{P}(\boldsymbol{V}, Q)$ yields a surjective homomorphism of groups

$$\vartheta \colon \mathcal{G}(\boldsymbol{V}, Q) \to \mathrm{PO}'(\boldsymbol{V}, Q)$$

with

$$\ker \vartheta = \big\{F\boldsymbol{p} \in \mathcal{G}(\boldsymbol{V}, Q) \mid \xi_{\boldsymbol{p}} \in \mathrm{I}'(\boldsymbol{V}, Q)\big\}.$$

# Faithful action of $\mathcal{G}(\boldsymbol{V}, Q)$ on $\mathbb{P}(\boldsymbol{V}, Q)$

All instances where $\mathcal{G}(\boldsymbol{V}, Q) \cong \mathrm{PO}'(\boldsymbol{V}, Q)$ (via $\vartheta$) are as follows:

| $\dim \boldsymbol{V}^{\perp}$ | $Q(\boldsymbol{V}^{\perp})$ | $\dim \boldsymbol{V}$ | Char $F$ |
|---|---|---|---|
| | | $= 0$ | |
| $= 0$ | | $> 0$ | $= 2$ |
| $= 1$ | $= \{0\}$ | | |

# Faithful action of $\mathcal{G}_0(\boldsymbol{V}, Q)$ on $\mathbb{P}(\boldsymbol{V}, Q)$

All instances where $\mathcal{G}_0(\boldsymbol{V}, Q) \cong \mathrm{PO}'(\boldsymbol{V}, Q)$ (via $\vartheta|\mathcal{G}_0(\boldsymbol{V}, Q)$) are as follows:

| $\dim \boldsymbol{V}^\perp$ | $Q(\boldsymbol{V}^\perp)$ | $\dim \boldsymbol{V}$ | Char $F$ | Remark |
|---|---|---|---|---|
| | | $= 0$ | | $\mathcal{G}(\boldsymbol{V}, Q) = \mathcal{G}_0(\boldsymbol{V}, Q)$ |
| $= 0$ | | odd | $\neq 2$ | $\mathcal{G}(\boldsymbol{V}, Q) \neq \mathcal{G}_0(\boldsymbol{V}, Q)$ |
| $= 1$ | $= \{0\}$ | $= 1$ | | $\mathcal{G}(\boldsymbol{V}, Q) = \mathcal{G}_0(\boldsymbol{V}, Q)$ |
| $= 1$ | $\neq \{0\}$ | | | $\mathcal{G}(\boldsymbol{V}, Q) \neq \mathcal{G}_0(\boldsymbol{V}, Q)$ |

# A non-faithful action of $\mathcal{G}(\boldsymbol{V}, Q)$ on $\mathbb{P}(\boldsymbol{V}, Q)$

There is one instance, where each element of $\mathrm{PO}'(\boldsymbol{V}, Q)$ is represented by an unordered pair of distinct points from $\mathcal{G}(\boldsymbol{V}, Q)$:

| $\dim \boldsymbol{V}^\perp$ | $Q(\boldsymbol{V}^\perp)$ | $\dim \boldsymbol{V}$ | Char $F$ |
|---|---|---|---|
| $= 0$ | | $> 0$ and even | $\neq 2$ |

To be more precise, let us take an arbitrary orthogonal basis $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of $\boldsymbol{V}$ and write

$$\boldsymbol{e} := \boldsymbol{e}_0 \boldsymbol{e}_1 \cdots \boldsymbol{e}_n \in \mathrm{Lip}_0^\times(\boldsymbol{V}, Q).$$

From (3), $F1 \neq F\boldsymbol{e}$. Then $\ker \vartheta = \{F1, F\boldsymbol{e}\}$ and so

$$\mathcal{G}(\boldsymbol{V}, Q)/\{F1, F\boldsymbol{e}\} \cong \mathrm{PO}'(\boldsymbol{V}, Q).$$

# Final remarks

- If the quadratic form $Q$ is replaced by a non-zero multiple, say $cQ$ with $c \in F^\times$, then this does not affect the geometry of $\mathbb{P}(\boldsymbol{V}, Q)$, but neither the Clifford algebras $\mathrm{Cl}(\boldsymbol{V}, Q)$ and $\mathrm{Cl}(\boldsymbol{V}, cQ)$ nor the Lipschitz groups $\mathrm{Lip}^\times(\boldsymbol{V}, Q)$ and $\mathrm{Lip}^\times(\boldsymbol{V}, cQ)$ need to be isomorphic.
- There exists, however, a specific linear bijection $\mathrm{Cl}(\boldsymbol{V}, Q) \to \mathrm{Cl}(\boldsymbol{V}, cQ)$ that maps $\mathcal{M}(\boldsymbol{V}, Q)$ onto $\mathcal{M}(\boldsymbol{V}, cQ)$ and induces an isomorphism of groups $\mathcal{G}(\boldsymbol{V}, Q) \to \mathcal{G}(\boldsymbol{V}, cQ)$.
- By virtue of this isomorphism, we obtain equivalent actions of $\mathcal{G}(\boldsymbol{V}, Q)$ and $\mathcal{G}(\boldsymbol{V}, cQ)$ on $\mathbb{P}(\boldsymbol{V}, Q)$ according to (6).

# References


[1] M. F. Atiyah, R. Bott, A. Shapiro, Clifford modules. *Topology* **3** (1964), 3–38.

[2] E. W. Ellers, Decomposition of orthogonal, symplectic, and unitary isometries into simple isometries. *Abh. Math. Sem. Univ. Hamburg* **46** (1977), 97–127.

[3] M. Götzky, Über die Erzeugenden der engeren unitären Gruppen. *Arch. Math. (Basel)* **19** (1968), 383–389.

[4] M. Götzky, Unverkürzbare Produkte und Relationen in unitären Gruppen. *Math. Z.* **104** (1968), 1–15.

[5] H. Havlicek, Projective metric geometry and Clifford algebras. *Results Math.* **76** (2021), Paper No. 219, 22 pp. Corrected version: https://arxiv.org/abs/2109.11470v2

# References (cont.)


[6] J. Helmstetter, Lipschitz monoids and Vahlen matrices. *Adv. Appl. Clifford Algebr.* **15** (2005), 83–122.

[7] J. Helmstetter, A survey of Lipschitz monoids. *Adv. Appl. Clifford Algebr.* **22** (2012), 665–688.

[8] J. Helmstetter, A. Micali, *Quadratic Mappings and Clifford Algebras*. Birkhäuser Verlag, Basel 2008.

[9] R. Jurk, Zur Darstellung der klassischen Gruppen durch Clifford-Algebren. *J. Geom.* **16** (1981), 72–82.

[10] M. Kneser, Witts Satz über quadratische Formen und die Erzeugung orthogonaler Gruppen durch Spiegelungen. *Math.-Phys. Semesterber.* **17** (1970), 33–45.

## References (cont.)

[11] E. M. Schröder, Metric geometry. In: F. Buekenhout, editor, *Handbook of Incidence Geometry*, 945–1013, North-Holland, Amsterdam 1995.