Parallelisms and Algebras: A Tribute to Silvia Pianta

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Geometric Structures and Loops
A day in honor of Silvia Pianta & Elena Zizioli
Università Cattolica del Sacro Cuore
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The kinematic mapping of Blaschke and Grünwald

111 years ago . . .


Grünwald J.

Überreicht vom Verfasser.

Ein Abbildungsprinzip, welches die ebene Geometrie und Kinematik mit der räumlichen Geometrie verknüpft

von

Josef Grünwald in Prag.

(Vorgelegt in der Sitzung am 4. Mai 1911.)

Aus den Sitzungsberichten der kaiserl. Akademie der Wissenschaften in Wien. Mathem.-naturw. Klasse; Bd. CXX. Abt. IIa. Mai 1911.

WIEN, 1911.

AUS DER KAISERLICH-KÖNIGLICHEN HOF- UND STAATSDRUCKEREI.

IN KOMMISSION BEI ALFRED HÖLDER,

K. U. K. HOF- UND UNIVERSITÄTSBUCHHÄNDLER,

BUCHHÄNDLER DER KAISERLICHEN AKADEMIE DER WISSENSCHAFTEN.

In 1911, W. Blaschke [3] and, independently, J. Grünwald [10] established a seminal result, which since then is known as the kinematic mapping of Blaschke and Grünwald.

W. Blaschke refers to Grünwald's work in an erratum [4] to his article, which appeared in 1912.

The kinematic mapping

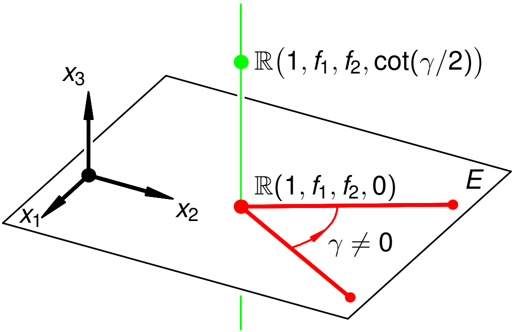
We consider the Euclidean space $\mathbb{R}^3$, which is embedded in the projective space $\mathbb{P}(\mathbb{R}^4)$ via

$$(x_1, x_2, x_3) \mapsto \mathbb{R}(1, x_1, x_2, x_3).$$

The kinematic mapping of Blaschke and Grünwald assigns to each direct motion of the Euclidean plane E, given as $x_3 = 0$, a point of the projective space $\mathbb{P}(\mathbb{R}^4)$.

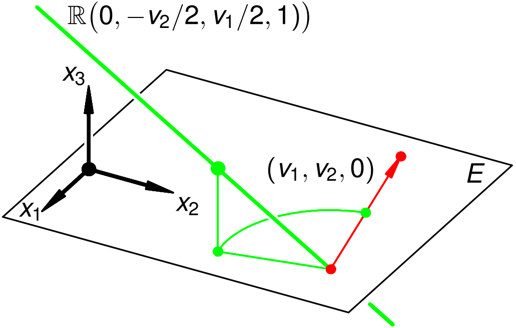
Such a direct motion is either a non-trivial rotation or a translation.

Image of a non-trivial rotation



The rotation about the fixed point $\mathbb{R}(1, f_1, f_2, 0)$ through the angle $\gamma \neq 0$ is mapped to the point $\mathbb{R}\big(1, f_1, f_2, \cot(\gamma/2)\big)$.

Image of a translation



The translation along the vector $(v_1, v_2, 0)$ is mapped to the point $\mathbb{R}(0, -v_2/2, v_1/2, 1)$, which belongs to the plane at infinity.

The slit space $\mathbb{P}(\mathbb{R}^4) \setminus S$

Removing the line S with equation $x_0 = x_3 = 0$ makes the projective space $\mathbb{P}(\mathbb{R}^4)$ into a *slit space* $\mathbb{P}(\mathbb{R}^4) \setminus S$.

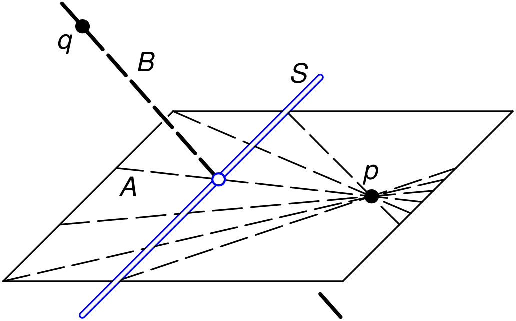
The kinematic mapping is a bijection of the group of direct motions of the Euclidean plane E ($x_3 = 0$) onto the point set $\mathcal{P}$ of the slit space $\mathbb{P}(\mathbb{R}^4) \setminus S$.

The lines of this slit space fall into two classes:

- a projective line is skew to S;
- an affine line meets S at a unique point.

Affine lines that meet S at the same point are called *parallel*.

Affine lines



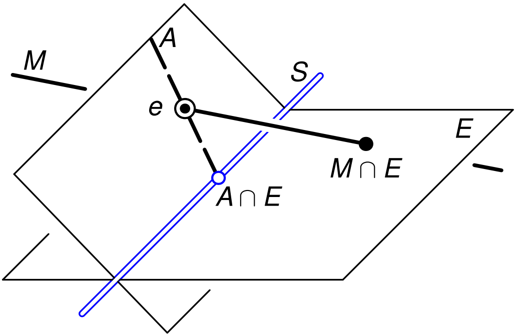
All affine lines through a point $p \in \mathcal{P}$ determine an affine plane with S being its line at infinity.

The affine line B is the only line through q being parallel to the affine line A.

Properties

- The kinematic mapping makes the point set $\mathcal{P}$ into a group $(\mathcal{P}, \cdot)$, which is isomorphic to the group of direct motions of the Euclidean plane E.
- For all $a \in \mathcal{P}$, the left translation $\lambda_a \colon \mathcal{P} \to \mathcal{P} \colon x \mapsto ax$ extends to a collineation of $\mathbb{P}(\mathbb{R}^4)$.
- For all $a \in \mathcal{P}$, the right translation $\rho_a \colon \mathcal{P} \to \mathcal{P} \colon x \mapsto xa$ extends to a collineation of $\mathbb{P}(\mathbb{R}^4)$.
- All lines through the point $e := \mathbb{R}(0, 0, 0, 1)$, which is the neutral element of the group $(\mathcal{P}, \cdot)$, give rise to subgroups of $(\mathcal{P}, \cdot)$.

Lines through e



A projective line M through e represents all rotations about the point $M \cap E$.

An affine line A through e represents all translations in the direction orthogonal to $A \cap E = A \cap S$.

Generalisations

There exists a wealth of results about the kinematic mapping of Blaschke and Grünwald and its generalisations.

- For example, there are kinematic mappings for the pseudo-Euclidean plane (Minkowski plane), non-Euclidean planes and their higher-dimensional analogues.
- Further aspects show up in the context of differential geometry and the theory of Lie groups.

For extensive bibliographies we refer to O. Bottema and B. Roth [6], O. Giering [9], H. Karzel and G. Kist [17], H. Karzel and H.-J. Kroll [18], M. Husty, A. Karger, H. Sachs and W. Steinhilper [14], A. Karger and J. Novák [15], D. Klawitter [20], J. Selig [29].

Partial parallelism spaces

Linear spaces

Definition

Let $\mathcal{P}$ be a set of *points* and let $\mathcal{L}$ be a subset of the power set of $\mathcal{P}$; the elements of $\mathcal{L}$ are called *lines*. The pair $(\mathcal{P}, \mathcal{L})$ is said to be a *linear space*, if it satisfies the following axioms:

(L1) Any two distinct points are contained in a unique line.

(L2) Any line contains at least two points.

Linear spaces are also known under the name incidence spaces.

The automorphisms of a linear space will be addressed as collineations.

Partial parallelisms

Definition (M. Marchi and S. Pianta [22], [23])

Let $(\mathcal{P}, \mathcal{L})$ be a linear space and let $\mathcal{L}'$ be a distinguished subset of $\mathcal{L}$. An equivalence relation $\parallel$ on $\mathcal{L}'$ is called a *partial parallelism* of $(\mathcal{P}, \mathcal{L})$, if it satisfies the following condition:

(PP) Any point of $\mathcal{P}$ is incident with a unique line from each equivalence class of $\parallel$.

(PP) is an analogue of "Euclid's axiom".

We shall refer to $\mathcal{L}'$ as the domain of $\parallel$.

If $\mathcal{L}' = \mathcal{L}$, then $\parallel$ turns into a *parallelism* of $(\mathcal{P}, \mathcal{L})$.

Partial parallelism spaces

Definition (M. Marchi and S. Pianta [22], [23])

A *partial parallelism space* is a quadruple $(\mathcal{P}, \mathcal{L}, \mathcal{L}_{\text{aff}}, \parallel)$ satisfying the following conditions:

- $(\mathcal{P}, \mathcal{L})$ is a linear space.
- $\mathcal{L}_{\text{aff}}$ is a distinguished subset of $\mathcal{L}$, the set of affine lines.
- $\parallel$ is a partial parallelism of $(\mathcal{P}, \mathcal{L})$ with domain $\mathcal{L}_{\text{aff}}$.
- There exist at least two lines.

Examples

- The slit space $\mathbb{P}(\mathbb{R}^4) \setminus S$ is an example of a partial parallelism space.
- Further examples can be obtained from arbitrary *slit spaces*. Such a space arises, by analogy to the above, from a projective space by deleting one of its proper subspaces.
- Any *affine parallel structure*, as introduced by J. André [1], yields an example where $\mathcal{L} = \mathcal{L}_{\text{aff}}$.

A characterisation

In their paper [23], M. Marchi and S. Pianta gave an elegant characterisation of slit spaces as partial parallelism spaces satisfying a few extra conditions.

Before, H. Karzel and H. Meißner [19] had also given such a characterisation. However, they adopted a quite different formalism. For example, among their basic notions there is nothing like a partial parallelism.

Recent work by K. Petelczyc and M. Żynel [24] deals with generalisations to polar spaces.

Kinematic spaces

Kinematic spaces

Definition (H. Karzel [16])

Let $(\mathcal{P}, \mathcal{L})$ be a linear space and let $(\mathcal{P}, \cdot)$ be a group with neutral element e. The triple $(\mathcal{P}, \mathcal{L}, \cdot)$ is said to be a *kinematic space* if the following axioms hold:

(K1) For all $a \in \mathcal{P}$, the left translation $\lambda_a \colon \mathcal{P} \to \mathcal{P} \colon x \mapsto ax$ is a collineation of the linear space $(\mathcal{P}, \mathcal{L})$.

(K2) For all $a \in \mathcal{P}$, the right translation $\rho_a \colon \mathcal{P} \to \mathcal{P} \colon x \mapsto xa$ is a collineation of the linear space $(\mathcal{P}, \mathcal{L})$.

(K3) All lines through the point e are subgroups of $(\mathcal{P}, \cdot)$.

Quadratic algebras

Let $\boldsymbol{A}$ be a associative unital non-zero algebra over a commutative field F; we thereby suppose $F \subseteq \boldsymbol{A}$.

If $\boldsymbol{A}$ satisfies the condition $\boldsymbol{a}^2 \in F + F\boldsymbol{a}$ for all $\boldsymbol{a} \in \boldsymbol{A}$, then $\boldsymbol{A}$ is called a *quadratic algebra* (or: *kinematic algebra*).

Any quadratic F-algebra $\boldsymbol{A}$ determines a kinematic space $(\mathcal{P}, \mathcal{L}, \cdot)$, which is embedded in the projective space $\mathbb{P}(\boldsymbol{A})$:

- $\mathcal{P} := \{F\boldsymbol{p} \mid \boldsymbol{p} \in \boldsymbol{A}^*\}$, where $\boldsymbol{A}^*$ denotes the group of invertible elements of $\boldsymbol{A}$.
- $\mathcal{L} := \{X \cap \mathcal{P} \mid X$ is a line of $\mathbb{P}(\boldsymbol{A})$ and $|X \cap \mathcal{P}| \geq 2\}$.
- The product on $\mathcal{P}$ is given as $F\boldsymbol{p} \cdot F\boldsymbol{q} := F(\boldsymbol{pq})$ for all $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{A}^*$.

Examples

- Any quaternion skew field is a 4-dimensional quadratic algebra over its centre.
- The algebra of 2×2 matrices over any commutative field F, in symbols $F^{2\times 2}$, is a 4-dimensional quadratic F-algebra.
- Study's quaternions are a 4-dimensional quadratic $\mathbb{R}$-algebra with basis $\{1, i, \varepsilon_1, \varepsilon_2\}$; multiplication is given by:

	i	ε_1	ε_2
i	-1	ε_2	$-\varepsilon_1$
ε_1	$-\varepsilon_2$	0	0
ε_2	ε_1	0	0

The corresponding kinematic space is the one of Blaschke and Grünwald.

Algebraic properties of a kinematic space

- All lines through e constitute a *fibration* $\mathcal{F}$ of the group $(\mathcal{P}, \cdot)$, that is, each $x \in \mathcal{P} \setminus \{e\}$ belongs to precisely one subgroup from $\mathcal{F}$.
- This fibration $\mathcal{F}$ is invariant under all inner automorphisms of $(\mathcal{P}, \cdot)$; in symbols: $a^{-1}\mathcal{F}a = \mathcal{F}$ for all $a \in \mathcal{P}$.

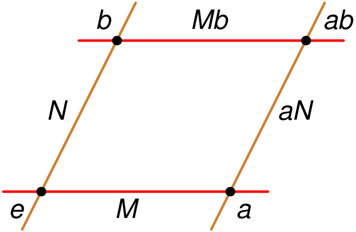
Some authors use the term group partition rather than group fibration.

Geometric properties of a kinematic space

- Under the action of the group $\{\lambda_a \mid a \in \mathcal{P}\}$ of all left translations the line set $\mathcal{L}$ splits into orbits. The corresponding equivalence relation is a parallelism, which is called the *left parallelism* $\parallel_\ell$.
- The *right parallelism* $\parallel_r$ is defined in an analogous way.

Geometric properties of a kinematic space

- If $|\mathcal{F}| > 1$, then there exists "mixed" parallelograms: $M \parallel_r Mb$, $N \parallel_\ell aN$.



- Each kinematic space $(\mathcal{P}, \mathcal{L}, \cdot)$ determines the geometric structure
$$(\mathcal{P}, \mathcal{L}, \parallel_\ell, \parallel_r),$$
that is, a linear space with two parallelisms (not necessarily distinct).

A geometric problem

A crucial problem is to describe the group, say Γ, comprising those collineations of $(\mathcal{P}, \mathcal{L})$ which preserve $\parallel_\ell$ and $\parallel_r$ (in both directions).

The group $\{\lambda_a \mid a \in \mathcal{P}\}$ of all left translations is easily seen to be a subgroup of Γ, and it acts regularly on $\mathcal{P}$.

Thus, in order to describe Γ, it suffices to determine the stabiliser of e in Γ, in symbols:

$$\Gamma_e := \{\varkappa \in \Gamma \mid \varkappa(e) = e\}.$$

Algebra vs. geometry

Theorem (S. Pianta [25])

If $|\mathcal{F}| > 1$, then Γ_e comprises precisely those automorphisms of the group $(\mathcal{P}, \cdot)$ that stabilise the fibration $\mathcal{F}$ as a set.

Corollary (S. Pianta [25])

If $|\mathcal{F}| > 1$, then the group $\{\lambda_a \mid a \in \mathcal{P}\}$ of all left translations is a normal subgroup of Γ. Furthermore,

$$\Gamma = \{\lambda_a \mid a \in \mathcal{P}\} \rtimes \Gamma_e.$$

Applications

Pianta's theorem ($\Gamma = \{\lambda_a \mid a \in \mathcal{P}\} \rtimes \Gamma_e$) turned out as a powerful tool in order to explicitly describe the group Γ for specific classes of kinematic spaces:

- S. Pianta, *Non-commutative affine kinematic spaces and their automorphism group* [26].
  Background: near vector spaces.
- S. Pianta and E. Zizioli, *Collineations of geometric structures derived from quaternion algebras* [27].
  Among other results, Γ_e is determined for the kinematic space on any matrix algebra $F^{2\times 2}$.
- S. Pianta and E. Zizioli, *Split extensions of kinematic spaces and their automorphisms* [28].
  Background: planar nearfields.

Outlook

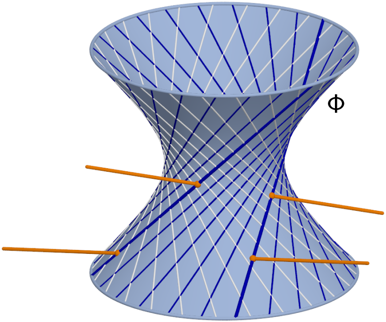
Real quaternions

The kinematic space on the real quaternions $\mathbb{H}$ provides a point model for the motion group of the elliptic plane. Lines of this kinematic space are left or right parallel precisely when they are parallel in the sense of W. K. Clifford [7].

A classical way to understand the properties of this kinematic space makes use of its embedding in a complex projective space:

1. Complexification of $\mathbb{H}$ gives a 4-dimensional quadratic $\mathbb{C}$-algebra, which is isomorphic (as a $\mathbb{C}$-algebra) to $\mathbb{C}^{2\times 2}$.
2. The kinematic space on $\mathbb{C}^{2\times 2}$ is, loosely speaking, a three-dimensional projective space over $\mathbb{C}$ from which one hyperbolic quadric Φ has been removed.

F. Klein's perspective of Clifford's parallelism [21]



Two distinct lines of the kinematic space on $\mathbb{H}$ are left or right parallel if, and only if, the corresponding lines of $\mathbb{P}(\mathbb{C}^{2\times 2})$ meet complex conjugate (and hence skew) generators of Φ.
One regulus of Φ yields the left parallelism, the other one the right parallelism; see, among others, D. Betten and R. Riesinger [2], A. Cogliati [8].

Clifford-like parallelisms

Finally, let us take a glance at the following paper:

A. Blunck, S. Pasotti and S. Pianta, *Generalized Clifford parallelisms* [5].

It is shown there that left and right parallelism in a kinematic space on an arbitrary quaternion skew field can be described in the spirit of the previous slides. However, in general it is no longer enough to consider a single quadratic extension in order to accomplish this task.

The last observation paves the way for the notion of a Clifford-like parallelism, which fails to have a (non-trivial) analogue in the case of real quaternions.

See also H. H., S. Pasotti and S. Pianta [11], [12], [13].

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