Projective metric geometry and Clifford algebras

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Combinatorics 2024
Carovigno, June 4th, 2024

Metric vector spaces

- Let $\boldsymbol{V}$ be a vector space over a (commutative) field F, and $Q\colon \boldsymbol{V} \to F$ be a quadratic form. Then $(\boldsymbol{V}, Q)$ is called a *metric vector space* (E. M. Schröder [8]).
- Throughout, we assume $\dim \boldsymbol{V} =: n+1$ to be finite.
- A vector $\boldsymbol{r} \in \boldsymbol{V}$ is called *regular* if $Q(\boldsymbol{r}) \neq 0$.
- The *polar form* of Q is the symmetric bilinear form

$$B\colon \boldsymbol{V} \times \boldsymbol{V} \to F\colon (\boldsymbol{x}, \boldsymbol{y}) \mapsto Q(\boldsymbol{x}+\boldsymbol{y}) - Q(\boldsymbol{x}) - Q(\boldsymbol{y}).$$

- Vectors $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$ are *orthogonal*, in symbols $\boldsymbol{x} \perp \boldsymbol{y}$, precisely when $B(\boldsymbol{x}, \boldsymbol{y}) = 0$.
- The *radical* of B is a subspace of $\boldsymbol{V}$, namely

$$\boldsymbol{V}^{\perp} := \{\boldsymbol{x} \in \boldsymbol{V} \mid \boldsymbol{x} \perp \boldsymbol{y} \text{ for all } \boldsymbol{y} \in \boldsymbol{V}\}.$$

The Clifford algebra of $(\boldsymbol{V}, Q)$

Each metric vector space $(\boldsymbol{V}, Q)$ determines its Clifford algebra $\mathrm{Cl}(\boldsymbol{V}, Q)$, which has the following properties:

- $\mathrm{Cl}(\boldsymbol{V}, Q)$ is an associative unital F-algebra containing $\boldsymbol{V}$ as a subspace.
- By identifying $1 \in F$ with the unit element of $\mathrm{Cl}(\boldsymbol{V}, Q)$, we obtain $F \leq \mathrm{Cl}(\boldsymbol{V}, Q)$.
- For all $\boldsymbol{x} \in \boldsymbol{V}$, we have $Q(\boldsymbol{x}) = \boldsymbol{x}^2$.
- For all $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$, we have $B(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}\boldsymbol{y} + \boldsymbol{y}\boldsymbol{x}$.
- If $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is a basis of $\boldsymbol{V}$, then

  $$\{\boldsymbol{e}_{j_1}\boldsymbol{e}_{j_2}\cdots\boldsymbol{e}_{j_k} \mid 0 \leq j_1 < j_2 < \cdots < j_k \leq n\},$$

  is a basis of $\mathrm{Cl}(\boldsymbol{V}, Q)$; thereby an empty product is understood to be $1 \in \mathrm{Cl}(\boldsymbol{V}, Q)$.
- The dimension of $\mathrm{Cl}(\boldsymbol{V}, Q)$ equals 2^{n+1}.

The Clifford algebra of $(\boldsymbol{V}, Q)$ (cont.)

- The Clifford algebra $\mathrm{Cl}(\boldsymbol{V}, Q)$ is $\mathbb{Z}/(2\mathbb{Z})$-graded and so it is the direct sum of the *even part* $\mathrm{Cl}_0(\boldsymbol{V}, Q)$, which is a subalgebra of $\mathrm{Cl}(\boldsymbol{V}, Q)$, and the *odd part* $\mathrm{Cl}_1(\boldsymbol{V}, Q)$.
- In particular, $F \leq \mathrm{Cl}_0(\boldsymbol{V}, Q)$ and $\boldsymbol{V} \leq \mathrm{Cl}_1(\boldsymbol{V}, Q)$.
- If $\boldsymbol{h} \in \mathrm{Cl}_i(\boldsymbol{V}, Q)$, $i \in \{0, 1\}$, then we say that $\boldsymbol{h}$ is *homogeneous* of *degree* i and write $\partial \boldsymbol{h} = i$.
- The *main involution* σ is that algebra automorphism of $\mathrm{Cl}(\boldsymbol{V}, Q)$ which sends any $\boldsymbol{h} \in \mathrm{Cl}_i(\boldsymbol{V}, Q)$, $i \in \{0, 1\}$ to $(-1)^{\partial \boldsymbol{h}} \boldsymbol{h} \in \mathrm{Cl}_i(\boldsymbol{V}, Q)$.

The weak orthogonal group of $(\boldsymbol{V}, Q)$

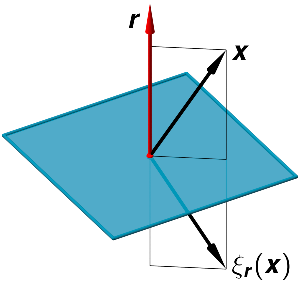
- A mapping $\psi \in \mathrm{GL}(\boldsymbol{V})$ is called an *isometry* if $Q = Q \circ \psi$.
- All isometries of $(\boldsymbol{V}, Q)$ constitute the *orthogonal group* $\mathrm{O}(\boldsymbol{V}, Q)$.
- The *weak orthogonal group* $\mathrm{O}'(\boldsymbol{V}, Q)$ consists of all isometries of $(\boldsymbol{V}, Q)$ that fix the radical $\boldsymbol{V}^{\perp}$ elementwise (E. Ellers [2]).

Reflections

Let $\boldsymbol{r} \in \boldsymbol{V}$ be regular. Then the mapping

$$\xi_{\boldsymbol{r}} \colon \boldsymbol{V} \to \boldsymbol{V} \colon \boldsymbol{x} \mapsto \boldsymbol{x} - B(\boldsymbol{r}, \boldsymbol{x}) Q(\boldsymbol{r})^{-1} \boldsymbol{r}$$

is called the *reflection* of $(\boldsymbol{V}, Q)$ in the direction of $\boldsymbol{r}$.



- A vector $\boldsymbol{y} \in \boldsymbol{V}$ is fixed under $\xi_{\boldsymbol{r}}$ if, and only if, $\boldsymbol{y} \perp \boldsymbol{r}$.
- We have $\xi_{\boldsymbol{r}} \in \mathrm{O}'(\boldsymbol{V}, Q)$.

Reflections in terms of $\mathrm{Cl}(\boldsymbol{V}, Q)$

Let $\xi_{\boldsymbol{r}}$ be the reflection in the direction of a regular vector $\boldsymbol{r} \in \boldsymbol{V}$.

Then, for all $\boldsymbol{x} \in \boldsymbol{V}$,

$$\begin{aligned}
\xi_{\boldsymbol{r}}(\boldsymbol{x}) &= \boldsymbol{x} - B(\boldsymbol{r}, \boldsymbol{x}) \cdot Q(\boldsymbol{r})^{-1} \cdot \boldsymbol{r} \\
&= \boldsymbol{x} - (\boldsymbol{r}\boldsymbol{x} + \boldsymbol{x}\boldsymbol{r}) \cdot \boldsymbol{r}^{-2} \cdot \boldsymbol{r} \\
&= \boldsymbol{x} - \boldsymbol{r}\boldsymbol{x}\boldsymbol{r}^{-1} - \boldsymbol{x} \\
&= -\boldsymbol{r}\boldsymbol{x}\boldsymbol{r}^{-1} \\
&= \boldsymbol{r}\boldsymbol{x}\sigma(\boldsymbol{r})^{-1},
\end{aligned}$$

where σ denotes the main involution.

The Lipschitz group $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$

Below we follow J. Helmstetter [5].

The *Lipschitz group* $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$ is the multiplicative group in $\mathrm{Cl}(\boldsymbol{V}, Q)$ generated by the set comprising all non-zero scalars in F, all regular vectors in $\boldsymbol{V}$ and all elements

$$1 + \boldsymbol{st} \quad \text{with} \quad \boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{V} \quad \text{and} \quad Q(\boldsymbol{s}) = Q(\boldsymbol{t}) = B(\boldsymbol{s}, \boldsymbol{t}) = 0.$$

- Up to some exceptional cases, the Lipschitz group $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$ is already generated by the set of regular vectors in $\boldsymbol{V}$.
- $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$ contains only homogeneous elements.

The Lipschitz group $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$ (cont.)

The mapping

$$\xi\colon\ \mathrm{Lip}^{\times}(\boldsymbol{V}, Q) \to \mathrm{O}'(\boldsymbol{V}, Q)\colon\ \boldsymbol{p} \mapsto \left(\xi_{\boldsymbol{p}}\colon\ \boldsymbol{x} \mapsto \boldsymbol{p}\boldsymbol{x}\sigma(\boldsymbol{p})^{-1}\right) \tag{1}$$

is a surjective homomorphism of groups, known as the *twisted adjoint representation* of $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$ (M. F. Atiyah, R. Bott and A. Shapiro [1]).

Main issue

In projective metric geometry one deals with $\mathbb{P}(\boldsymbol{V}, Q)$, the projective space on $(\boldsymbol{V}, Q)$ (E. M. Schröder [8]).

- If the quadratic form Q is replaced by a non-zero multiple, say cQ with $c \in F^{\times} := F \setminus \{0\}$, then this does not affect the geometry of $\mathbb{P}(\boldsymbol{V}, Q)$.
- On the other hand, the Clifford algebras $\mathrm{Cl}(\boldsymbol{V}, Q)$ and $\mathrm{Cl}(\boldsymbol{V}, cQ)$ need not be isomorphic. Likewise, the Lipschitz groups $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q)$ and $\mathrm{Lip}^{\times}(\boldsymbol{V}, cQ)$ need not be isomorphic.

Example

Let $|F| = 3$ and $\dim \boldsymbol{V} = 1$. We pick a basis vector $\boldsymbol{e}_0 \in \boldsymbol{V}$ and define $Q\colon \boldsymbol{V} \to F$ by $Q(\boldsymbol{e}_0) = 1$.

- The Clifford algebra $\mathrm{Cl}(\boldsymbol{V}, Q)$ contains zero divisors, since

$$1 - \boldsymbol{e}_0 \neq 0 \text{ and } (1 - \boldsymbol{e}_0)(1 + \boldsymbol{e}_0) = 1 - \boldsymbol{e}_0^2 = 1 - 1 = 0.$$

- $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q) = \{1, -1, \boldsymbol{e}_0, -\boldsymbol{e}_0\}$, where

$$1^2 = (-1)^2 = \boldsymbol{e}_0^2 = (-\boldsymbol{e}_0)^2 = 1.$$

Example (cont.)

Next, we replace Q with $-Q$.

- The Clifford algebra $\mathrm{Cl}(\boldsymbol{V}, -Q)$ is a field with 9 elements. Indeed, now $\boldsymbol{e}_0^2 = -1$ gives that

$$
\begin{aligned}
& & 1 - \boldsymbol{e}_0, \qquad & (1-\boldsymbol{e}_0)^2 = \boldsymbol{e}_0, \\
(1-\boldsymbol{e}_0)^3 &= & 1 + \boldsymbol{e}_0, \qquad & (1-\boldsymbol{e}_0)^4 = -1, \\
(1-\boldsymbol{e}_0)^5 &= & -1 + \boldsymbol{e}_0, \qquad & (1-\boldsymbol{e}_0)^6 = -\boldsymbol{e}_0, \\
(1-\boldsymbol{e}_0)^7 &= & -1 - \boldsymbol{e}_0, \qquad & (1-\boldsymbol{e}_0)^8 = 1
\end{aligned}
$$

  are all non-zero elements of $\mathrm{Cl}(\boldsymbol{V}, -Q)$.

- $\mathrm{Lip}^{\times}(\boldsymbol{V}, Q) = \{1, -1, \boldsymbol{e}_0, -\boldsymbol{e}_0\}$, where

$$1^2 = (-1)^2 = 1 \neq -1 = \boldsymbol{e}_0^2 = (-\boldsymbol{e}_0)^2.$$

A Clifford algebra for $(\boldsymbol{V}, cQ)$

Theorem ([3, Sect. 6]).

Let $(\boldsymbol{V}, Q)$ be a metric vector space and $c \in F^\times$. The vector space underlying $\mathrm{Cl}(\boldsymbol{V}, Q)$ can be made into a Clifford algebra for $(\boldsymbol{V}, cQ)$ by defining a multiplication $\odot_c$ as follows: Given any $\boldsymbol{f}, \boldsymbol{g} \in \mathrm{Cl}(\boldsymbol{V}, Q)$ write $\boldsymbol{f} = \boldsymbol{f}_0 + \boldsymbol{f}_1$ and $\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{g}_1$, where $\boldsymbol{f}_i, \boldsymbol{g}_i \in \mathrm{Cl}_i(\boldsymbol{V}, Q)$ for $i \in \{0, 1\}$ and put

$$\boldsymbol{f} \odot_c \boldsymbol{g} := \underbrace{\boldsymbol{f}_0\boldsymbol{g}_0 + c\boldsymbol{f}_1\boldsymbol{g}_1}_{\in \mathrm{Cl}_0(\boldsymbol{V}, Q)} + \underbrace{\boldsymbol{f}_0\boldsymbol{g}_1 + \boldsymbol{f}_1\boldsymbol{g}_0}_{\in \mathrm{Cl}_1(\boldsymbol{V}, Q)}.$$

Our proof is based upon a result by M.-A. Knus [6, Ch. IV (7.1.1)].

A Clifford algebra for $(\boldsymbol{V}, cQ)$ (cont.)

Definition.

We denote the Clifford algebra for $(\boldsymbol{V}, cQ)$, as defined in the previous theorem, as $\mathrm{Cl}(\boldsymbol{V}, Q, \odot_c)$.

- The even Clifford algebras $\mathrm{Cl}_0(\boldsymbol{V}, Q)$ and $\mathrm{Cl}_0(\boldsymbol{V}, Q, \odot_c)$ are identical (as algebras).
- The subspaces $\mathrm{Cl}_1(\boldsymbol{V}, Q)$ and $\mathrm{Cl}_1(\boldsymbol{V}, Q, \odot_c)$ are identical.
- Let $\boldsymbol{p}, \boldsymbol{q}$ be homogeneous elements of $\mathrm{Cl}(\boldsymbol{V}, Q)$. Then $\boldsymbol{p} \odot_c \boldsymbol{q} = c^{\partial \boldsymbol{p} \, \partial \boldsymbol{q}} \boldsymbol{p}\boldsymbol{q}$.

The group $\mathcal{G}(\boldsymbol{V}, Q)$

The Lipschitz group $\mathrm{Lip}^\times(\boldsymbol{V}, Q)$ gives rise to the point set

$$\mathcal{G}(\boldsymbol{V}, Q) := \big\{F\boldsymbol{p} \mid \boldsymbol{p} \in \mathrm{Lip}^\times(\boldsymbol{V}, Q)\big\}$$

in $\mathbb{P}\big(\mathrm{Cl}(\boldsymbol{V}, Q)\big)$, which can be made into (multiplicative) group in the following way:

$$(F\boldsymbol{p})(F\boldsymbol{q}) := F(\boldsymbol{pq}) \ \text{ for all } \ F\boldsymbol{p}, F\boldsymbol{q} \in \mathcal{G}(\boldsymbol{V}, Q).$$

- $\mathcal{G}(\boldsymbol{V}, Q) \cong \mathrm{Lip}^\times(\boldsymbol{V}, Q)/F^\times$.
- $\mathcal{G}(\boldsymbol{V}, Q) = \mathcal{G}(\boldsymbol{V}, Q, \odot_c)$ for all $c \in F^\times$ [3, Cor. 6.6 (e)].

Action of $\mathcal{G}(\boldsymbol{V}, Q)$ on $\mathbb{P}(\boldsymbol{V}, Q)$

- From (1), the group $\mathcal{G}(\boldsymbol{V}, Q)$ acts on the projective space $\mathbb{P}(\boldsymbol{V}, Q)$ as follows: For all points $F\boldsymbol{p} \in \mathcal{G}(\boldsymbol{V}, Q)$ and all flats $\boldsymbol{X} \in \mathbb{P}(\boldsymbol{V}, Q)$, we have

$$F\boldsymbol{p} \mapsto \left(\boldsymbol{X} \mapsto \xi_{\boldsymbol{p}}(\boldsymbol{X}) = \boldsymbol{p}\boldsymbol{X}\sigma(\boldsymbol{p})^{-1}\right). \tag{2}$$

- This action of $\mathcal{G}(\boldsymbol{V}, Q)$ on $\mathbb{P}(\boldsymbol{V}, Q)$ yields a surjective homomorphism of groups

$$\mathcal{G}(\boldsymbol{V}, Q) \to \mathrm{PO}'(\boldsymbol{V}, Q),$$

  where $\mathrm{PO}'(\boldsymbol{V}, Q)$ denotes the image of $\mathrm{O}'(\boldsymbol{V}, Q)$ under the canonical homomorphism $\mathrm{GL}(\boldsymbol{V}) \to \mathrm{PGL}(\boldsymbol{V})$.
- The group action (2) remains unaltered when going over to any $\mathrm{Cl}(\boldsymbol{V}, Q, \odot_c)$ with $c \in F^\times$ [3, Cor. 6.6 (f)].

Final remarks

- There are several other notions that remain unchanged under the transition from $\mathrm{Cl}(\boldsymbol{V}, Q)$ to $\mathrm{Cl}(\boldsymbol{V}, Q, \odot_c)$; see [3, Cor. 6.6].
- Among these notions is the point set arising from the Lipschitz monoid. This point set is the union of two algebraic varieties—one in $\mathbb{P}\big(\mathrm{Cl}_0(\boldsymbol{V}, Q)\big)$ and one in $\mathbb{P}\big(\mathrm{Cl}_1(\boldsymbol{V}, Q)\big)$ (J. Helmstetter [5]).

References

For related work see [3], [4], [5], [7], [8] and the references therein.

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