GEOMETRY OF FIELD EXTENSIONS

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K, L fields $\quad K \subset L, \; K \neq L$

K is not necessarily a part of the centre of L

(W. BENZ)

$L \oplus L$ vector space over $\left< \begin{array}{l} L \\ K \end{array} \right.$

$\underline{\underline{L}}$: projective line over L

K-sublines (chains)

$\underline{\underline{K}}$: spread of $\mathcal{P}_K(L \oplus L)$

chains

$\downarrow$

SEGRE-manifold if $K \subset Z_L(L)$

(Geometry of algebras!)

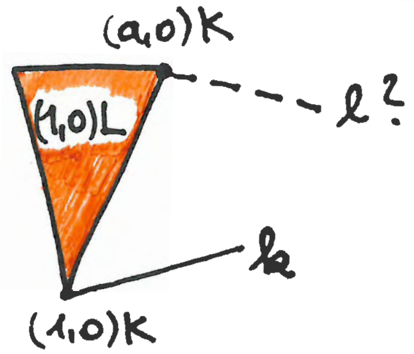
k... chain
ℓ... line

ℓ is called a transversal line of k if

$$\begin{array}{ccc} k & \longrightarrow & \ell \\ y & \longmapsto & y\cap\ell \end{array} \quad \text{is bijective}$$

standard chain c... $\{(k_0,k_1)L \mid (0,0)\neq(k_0,k_1)\in K\times K\}$

standard transversal line $\mathfrak{k}$... K



ℓ is transversal $\iff$ $\boxed{a^{-1}Ka = K}$ $\iff$ $\left\{\begin{array}{l}\varphi_a: K\to K\\ \quad x\mapsto a^{-1}xa\end{array}\right\}\in \mathrm{Aut}(K)$

$$\left.\begin{array}{l}\alpha: \mathfrak{k}\to\ell\\ \quad y\cap\mathfrak{k}\mapsto y\cap\ell\end{array}\right\}$$ is a bijection (φ_a-semilinear!)

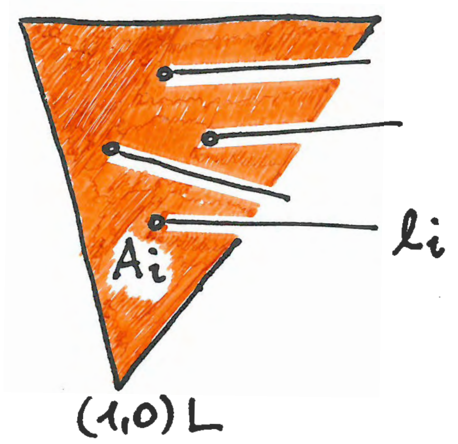
φ_a is inner $\iff \exists\, u\in K^\times : a^{-1}xa = u^{-1}xu \quad \forall x\in K$

$\iff au^{-1}\in Z_L(K)^\times \quad (u\in K^\times)$

$\iff \boxed{a \in Z_L(K)^\times\cdot K^\times}$

$\iff \alpha$ is projective

$:\iff \mathfrak{k}$ and ℓ are projectively linked.



- $\{A_i \mid i \in I\}$ is an r-frame $(r \geq 1)$ $\Longrightarrow$
  ℓ_i's are projectively linked
- different ℓ_i's not projectively linked $\Longrightarrow$
  $\{A_i \mid i \in I\}$ is independent and no other point of
  span$\{A_i \mid i \in I\}$ is on a transversal line
- $\{\ell_i \mid i \in I\}$ maximal set of pairwise projectively linked transversal lines $\Longrightarrow$
  $\mathcal{T} := \{A_i \mid i \in I\}$ as a trace space is isomorphic
  to the projective space on $Z_L(K)$ over $Z_K(K)$.

Theorem of CARTAN-BRAUER-HUA

$$a^{-1}Ka = K \quad \forall a \in L^{\times} \iff K \subset Z_L(L)$$

Example:

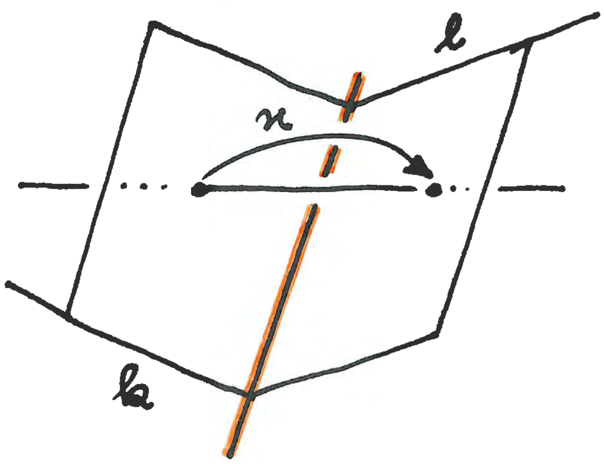
$L = \mathbb{H} \ \dots \ \{1, i, j, k\}$

$K = \mathbb{C} \ \dots \ \{1, i\}$

Spread of $PG(3,\mathbb{C})$

$$\left.\begin{array}{l} j^{-1}\mathbb{C}\, j = \mathbb{C} \quad (\varphi_j \text{ is conjugation in } \mathbb{C}) \\ j \notin Z_{\mathbb{H}}(\mathbb{C}) \end{array}\right\} \Rightarrow$$

Every chain has exactly two transversal lines (linked by an anti-projectivity).



$\varkappa$.. non-projective collineation generating the spread