170 Years of Harmonicity Preservers

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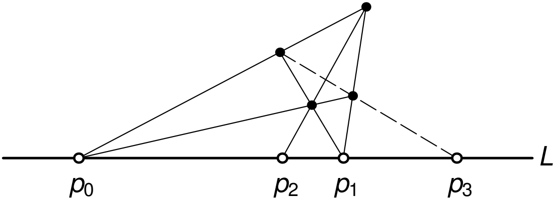
XI Północne Spotkania Geometryczne,
Gdańsk, June 21, 2017

Background

Harmonic quadruples



In a projective plane, a quadruple of points (p_0, p_1, p_2, p_3) on a line L is harmonic if there exists a quadrangle such that one pair of opposite sides intersects at p_0, and a second pair at p_1, while the third pair meets L at p_2 and p_3.

K. G. Chr. von Staudt, Geometrie der Lage [53] (1847)

Zwei einförmige Grundgebilde heissen zu einander projektivisch ($\barwedge$), wenn sie so auf einander bezogen sind, dass jedem harmonischen Gebilde in dem einen ein harmonisches Gebilde im andern entspricht.

Next, after defining perspectivities, the following theorem is established:

Any projectivity is a finite composition of perspectivities and vice versa.

Any result in this spirit now is called a *von Staudt's theorem*.

Remark

It was noticed by M. G. Darboux [23] in 1880 that there is a small gap in von Staudt's reasoning. See also F. Klein [38], F. Schur [52], and the survey by J.-D. Voelke [54].

The Projective Line over a Ring

Our rings

All our rings are associative, with a unit element $1 \neq 0$ which is preserved by homomorphisms, inherited by subrings, and acts unitally on modules.

Free modules

- Let $(R, +, \cdot)$ be a ring.
  The set of units (invertible elements) of R is is a group under multiplication and will be denoted by R^*.
- Let M be a free left R-module of rank 2. So, M has at least one basis with two elements.
  For any basis (e_0, e_1) of M the mapping
  $$R^2 \to M\colon (x_0, x_1) \mapsto x_0 e_0 + x_1 e_1 \tag{1}$$
  is bijective.
- We may consider R^2 a left R-module in the usual way.
  Then the mapping (1) is an isomorphism of left R-modules.
- We do not require that all bases of M have the same number of elements.

The projective line on M

Definition

An element $a \in M$ is called *admissible* if there exists $b \in M$ such that (a, b) is a basis of M (with two elements).
The *projective line on M* is the set $\mathbb{P}(M)$ of all cyclic submodules Ra, where $a \in M$ is admissible. The elements of $\mathbb{P}(M)$ are called *points*.

$\mathbb{P}(M)$ is also called the *projective line over the ring R*. Many authors confine themselves to the case $M = R^2$.

See A. Blunck, A. Herzer: Kettengeometrien [17], A. Herzer [31], and H. H. [29] for a detailed exposition of the results that are presented on the next slides.

Remarks

- Admissible elements of M generate the same point if, and only if, they are left proportional by a unit in R.
- Let (a, b) be a basis of M. Then $\mathbb{P}(M)$ may also be described as the orbit of the "starter point" Ra under the natural action of the general linear group $\mathrm{GL}(M)$ on $\mathbb{P}(M)$.
- $\mathbb{P}(M)$ is the set of all free cyclic submodules p, such that there is a free cyclic submodule q with $p \oplus q = M$.
- Some authors use different definitions: For example, in projective lattice geometry all cyclic submodules of M are considered as "points" (U. Brehm, M. Greferath, S. E. Schmidt [18], C. A. Faure [24]).

The distant relation

Definition

Two points p and q of $\mathbb{P}(M)$ are called *distant*, in symbols $p \bigtriangleup q$, if $M = p \oplus q$.

The distant relation (cont.)

Lemma

Suppose that admissible elements $a, b \in M$ have coordinates

$$(x_0, x_1) \;\; \textit{resp.} \;\; (y_0, y_1)$$

w. r. t. some basis (e_0, e_1) of M. Then the following are equivalent:

1. *$Ra \,\triangle\, Rb$ are distant points of $\mathbb{P}(M)$.*
2. *(a, b) is a basis of M.*
3. *The matrix*
$$\begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix} \in R^{2\times 2}$$
is invertible.

Properties

- The relation $\vartriangle$ is symmetric and antireflexive.
- The relation $\vartriangle$ is invariant under the action of $\mathrm{GL}(M)$ on $\mathbb{P}(M)$.
- The group $\mathrm{GL}(M)$ acts transitively on the triples of mutually distant points of $\mathbb{P}(M)$.
- Non-distant points are also called *neighbouring*.
- The relation $\not\vartriangle$ equals the identity relation on $\mathbb{P}(M) \Leftrightarrow$ R is a field.
- The relation $\not\vartriangle$ is an equivalence relation $\Leftrightarrow$ R is a *local ring*, i.e., $R \setminus R^*$ is an ideal of R.

The distant graph

- $(\mathbb{P}(M), \triangle)$ is called the *distant graph* of $\mathbb{P}(M)$. It is an undirected graph without loops.
- Let (e_0, e_1) be a basis of M. Then the mapping

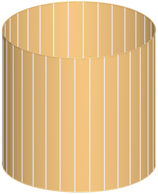
$$R \to \{p \in \mathbb{P}(M) \mid p \triangle Re_0\} : x \mapsto R(xe_0 + 1e_1) \qquad (2)$$

  is bijective.
- From (2), each vertex (point) of the distant graph is on $\#R$ edges.

Examples

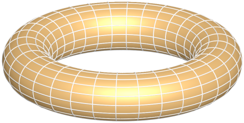
The projective line over some rings can be modelled as surfaces with a system of distinguished curves that illustrate the non-distant relation.

Cylinder:



Real dual numbers $\mathbb{R}[\varepsilon]$, $\varepsilon^2 = 0$.

Torus:



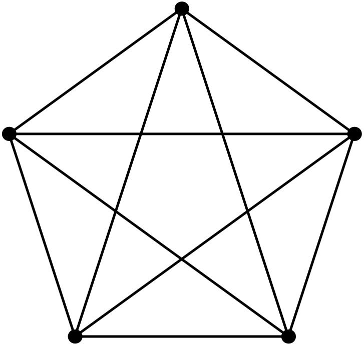
Real double numbers $\mathbb{R} \times \mathbb{R}$.

Examples: Rings with four elements

Ring

- $R = \mathrm{GF}(4)$ (Galois field).
- $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.
- $R = \mathbb{Z}_4$.
- $R = \mathbb{Z}_2[\varepsilon]$, $\varepsilon^2 = 0$ (dual numbers over $\mathbb{Z}_2$).

Distant graph



$\#\mathbb{P}(R) = 5$

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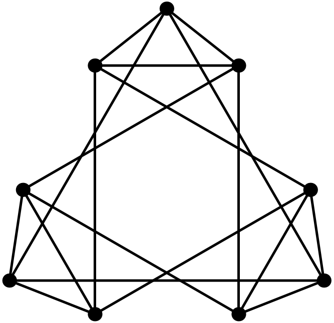
Distant graph



$\#\mathbb{P}(R) = 9$

Examples: Rings with four elements

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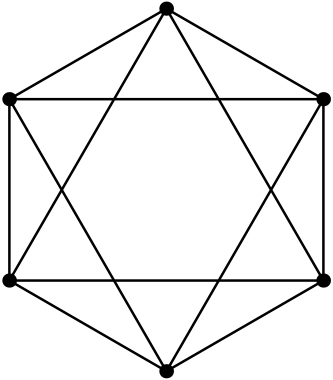
Distant graph



$\#\mathbb{P}(R) = 6$

Examples: Rings with four elements

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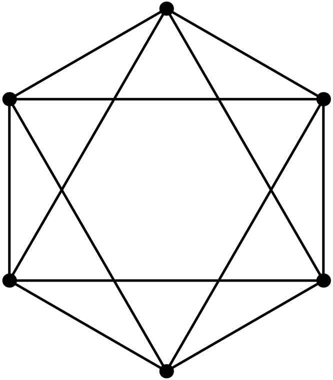
Distant graph



$\#\mathbb{P}(R) = 6$

Harmonic quadruples

Definition

A quadruple $(p_0, p_1, p_2, p_3) \in \mathbb{P}(M)^4$ is *harmonic* if there exists a basis (g_0, g_1) of M such that

$$p_0 = Rg_0, \quad p_1 = Rg_1, \quad p_2 = R(g_0 + g_1), \quad p_3 = R(g_0 - g_1).$$

In this case we write $\mathrm{H}(p_0, p_1, p_2, p_3)$.

Properties

From $\mathrm{H}(p_0, p_1, p_2, p_3)$ we obtain:

- $\mathrm{H}(p_1, p_0, p_2, p_3)$.
- $\mathrm{H}(p_0, p_1, p_3, p_2)$.
- $p_0 \vartriangle p_1$ and $\{p_0, p_1\} \vartriangle \{p_2, p_3\}$.
- $p_2 \neq p_3 \Leftrightarrow 1 \neq -1$ in $R \Leftrightarrow 1 + 1 = 2 \neq 0$ in R.
- $p_2 \vartriangle p_3 \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in \mathrm{GL}_2(R) \Leftrightarrow 2$ is a unit in R.
- $\mathrm{H}(p_3, p_2, p_1, p_0) \Leftrightarrow 2$ is a unit in R.

Existence and uniqueness

Proposition

Given three mutually distant points $p_0, p_1, p_2 \in \mathbb{P}(M)$ there exists a unique point $p_3 \in \mathbb{P}(M)$ such that (p_0, p_1, p_2, p_3) is a harmonic quadruple.

An affine description

Let (g_0, g_1) be a basis of M. For all $x \in R$ we do not distinguish between the point $R(xg_0 + 1g_1) \in \mathbb{P}(M)$ and the ring element x. (See the bijection in equation (2)). Also, we let

$$\infty := Rg_0 \in \mathbb{P}(M).$$

Lemma

For all $x, y, z \in R$ the following are equivalent:

1. $\mathrm{H}(\infty, x, y, z)$.
2. $z - x = -(y - x) \in R^*$.

Furthermore, if $2 \in R^*$ then each of these conditions is equivalent to

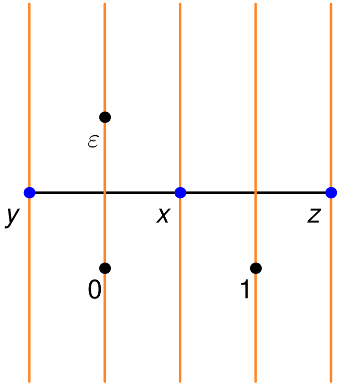
$$x = \frac{y+z}{2} \quad \text{and} \quad y - z \in R^*.$$

Examples

Ring

- $R = \mathbb{R}[\varepsilon]$
  (real dual numbers).
- $R = \mathbb{R} \times \mathbb{R}$
  (real double numbers).

$H(\infty, x, y, z)$



$$z - x = -(y - x) \in (\mathbb{R}[\varepsilon])^*$$

Examples

Ring

- $R = \mathbb{R}[\varepsilon]$ (real dual numbers).
- $R = \mathbb{R} \times \mathbb{R}$ (real double numbers).

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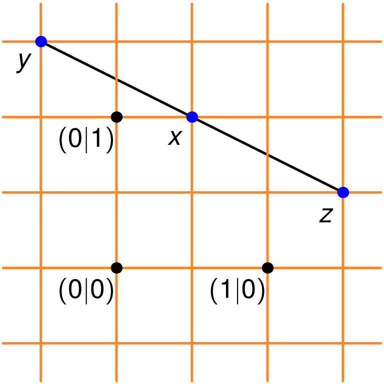
$H(\infty, x, y, z)$



Examples: Rings with four elements

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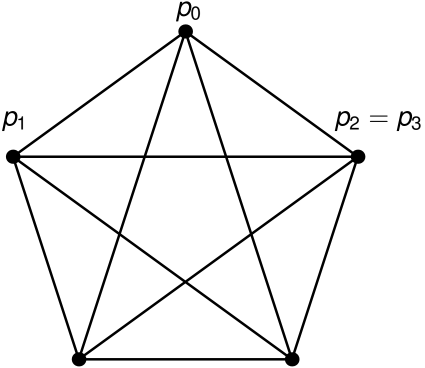
$H(p_0, p_1, p_2, p_3)$



$1 + 1 = 0 \in \mathrm{GF}(4)$

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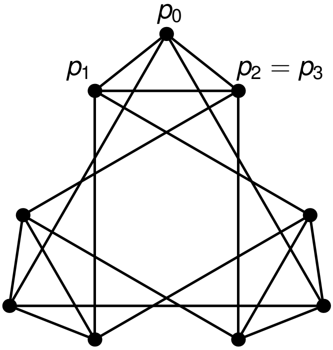

$(1|1) + (1|1) = (0|0) \in \mathbb{Z}_2 \times \mathbb{Z}_2$

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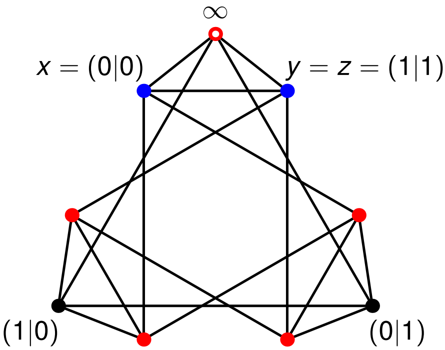
$H(\infty, x, y, z)$



$$z - x = (1|1) - (0|0) = -\big((1|1) - (0|0)\big) = -(y - x) \in (\mathbb{Z}_2 \times \mathbb{Z}_2)^*$$

Examples: Rings with four elements

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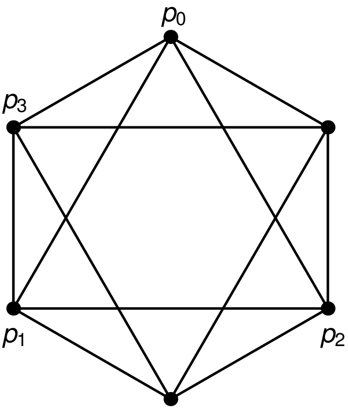
$1 + 1 = 2 \in \mathbb{Z}_4 \setminus \mathbb{Z}_4^*$

$H(p_0, p_1, p_2, p_3)$



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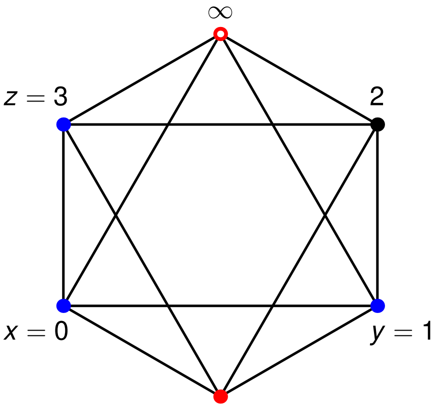
$H(\infty, x, y, z)$



$$z - x = 3 - 0 = -(1 - 0) = -(y - x) \in \mathbb{Z}_4^*$$

Examples: Rings with four elements

Ring

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- $R = \mathbb{Z}_4$.
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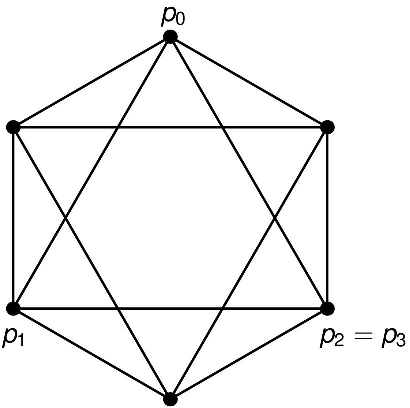
$1 + 1 = 0 \in \mathbb{Z}_2[\varepsilon]$

$H(p_0, p_1, p_2, p_3)$



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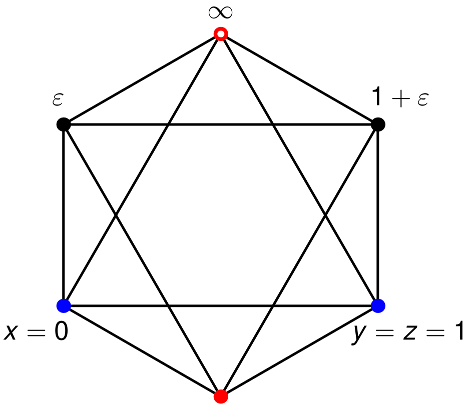
$$1 - 0 = -(1 - 0) \in \mathbb{Z}_2[\varepsilon]^*$$

$H(\infty, x, y, z)$



Harmonicity Preservers

Basics

Let M and M' be free left modules of rank 2 over rings R and R', respectively.

Definition

A mapping $\mu : \mathbb{P}(M) \to \mathbb{P}(M')$ is said to be a *harmonicity preserver* if it takes all harmonic quadruples of $\mathbb{P}(M)$ to harmonic quadruples of $\mathbb{P}(M')$.

No further assumptions, like injectivity or surjectivity of μ will be made.

Main problem

Give an algebraic description of all harmonicity preservers between projective lines over rings R and R'.

Solutions and contributions

- **Commutative Fields** with characteristic $\neq 2$ (1935): O. Schreier and E. Sperner [51].
- **Skew Fields** with characteristic $\neq 2$ (1941–1953): G. Ancochea [1], [2], [3], L.-K. Hua [33], [34], [35].

Theorem (v. Staudt–Hua)

Let R and R' be skew fields of characteristic $\neq 2$. The harmonicity preservers $\mathbb{P}(M) \to \mathbb{P}(M')$ are precisely the mappings that arise from semilinear monomorphisms $M \to M'$ with respect to a homomorphism $R \to R'$ or semilinear monomorphisms of M to the dual of M' with respect to an antihomomorphism of $R \to R'$.

Solutions and contributions (cont.)

- **(Non) commutative rings** subject to varying extra assumptions (1964–2015):
  W. Benz [8], [9],
  H. Schaeffer [50],
  B. V. Limaye and N. B. Limaye [44], [45], [46],
  N. B. Limaye [47], [48],
  B. R. McDonald [49],
  C. Bartolone and F. Di Franco [7],
  C. Bartolone and F. Bartolozzi [6],
  A. Blunck and H. H. [14], [15], [16],
  H. H. [30].

Variations

There is a wealth of results based on . . .

- **a different definition of harmonic quadruples**,
- **another invariance property**,
- **quadruples with a fixed cross ratio**,
- **other geometric structures (e.g. Moufang planes)**.

R. Baer [4], C. Bartolone and F. Di Franco [7], C. Bartolone and F. Bartolozzi [6], W. Bertram [12], A. Blunck [13], F. Buekenhout [19], D. Chkhatarashvili [20], L. Cirlincione and M. Enea [21], St. P. Cojan [22], J. C. Ferrar [25], V. Havel [26], [27], [28], A. J. Hoffman [32], D. G. James [36], B. Klotzek [39], M. Kulkarni [40], A. A. Lashkhi [42], [43], A. Lashkhi [41], B. V. Limaye and N. B. Limaye [44], [46].

See also the following surveys: W. Benz, W. Leissner, and H. Schaeffer [10]; W. Benz, H. J. Samaga, and H. Schaeffer [11]; H. Karzel and H. Kroll [37].

Jordan homomorphisms of rings

Definition

A mapping $\alpha : R \to R'$ is a *Jordan homomorphism* if for all $x, y \in R$ the following conditions are satisfied:

1. $(x + y)^\alpha = x^\alpha + y^\alpha$,
2. $1^\alpha = 1'$,
3. $(xyx)^\alpha = x^\alpha y^\alpha x^\alpha$.

Jordan homomorphisms of rings

Examples

- All homomorphisms of rings, in particular $\mathrm{id}_R : R \to R$.
- All antihomomorphisms of rings; e. g. the conjugation of real quaternions: $\mathbb{H} \to \mathbb{H}$ with $z \mapsto \overline{z}$.
- The mapping $\mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H} : (z, w) \mapsto (\overline{z}, w)$ which is neither homomorphic nor antihomomorphic.

Jordan homomorphisms of rings

Examples (cont.)

- Let V be a two-dimensional vector space over a commutative field $\mathbb{F}$. We decompose its exterior algebra in the form

$$\bigwedge V = \mathbb{F} \oplus W \text{ with } W := V \oplus (V \wedge V).$$

Choose any $\mathbb{F}$-linear endomorphism λ of W. Then

$$\alpha\colon \mathbb{F} \oplus W \to \mathbb{F} \oplus W\colon (f, w) \mapsto (f, w^\lambda)$$

is a Jordan homomorphism.

There is a choice of λ such that α is neither homomorphic nor antihomomorphic.

Beware of Jordan homomorphisms

Beware of Jordan homomorphisms

Beware of Jordan homomorphisms

Let $\alpha : R \to R'$ be a Jordan homomorphism. Given bases (e_0, e_1) of M and (e'_0, e'_1) of M' the mapping $M \to M'$ defined by

$$x_0 e_0 + x_1 e_1 \mapsto x_0^\alpha e'_0 + x_1^\alpha e'_1 \text{ for all } x_0, x_1 \in R$$

need not take submodules to submodules (let alone points to points).

Let $\mu : \mathbb{P}(M) \to \mathbb{P}(M')$ be a harmonicity preserver. Furthermore, let R satisfy the following two conditions:

(i) Given $x_1, x_2, \ldots, x_5 \in R$ there exists $x \in R$ such that $x - x_1, x - x_2, \ldots, x - x_5$ are units in R.

(ii) 2 is a unit in R.

Step 1: A local coordinate representation of μ

There are bases (e_0, e_1) of M and (e'_0, e'_1) of M' such that

$$(Re_0)^\mu = R'e'_0, \quad (Re_1)^\mu = R'e'_1, \quad \big(R(e_0 \pm e_1)\big)^\mu = R'(e'_0 \pm e'_1).$$

Then there exists a unique mapping $\beta : R \to R'$ with the property

$$\big(R(xe_0 + 1e_1)\big)^\mu = R'(x^\beta e'_0 + 1'e'_1) \quad \text{for all} \quad x \in R.$$

This β is additive and satisfies $1^\beta = 1'$.

Step 2: Change of coordinates

We may repeat Step 1 for the new bases

$$(f_0, f_1) := (te_0 + e_1, -e_0) \quad \text{and} \quad (f_0', f_1') := (t^\beta e_0' + e_1', -e_0'),$$

where $t \in R$ is arbitrary. So the transition matrices are

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad E(t^\beta) := \begin{pmatrix} t^\beta & 1' \\ -1' & 0' \end{pmatrix}.$$

Then the new local representation of μ yields the same mapping β as in Step 1.

Step 3: β is a Jordan homomorphism

By combining Step 1 and Step 2 (for $t = 0$) one obtains:

The mapping β from Step 1 is a Jordan homomorphism.

Part of the proof relies on previous work.

Step 4: Induction

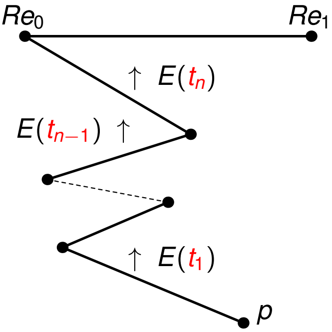
Suppose that $p \in \mathbb{P}(M)$ can be written as

$$p = R(x_0 e_0 + x_1 e_1)$$

with

$$(x_0, x_1) = (1, 0) \cdot E(t_1) \cdot E(t_2) \cdots E(t_n)$$

for some $t_1, t_2, \ldots, t_n \in R$, where n is variable.



Then the image point of p under μ is $R'(x'_0 e'_0 + x'_1 e'_1)$ with

$$(x'_0, x'_1) = (1', 0') \cdot E(t_1^\beta) \cdot E(t_2^\beta) \cdots E(t_n^\beta).$$

Step 5: Conclusion

Theorem (Main Result - Part 1)

If the distant graph $(\mathbb{P}(M), \triangle)$ is connected then the formula from Step 4 describes the harmonicity preserver μ in terms of a Jordan homomorphism.

Theorem (Main Result - Part 2)

If the distant graph $(\mathbb{P}(M), \triangle)$ is not connected then the formula from Step 4 yields just the restriction of μ to one connected component of the distant graph.
Under these circumstances μ can be described in terms of several Jordan homomorphisms (one for each connected component).

For a detailed proof see [30].

Final remarks

- Any Jordan homomorphism $R \to R'$ gives rise to a harmonicity preserver. This follows from previous work of C. Bartolone [5], A. Blunck and H. H. [15].
- For a wide class of rings, namely rings of stable rank 2, in order to reach all points of $\mathbb{P}(M)$ it suffices to let $n \leq 2$ in Step 4.

  In this case the formula for μ can be rewritten as

$$R\big((t_1 t_2 - 1)e_0 + t_1 e_1\big) \mapsto R'\big((t_1^\alpha t_2^\alpha - 1')e_0' + t_1^\alpha e_1'\big).$$

Final remarks and open problems

- For $R = R'$, $M = M'$, and $\alpha = \mathrm{id}_R$ the harmonicity preserver μ is a projectivity of $\mathbb{P}(M)$.
- Von Staudt's theorem from 1847 is based on the fact that the only Jordan homomorphism $\mathbb{R} \to \mathbb{R}$ is the identity mapping.
- Can the richness condition (i) be weakened in general or not?
- What can be said about harmonicity preservers in the case when $2 \in R$ is a non-unit different from zero?.

References

[1] G. Ancochea, Sobre el teorema fundamental de la geometria proyectiva. *Revista Mat. Hisp.-Amer. (4)* **1** (1941), 37–42.

[2] G. Ancochea, Le théorème de von Staudt en géométrie projective quaternionienne. *J. Reine Angew. Math.* **184** (1942), 193–198.

[3] G. Ancochea, On semi-automorphisms of division algebras. *Ann. of Math. (2)* **48** (1947), 147–153.

[4] R. Baer, *Linear Algebra and Projective Geometry*. Academic Press, New York 1952.

References (cont.)

[5] C. Bartolone, Jordan homomorphisms, chain geometries and the fundamental theorem. *Abh. Math. Sem. Univ. Hamburg* **59** (1989), 93–99.

[6] C. Bartolone, F. Bartolozzi, Topics in geometric algebra over rings. In: R. Kaya, P. Plaumann, K. Strambach, editors, *Rings and Geometry*, 353–389, Reidel, Dordrecht 1985.

[7] C. Bartolone, F. Di Franco, A remark on the projectivities of the projective line over a commutative ring. *Math. Z.* **169** (1979), 23–29.

[8] W. Benz, Das von Staudtsche Theorem in der Laguerregeometrie. *J. Reine Angew. Math.* **214/215** (1964), 53–60.

References (cont.)

[9] W. Benz, *Vorlesungen über Geometrie der Algebren*. Springer, Berlin 1973.

[10] W. Benz, W. Leissner, H. Schaeffer, Kreise, Zykel, Ketten. Zur Geometrie der Algebren. Ein Bericht. *Jber. Deutsch. Math. Verein.* **74** (1972), 107–122.

[11] W. Benz, H.-J. Samaga, H. Schaeffer, Cross ratios and a unifying treatment of von Staudt's notion of reeller Zug. In: P. Plaumann, K. Strambach, editors, *Geometry – von Staudt's Point of View*, 127–150, Reidel, Dordrecht 1981.

[12] W. Bertram, The geometry of null systems, Jordan algebras and von Staudt's theorem. *Ann. Inst. Fourier (Grenoble)* **53** (2003), 193–225.

References (cont.)

[13] A. Blunck, Cross-ratios in Moufang planes. *J. Geom.* **40** (1991), 20–25.

[14] A. Blunck, H. Havlicek, The dual of a chain geometry. *J. Geom.* **72** (2001), 27–36.

[15] A. Blunck, H. Havlicek, Jordan homomorphisms and harmonic mappings. *Monatsh. Math.* **139** (2003), 111–127.

[16] A. Blunck, H. Havlicek, On distant-isomorphisms of projective lines. *Aequationes Math.* **69** (2005), 146–163.

[17] A. Blunck, A. Herzer, *Kettengeometrien – Eine Einführung*. Shaker Verlag, Aachen 2005.

References (cont.)

[18] U. Brehm, M. Greferath, S. E. Schmidt, Projective geometry on modular lattices. In: F. Buekenhout, editor, *Handbook of Incidence Geometry*, 1115–1142, Elsevier, Amsterdam 1995.

[19] F. Buekenhout, Une généralisation du théorème de von Staudt-Hua. *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **51** (1965), 1282–1293.

[20] D. Chkhatarashvili, K. von Staudt's theorem over Ore domains. *Bull. Georgian Acad. Sci.* **158** (1998), 18–20.

[21] L. Cirlincione, M. R. Enea, Una generalizzazione del birapporto sopra un anello. *Rend. Circ. Mat. Palermo (II)* **39** (1990), 271–280.

References (cont.)

[22] S. P. Cojan, A generalization of the theorem of von Staudt-Hua-Buekenhout. *Mathematica (Cluj)* **27 (52)** (1985), 93–96.

[23] M. G. Darboux, Sur le théorème fondamental de la géométrie projective. (extrait d'une lettre à m. klein). *Math. Ann.* **17** (1880), 55–61.

[24] C.-A. Faure, Morphisms of projective spaces over rings. *Adv. Geom.* **4** (2004), 19–31.

[25] J. C. Ferrar, Cross-ratios in projective and affine planes. In: P. Plaumann, K. Strambach, editors, *Geometry – von Staudt's Point of View*, 101–125, Reidel, Dordrecht 1981.

References (cont.)

[26] V. Havel, On the theorem of Staudt in Moufang plane. *Czechoslovak Math. J.* **5 (80)** (1955), 83–90.

[27] V. Havel, Eine Bemerkung zum Staudtschen Satz in der Moufang-Ebene. *Czechoslovak Math. J.* **7(82)** (1957), 314–317.

[28] V. Havel, One generalization of the fourth harmonic point. *Czechoslovak Math. J.* **18 (93)** (1968), 294–300.

[29] H. Havlicek, Divisible designs, Laguerre geometry, and beyond. *J. Math. Sci. (N.Y.)* **186** (2012), 882–926.

[30] H. Havlicek, Von Staudt's theorem revisited. *Aequationes Math.* **89** (2015), 459–472.

References (cont.)

[31] A. Herzer, Chain geometries. In: F. Buekenhout, editor, *Handbook of Incidence Geometry*, 781–842, Elsevier, Amsterdam 1995.

[32] A. J. Hoffman, A note on cross ratio. *Amer. Math. Monthly* **58** (1951), 613–614.

[33] L.-K. Hua, On the automorphisms of a sfield. *Proc. Nat. Acad. Sci. U. S. A.* **35** (1949), 386–389.

[34] L.-K. Hua, Fundamental theorem of the projective geometry on a line and geometry of matrices. In: *Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950*, 317–325, Akadémiai Kiadó, Budapest 1952.

References (cont.)

[35] L.-K. Hua, On semi-homomorphisms of rings and their application in projective geometry (Russian). *Uspehi Matem. Nauk. (N.S.)* **8** (1953), 143–148.

[36] D. G. James, Projective homomorphisms and von Staudt's theorem. *Geom. Dedicata* **13** (1982), 291–294.

[37] H. Karzel, H.-J. Kroll, *Geschichte der Geometrie seit Hilbert*. Wissenschaftliche Buchgesellschaft, Darmstadt 1988.

[38] F. Klein, Ueber die geometrische Definition der Projectivität auf den Grundgebilden erster Stufe. *Math. Ann.* **17** (1880), 52–54.

References (cont.)

[39] B. Klotzek, Eine Verallgemeinerung des Satzes von v. Staudt-Hua. *Wiss. Z. Pädagog. Hochsch. "Karl Liebknecht" Potsdam* **32** (1988), 169–172.

[40] M. Kulkarni, Fundamental theorem of projective geometry over a commutative ring. *Indian J. Pure Appl. Math.* **11** (1980), 1561–1565.

[41] A. Lashkhi, Harmonic maps over rings. *Georgian Math. J.* **4** (1997), 41–64.

[42] A. A. Lashkhi, Harmonic mappings and collineations of modules (Russian). *Soobshch. Akad. Nauk Gruzin. SSR* **133** (1989), 497–500.

References (cont.)

[43] A. A. Lashkhi, Harmonic mappings of modules (Russian). *Mat. Zametki* **47** (1990), 161–163.

[44] B. V. Limaye, N. B. Limaye, Correction to: Fundamental theorem for the projective line over non-commutative local rings. *Arch. Math. (Basel)* **29** (1977), 672.

[45] B. V. Limaye, N. B. Limaye, The fundamental theorem for the projective line over commutative rings. *Aequationes Math.* **16** (1977), 275–281.

[46] B. V. Limaye, N. B. Limaye, Fundamental theorem for the projective line over non-commutative local rings. *Arch. Math. (Basel)* **28** (1977), 102–109.

References (cont.)

[47] N. B. Limaye, Projectivities over local rings. *Math. Z.* **121** (1971), 175–180.

[48] N. B. Limaye, Cross-ratios and projectivities of a line. *Math. Z.* **129** (1972), 49–53.

[49] B. R. McDonald, Projectivities over rings with many units. *Comm. Algebra* **9** (1981), 195–204.

[50] H. Schaeffer, Das von Staudtsche Theorem in der Geometrie der Algebren. *J. Reine Angew. Math.* **267** (1974), 133–142.

[51] O. Schreier, E. Sperner, *Einführung in die analytische Geometrie und Algebra. Bd. II*. B. G. Teubner, Leipzig 1935.

References (cont.)

[52] F. Schur, Ueber den Fundamentalsatz der projectivischen Geometrie. *Math. Ann.* **18** (1881), 252–254.

[53] K. G. C. v. Staudt, *Geometrie der Lage*. Bauer und Raspe (Julius Merz), Nürnberg 1847.

[54] J.-D. Voelke, Le théorème fondamental de la géométrie projective: évolution de sa preuve entre 1847 et 1900. *Arch. Hist. Exact Sci.* **62** (2008), 243–296.