

Higher Order Contact on Cayley's Surface

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Cayley's (ruled cubic) surface is, to within collineations of $\mathbb{P}_3(\mathbb{R})$, the surface F with equation

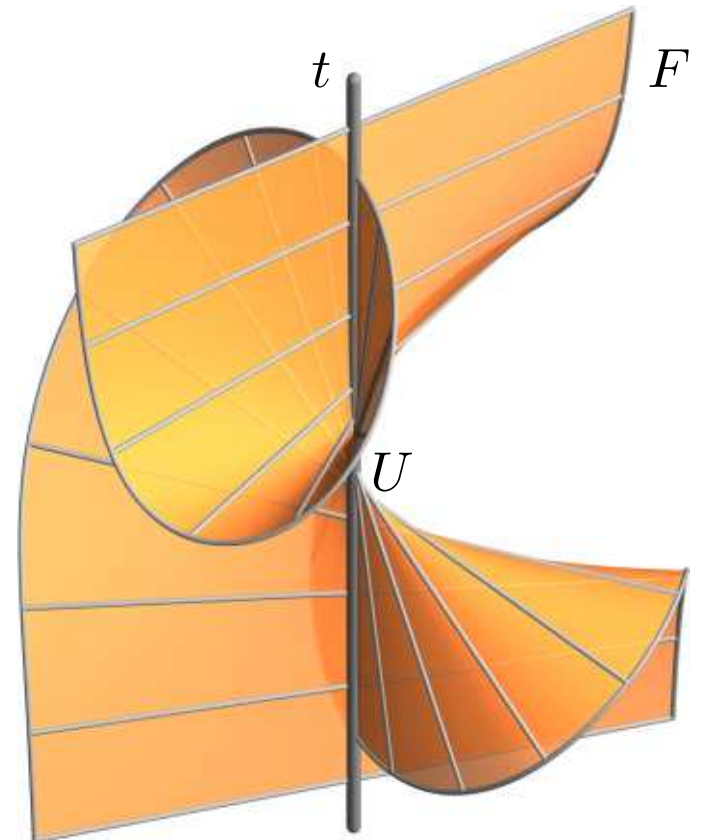
$$3x_0x_1x_2 - x_1^3 - 3x_3x_0^2 = 0.$$

The line $t : x_0 = x_1 = 0$ is a *torsal generator* of second order and a *directrix* for all other generators of F .

$U = \mathbb{R}(1, 0, 0, 0)^T$ is the *cuspidal point* on t .

The plane $\omega : x_0 = 0$ is the *tangent plane* at U . We consider it as *plane at infinity* (but not in all figures).

Affine chart for figure: $x_3 \neq 0$.



Each triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ with $\beta \neq 0$ gives rise to a function

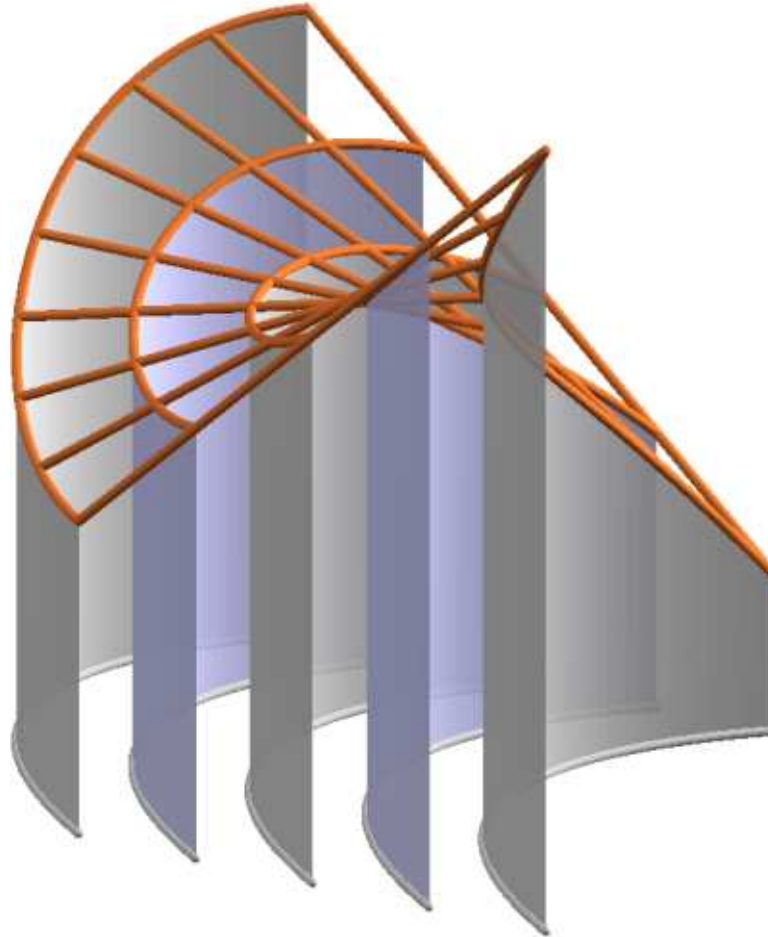
$$\Phi_{\alpha, \beta, \gamma} : \mathbb{R}^{2 \times 1} \rightarrow \mathbb{R}^{4 \times 1} : (u_0, u_1)^T \mapsto \left(u_0^3, u_0^2(u_1 - \gamma u_0), \frac{u_0(u_1^2 + \alpha u_0^2)}{\beta}, \frac{(u_1 - \gamma u_0)}{3\beta} (3(u_1^2 + \alpha u_0^2) - \beta(u_1 - \gamma u_0)^2) \right)^T.$$

In projective terms we obtain a curve $c_{\alpha, \beta, \gamma}$ on the Cayley surface, lying on the *parabolic cylinder* with equation $\alpha x_0^2 - \beta x_0 x_2 + (x_1 + \gamma x_0)^2 = 0$.

$\beta \neq 3$: *cubic parabola* $c_{\alpha, \beta, \gamma} \subset F$.

$\beta = 2, \gamma = 0$: *asymptotic curves* $c_{\alpha, 2, \gamma}$ of F .

$\beta = 3$: $\Phi_{\alpha, 3, \gamma}((1, u_1)^T)$. . . affine part of a *parabola* $c_{\alpha, 3, \gamma} \subset F$.



Cubic parabolas $c_{\alpha,\beta,0}$, where α ranges in $\{-\frac{3}{2}, -\frac{3}{4}, 0, \frac{3}{4}, \frac{3}{2}\}$, and $\beta = \frac{3}{2}$.

H. Neudorfer (1925): *The asymptotic curves of F (other than generators) have fourth order contact at U .*

H. Brauner (1964): *Given cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\bar{\alpha},\bar{\beta},\bar{\gamma}}$ the following assertions hold:*

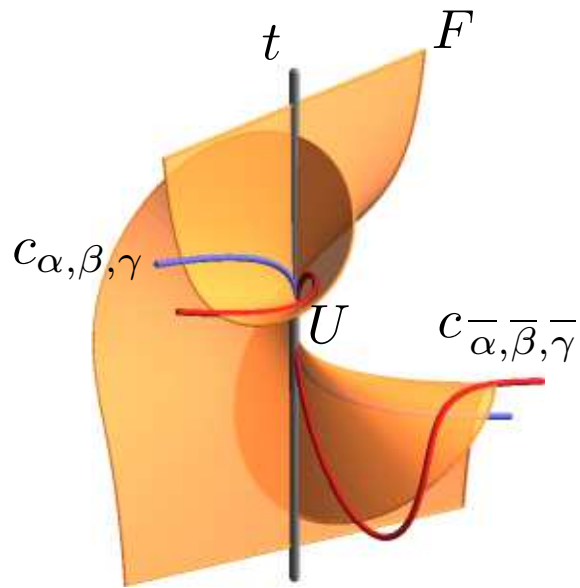
- $\beta = \bar{\beta} \quad \Rightarrow$ *second order contact at U ;*
- $\beta = \bar{\beta}$ and $\gamma = \bar{\gamma} \quad \Rightarrow$ *third order contact at U ;*
- *contact of order four at $U \Rightarrow$ curves are identical.*

Theorem. *Distinct cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\bar{\alpha},\bar{\beta},\bar{\gamma}}$ on Cayley's ruled surface have*

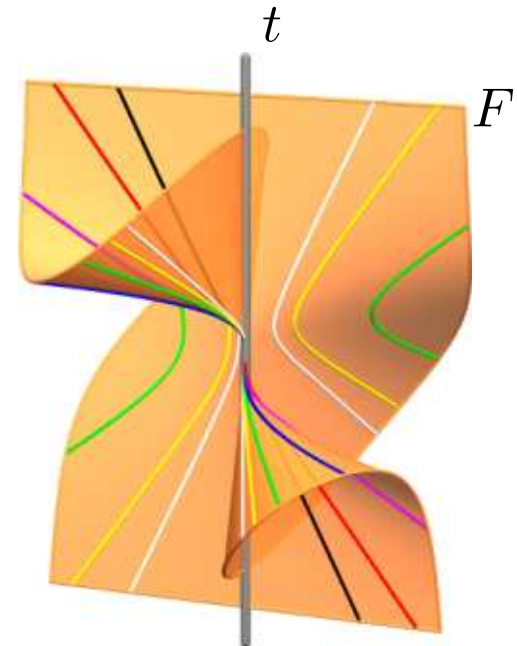
- *second order contact at $U \Leftrightarrow \beta = \bar{\beta}$ or $\beta = 3 - \bar{\beta}$;*
- *third order contact at $U \Leftrightarrow \beta = \bar{\beta}$ and $\gamma = \bar{\gamma}$, or $\beta = \bar{\beta} = \frac{3}{2}$;*
- *fourth order contact at $U \Leftrightarrow \beta = \bar{\beta} = \frac{3}{2}$ and $\gamma = \bar{\gamma}$.*

Proof. A long, even though straightforward calculation (Maple).

Affine chart for figures: $x_3 \neq 0$.



Second order contact at U :
 $(\alpha, \beta, \gamma) = (0, \frac{1}{10}, 0)$,
 $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = (1, 3 - \frac{1}{10}, \frac{1}{10})$.



Fourth order contact at U .
 $\alpha = -3, -2, \dots, 3$,
 $\beta = \frac{3}{2}, \gamma = 0$.

ω . . . *isotropic plane* with absolute flag (U, t) .

$\mathbb{P}_3(\mathbb{R})$. . . *two-fold isotropic space* with absolute flag (U, t, ω) .

Theorem. *Among all cubic parabolas $c_{\alpha, \beta, \gamma}$ on the Cayley surface F , the cubic parabolas with $\beta = \frac{3}{2}$ are precisely those with maximal twofold isotropic conical curvature.*

Proof. The *tangent surface* of a cubic parabola $c_{\alpha, \beta, \gamma}$ meets the plane ω in t and in an *isotropic circle* with *isotropic curvature*

$$\frac{1}{2} \beta(3 - \beta) \leq \frac{9}{8}.$$

Hence $c_{\alpha, \beta, \gamma}$ has the twofold isotropic conical curvature $\frac{1}{2} \beta(3 - \beta) \leq \frac{9}{8}$.

A plane with coordinates $\mathbb{R}(y_0, y_1, y_2, y_3)$ is tangent to the Cayley surface if, and only if,

$$3y_0y_3^2 - 3y_1y_2y_3 + y_2^3 = 0.$$

So, all tangent planes comprise a Cayley surface in the dual space.

All osculating planes of a twisted cubic form a twisted cubic in the dual space.

Dual contact of order $k \Leftrightarrow$ contact of order k of the dual curves.

Theorem. *Distinct cubic parabolas $c_{\alpha,\beta,\gamma}$ and $c_{\bar{\alpha},\bar{\beta},\bar{\gamma}}$ on Cayley's ruled surface have*

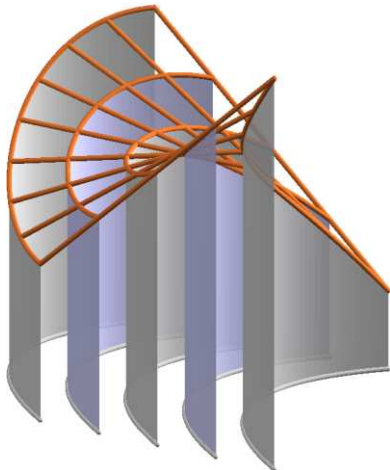
- *second order dual contact at ω $\Leftrightarrow \beta = \bar{\beta}$;*
- *third order dual contact at ω $\Leftrightarrow \beta = \bar{\beta}$ and $\gamma = \bar{\gamma}$, or $\beta = \bar{\beta} = \frac{5}{2}$;*
- *fourth order dual contact at ω $\Leftrightarrow \beta = \bar{\beta} = \frac{7}{3}$ and $\gamma = \bar{\gamma}$.*

In particular, two cubic parabolas of this kind, with fourth order contact at U and fourth order dual contact at ω , are identical.

Theorem. Let $\beta \neq 0, 3$ be a fixed real number. The image of the affine part of the Cayley surface F under the mapping

$$P \in c_{\alpha, \beta, 0} \setminus \{U\} \xrightarrow{\Sigma} \text{osculating plane of } c_{\alpha, \beta, 0} \text{ at } P$$

consists of tangent planes of a Cayley surface F' for $\beta \neq 0, 3, \frac{8}{3}$, and of tangent planes of a hyperbolic paraboloid for $\beta = \frac{8}{3}$.



Each $c_{\alpha, \beta, 0} \xrightarrow{\Sigma} D_{\beta}(c_{\alpha', \beta', 0})$ with D_{β} a duality,
 $\alpha' := \alpha(\beta - 3)$, and $\beta' := \frac{3\beta - 8}{\beta - 3}$,

whence

$$\beta = \frac{7}{3} \mapsto \beta' = \frac{3}{2}.$$

There remains the problem to find a geometric interpretation of the value

$$\beta = \bar{\beta} = \frac{5}{2}$$

which guarantees third order dual contact at U .