Veronese Varieties over Fields with non-zero Characteristic: A Survey

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Introduction

If F is a (commutative) field of characteristic 2 then all tangent lines of a conic in $\mathrm{PG}(2, F)$ are concurrent at a point called *nucleus*.

What happens in higher dimensions?

- Normal rational curves
- Veronese varieties

J.A. Thas, 1969. H. Timmermann, 1977, 1978.
A. Herzer, 1982. H. Karzel, 1987.

Part 1

Pascal's triangle modulo a prime

Representations in base p

Let p be a fixed prime. The *representation* of a non-negative integer $n \in \mathbb{N} := \{0, 1, 2, \dots\}$ *in base p* has the form

$$n = \sum_{\sigma=0}^{\infty} n_\sigma p^\sigma =: \langle n_\sigma \rangle$$

with only finitely many digits $n_\sigma \in \{0, 1, \dots, p-1\}$ different from 0.

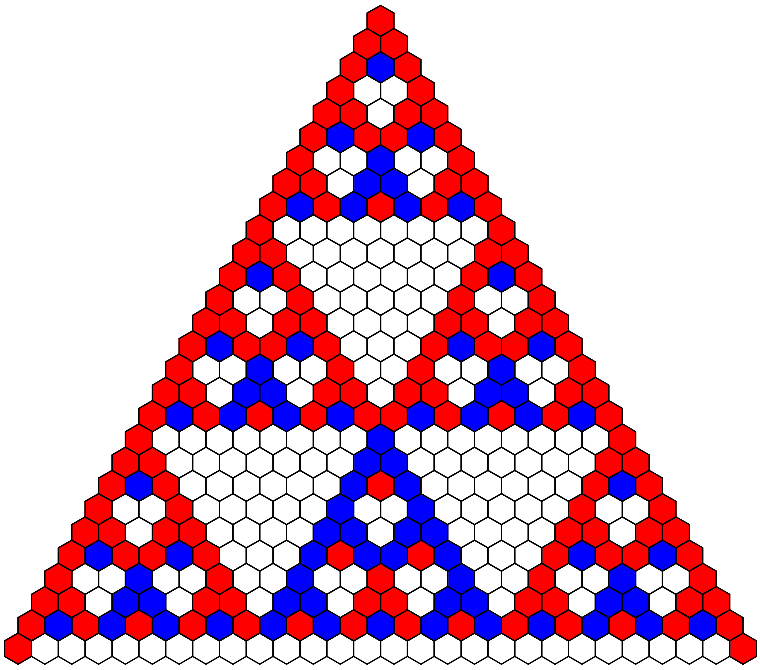
A theorem of Lucas

Let $\langle n_\sigma \rangle$ and $\langle j_\sigma \rangle$ be the representations of non-negative integers n and j in base p.

Then

$$\binom{n}{j} \equiv \prod_{\sigma=0}^{\infty} \binom{n_\sigma}{j_\sigma} \pmod{p}.$$

Pascal's triangle modulo 3



Pascal's triangle modulo p

Δ denotes *Pascal's triangle modulo p* and Δ^i is the *subtriangle* of Δ that is formed by the rows $0, 1, \ldots, p^i - 1$.

Each triangle Δ^{i+1} $(i \geq 0)$ has the following form, with products taken modulo p:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0}\Delta^i & & & & \\
 & & & \binom{1}{0}\Delta^i & \nabla^i & \binom{1}{1}\Delta^i & & & \\
 & & \binom{2}{0}\Delta^i & \nabla^i & \binom{2}{1}\Delta^i & \nabla^i & \binom{2}{2}\Delta^i & & \\
\multicolumn{9}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\
\binom{p-1}{0}\Delta^i & \nabla^i & & & \ldots & & & \nabla^i & \binom{p-1}{p-1}\Delta^i
\end{array}
$$

Here the ∇^i's are *subtriangles* with all entries equal to zero.

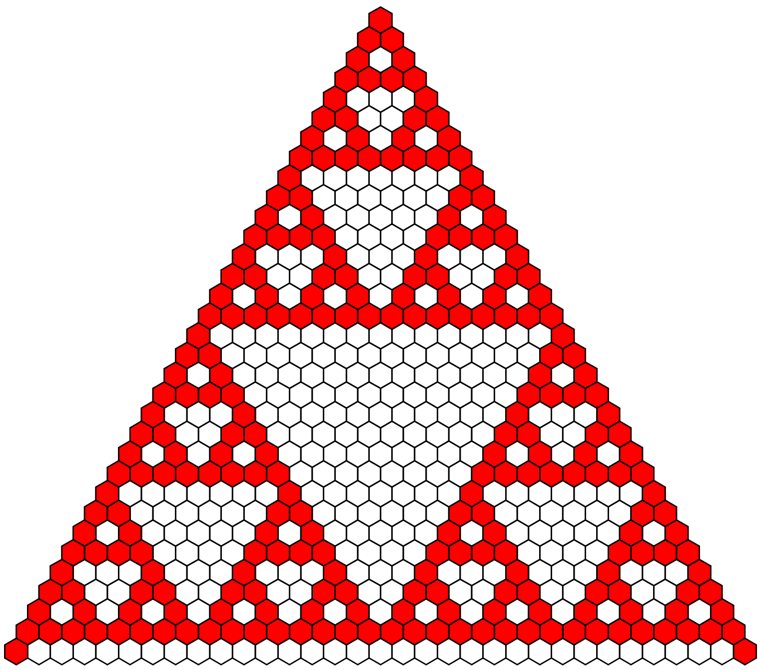
E. Hexel and H. Sachs, 1978. C.T. Long, 1981. N.A. Volodin, 1994.

A partition of zero entries

The zero entries of Pascal's triangle modulo p fall into (disjoint) maximal subtriangles ∇^i ($i \in \mathbb{N}^+$).

We get a *partition of all zero entries* of Δ by gluing together all triangles ∇^i of same size to one class $\bar{i}$, say.

Example. Pascal's triangle modulo 2



Counting zero entries

Then the number of entries in row n of Δ belonging to class $\bar{i}$ equals

$$\Phi(i,n) := \left(p^i - 1 - \sum_{\mu=0}^{i-1} n_\mu p^\mu\right) \cdot n_i \cdot \prod_{\sigma=i+1}^{\infty} (n_\sigma + 1).$$

The number of entries in row n of Δ belonging to classes $\bar{i}$, $\overline{(i+1)}$, ... is

$$\begin{aligned}\Sigma(i,n) &:= \sum_{\eta=i}^{\infty} \Phi(\eta,n) \\ &= n+1-\left(1+\sum_{\mu=0}^{i-1} n_\mu p^\mu\right)\prod_{\sigma=i}^{\infty}(n_\sigma+1).\end{aligned}$$

N.J. Fine, 1947. J. Gmainer, 1999.

The top line function $T(R, b)$

Given $b \in \mathbb{N}^+$ and $R \in \mathbb{N}$ then let

$$T(R, b) := \sum_{\sigma=R}^{\infty} b_\sigma p^\sigma.$$

This function has the following property:

If $(n, j) \in \bar{i}$ and

$$b := n + 1$$

then $T(i, b)$ gives the *top line* of the triangle ∇^i containing the (n, j)-entry of Δ, i.e.

$$0 \equiv \binom{n}{j} \equiv \binom{n-1}{j} \equiv \ldots \equiv \binom{T(i,b)}{j} \pmod{p},$$
$$0 \not\equiv \binom{T(i,b)-1}{j} \pmod{p}.$$

Part 2

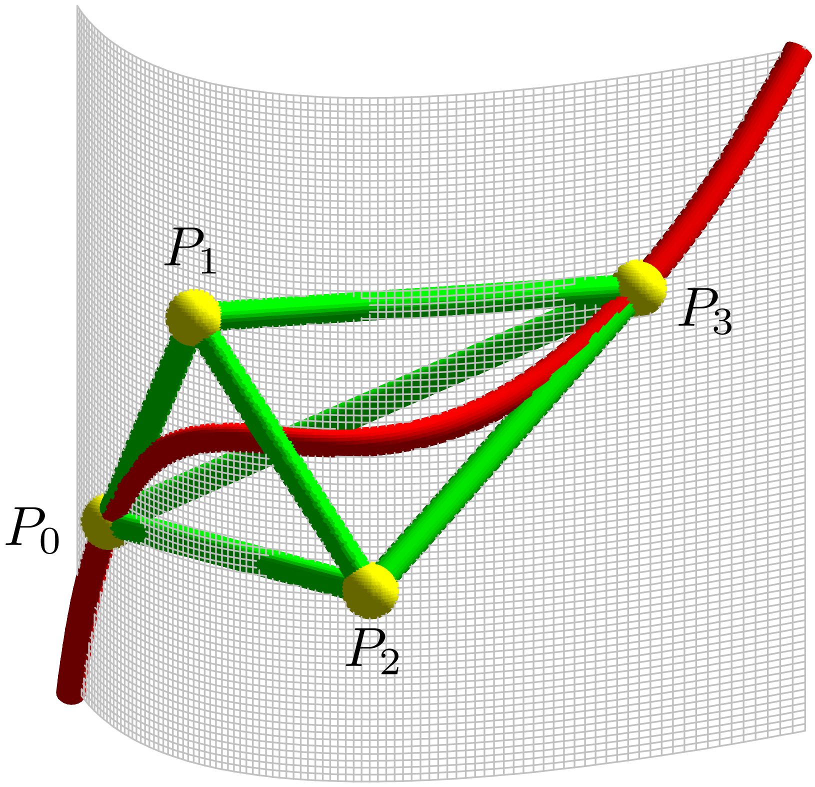
Nuclei of normal rational curves

Normal rational curves

In terms of (adequately chosen) coordinates and an inhomogeneous parameter a *normal rational curve* $\mathcal{V}_1^n$ in $\mathrm{PG}(n, F)$ is the point set

$$\{F(1, x, \ldots, x^n) \mid x \in F \cup \{\infty\}\}.$$

Example. *Twisted cubic*



Osculating subspaces

If we fix one $u \in F$ then columns of the regular matrix

$$\begin{pmatrix} \binom{0}{0} & 0 & 0 & \dots & 0 \\ \binom{1}{0}u & \binom{1}{1} & 0 & \dots & 0 \\ \binom{2}{0}u^2 & \binom{2}{1}u & \binom{2}{2} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ \binom{n}{0}u^n & \binom{n}{1}u^{n-1} & \binom{n}{2}u^{n-2} & \dots & \binom{n}{n} \end{pmatrix}$$

give, respectively, a point of the NRC and its *derivative points*.

The *k-osculating subspace* $(k \in \{-1, 0, \dots, n-1\})$ of $\mathcal{V}_1^n$ at the given point is the k-dimensional projective subspace spanned by the first $k+1$ columns of the matrix.

Nuclei

The *k-nucleus*

$$\mathcal{N}^{(k)}\mathcal{V}_1^n \quad (k \in \{-1, 0, \ldots, n-1\})$$

of a normal rational curve $\mathcal{V}_1^n$ is the intersection of all its k-osculating subspaces.

The main theorem

Theorem 1. *If* $\#F \geq k+1$*, then the* k*-nucleus* $\mathcal{N}^{(k)}\mathcal{V}_1^n$ *is spanned by those base points* P_j*, where* $j \in \{0, 1, \ldots, n\}$ *is subject to*

$$\binom{n}{j} \equiv \binom{n-1}{j} \equiv \ldots \equiv \binom{k+1}{j} \equiv 0 \pmod{\operatorname{char} F}.$$

Example

Let $n = 14$ and $p = 2$, whence $n + 1 = b = 15$.

$$
\begin{array}{rl}
 & 1 \\
 & 1\ 1 \\
 & 1\ 0\ 1 \\
 & 1\ 1\ 1\ 1 \\
 & 1\ 0\ 0\ 0\ 1 \\
 & 1\ 1\ 0\ 0\ 1\ 1 \\
 & 1\ 0\ 1\ 0\ 1\ 0\ 1 \\
 & 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\
T(3,15) = \langle 1,0,0,0 \rangle \rightarrow & 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\
 & 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \\
 & 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1 \\
 & 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1 \\
T(2,15) = \langle 1,1,0,0 \rangle \rightarrow & 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1 \\
 & 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \\
T(1,15) = \langle 1,1,1,0 \rangle \rightarrow & 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \\
T(0,15) = \langle 1,1,1,1 \rangle \rightarrow &
\end{array}
$$

k	$-1, 0, \ldots, 6$	$7, 8, 9, 10$	$11, 12$	13
dim	-1	0	2	6

A dimension formula

Theorem 2. *If* $\#F \geq k+1$, $\mathrm{char}\, F = p > 0$, *and*

$$T(R,b) = \sum_{\mu=R}^{\infty} b_\mu p^\mu \leq k+1 < \sum_{\sigma=Q}^{\infty} b_\sigma p^\sigma = T(Q,b)$$

with at most one $b_\sigma \neq 0$ *for* $\sigma \in \{Q, Q{+}1, \ldots, R{-}1\}$, *then the* k*-nucleus of* $\mathcal{V}_1^n$ *has dimension*

$$n - \left(1 + \sum_{\mu=0}^{R-1} n_\mu p^\mu\right) \prod_{\sigma=R}^{\infty} (n_\sigma + 1) = \Sigma(R,n) - 1.$$

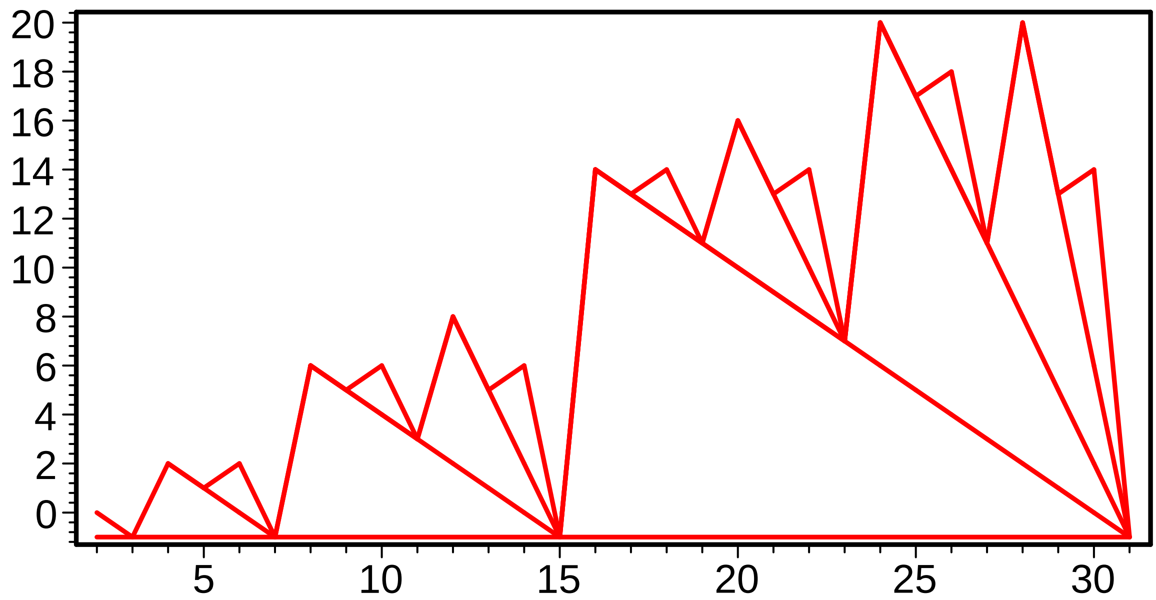
H. Timmermann, 1978. J. Gmainer, 1999.

Number of nuclei

Theorem 3. *If $\#F \geq n$, then there are as many distinct nuclei of $\mathcal{V}_1^n$ as non-zero digits in the representation of $b = n + 1$ in base p.*

J. Gmainer, 1999.

Example. $p = 2$, $n \in \{2, 3, \ldots, 31\}$



Other topics

- Geometric meaning of nuclei
- One-point nuclei
- Lattice of invariant subspaces
- Projection of a NRC from a nucleus or an invariant subspace

Part 3

Pascal's simplex modulo a prime

Multinomial coefficients

Let $t, e_0, e_1, \ldots, e_m \in \mathbb{N}$.

If $(e_0, e_1, \ldots, e_m) \in E_m^t$, i.e.,

$$e_0 + e_1 + \ldots + e_m = t$$

then

$$\binom{t}{e_0, e_1, \ldots, e_m} := \frac{t!}{e_0!\, e_1! \cdots e_m!}$$

otherwise

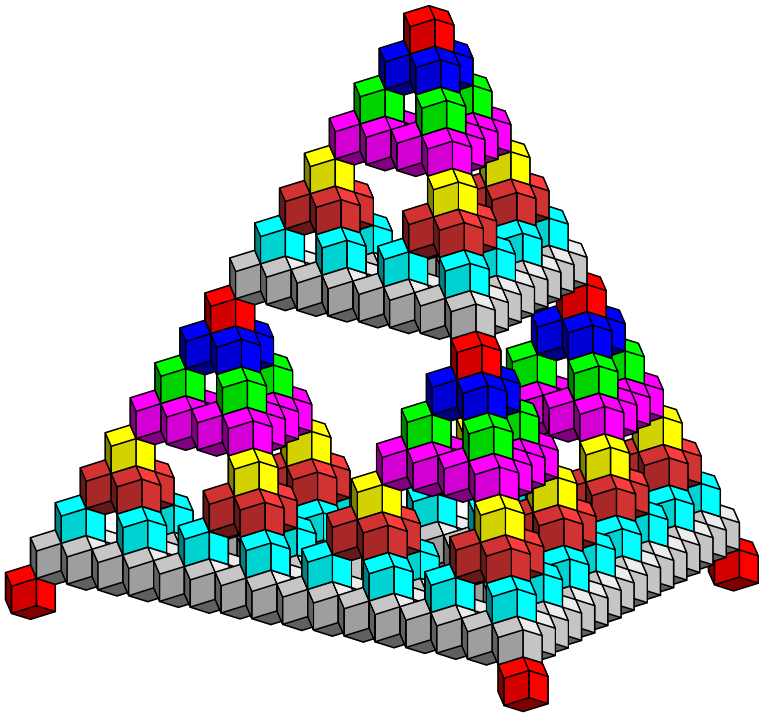
$$\binom{t}{e_0, e_1, \ldots, e_m} := 0.$$

Lucas revisited

If p is a prime then, in terms of digits in base p,

$$\binom{t}{e_0, \ldots, e_m} \equiv \prod_{\sigma \in \mathbb{N}} \binom{t_\sigma}{e_{0,\sigma}, \ldots, e_{m,\sigma}} \pmod{p}.$$

Pascal's pyramid modulo 2



P. Hilton and J. Pedersen, 1999. H. Walser.

Counting zero entries

Let p be a prime.

The number of $(m+1)$–tuples $(e_0, e_1, \ldots, e_m) \in E_m^t$ such that

$$\binom{t}{e_0, e_1, \ldots, e_m} \equiv 0 \pmod{p}$$

equals

$$\binom{m+t}{t} - \prod_{\sigma \in \mathbb{N}} \binom{m+t_\sigma}{t_\sigma}.$$

F.T. Howard, 1974. N.A. Volodin, 1989.

Part 4

Nuclei of Veronese varieties

Veronese varieties

In terms of (adequately chosen) coordinates the *Veronese mapping* is given by

$$F(x_0, x_1, \dots, x_m) \mapsto F(\dots, x_0^{e_0} x_1^{e_1} \dots x_m^{e_m}, \dots)$$

where $x_i \in F$ and $(e_0, e_1, \dots, e_m) \in E_m^t$.

Its image is a *Veronese variety* $\mathcal{V}_m^t$. Its ambient space has dimension

$$\binom{m+t}{t} - 1.$$

(By putting $m := 1$ and $n := t$ a NRC $\mathcal{V}_1^n$ is obtained.)

Osculating subspaces

The Veronese image of each r-dimensional subspace of the parameter space $(0 \leq r < m)$ is a sub-Veronesean $\mathcal{V}_r^t$ of $\mathcal{V}_m^t$.

There exists a *k-osculating subspace* of $\mathcal{V}_m^t$ along $\mathcal{V}_r^t$ for each $k \in \{-1, 0, \ldots, t-1\}$. We call it an *$(r,k)$-osculating subspace* of $\mathcal{V}_m^t$. Its dimension equals

$$\sum_{i=t-k}^{t} \binom{r+i}{i} \binom{m+t-r-i-1}{t-i} - 1.$$

In particular, each $(t-1, m-1)$-osculating subspace of a Veronese variety $\mathcal{V}_m^t$ is a hyperplane of the ambient space; it is called an *osculating hyperplane* or a *contact hyperplane*.

Nuclei

The (r, k)-*nucleus* of a Veronese variety $\mathcal{V}_m^t$ is the intersection of all its (r, k)-osculating subspaces.

Intersection of osculating hyperplanes

Theorem 4. *If $\#F \geq t$ then the $(m-1, t-1)$-nucleus of a Veronese variety $\mathcal{V}_m^t$ is spanned by those base points $P_{e_0,e_1,\ldots,e_m}$ satisfying*

$$\binom{t}{e_0, e_1, \ldots, e_m} \equiv 0 \mod \operatorname{char} F.$$

Theorem 5. *Let $\sum_{\sigma \in \mathbb{N}} t_\sigma p^\sigma$ be the representation of t in base $p = \operatorname{char} F > 0$.*

If $\#F \geq t$, then the $(m-1, t-1)$-nucleus of a Veronese variety $\mathcal{V}_m^t$ has dimension

$$\binom{m+t}{t} - \prod_{\sigma \in \mathbb{N}} \binom{m+t_\sigma}{t_\sigma} - 1.$$

J. Gmainer, H. H., 2000.

Remarks and open problems

- Connection to symmetric powers
- Coordinate-free definitions
- Nuclei of sub-Veroneseans
- A general dimension formula for nuclei ?
- Geometric meaning of nuclei ?
- Invariant subspaces?

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