Twelve points in PG(5,3) with 95040 self-transformations

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Remarks on a paper by Coxeter

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Designs:

$t-(v,k,\lambda)$ - design ... (P,B)

blocks per t points

points per block

number of points

blocks

points

Examples:

Projective planes : $2-(n^2+n+1,\, n+1,\, 1)$

Affine planes : $2-(n^2,\, n,\, 1)$

Witt's $5-(12,6,1)$ design W_{12}

12 points, 132 blocks

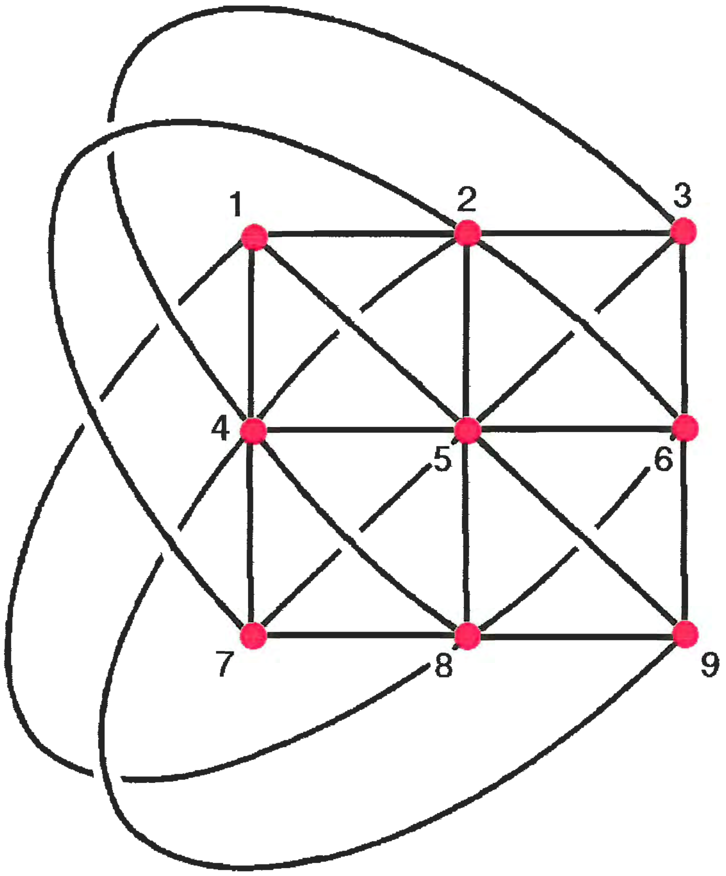
Choose 3 points $\longrightarrow$ 9 points remaining

... blocks through them $\longrightarrow$ 12 blocks $\longrightarrow$ 12 lines

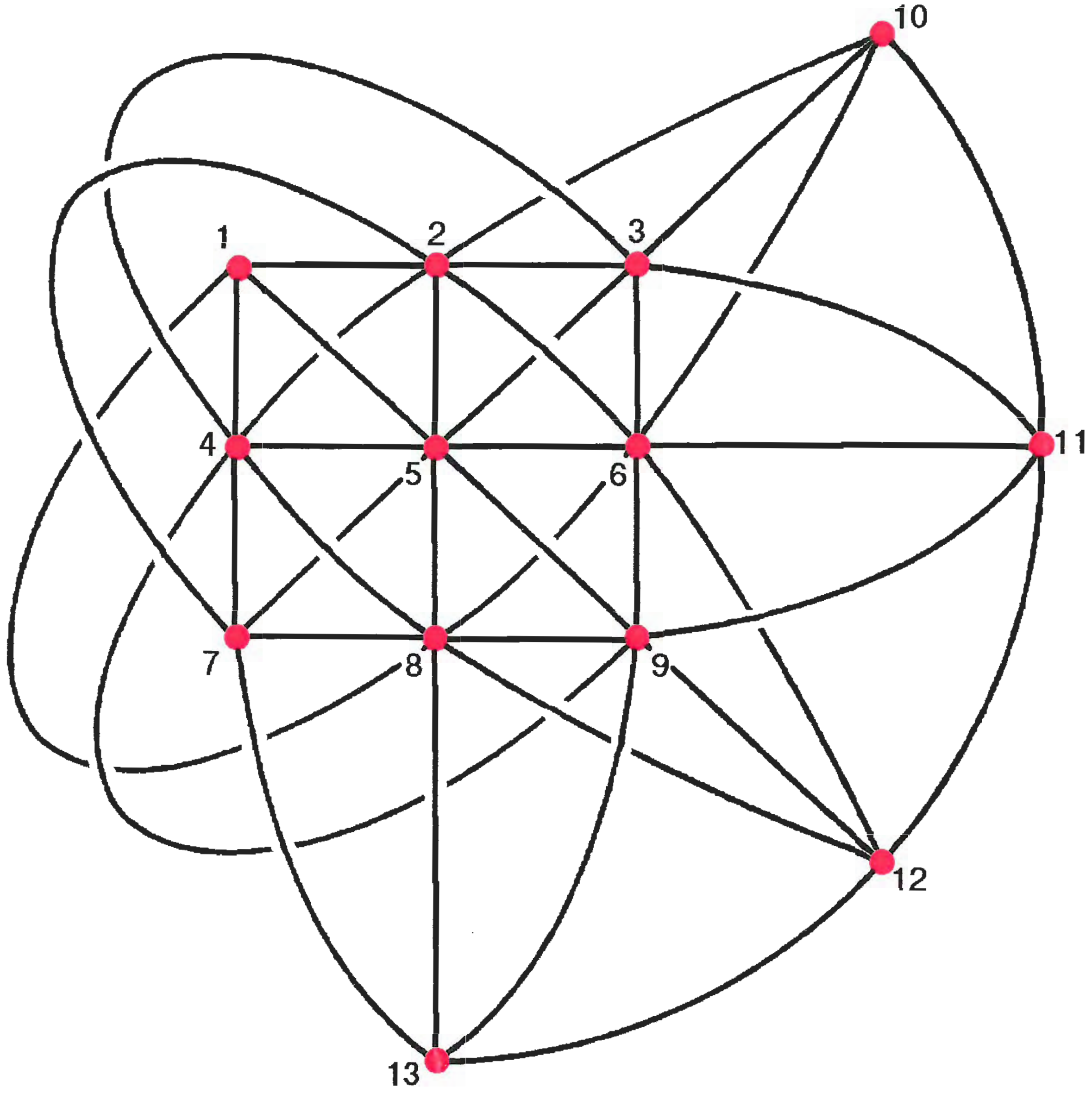
affine plane $AG(2,3)$

$\big\downarrow$ 3 fold extension

W_{12}



The affine plane AG(2,3)



The projective plane PG(2,3)

Point model of W_{12} in $PG(5,3)$

$\mathcal{K}$ ... 12 points

5 points in $\mathcal{K} \Rightarrow \exists^*$ hyperplane $\mathcal{H}$

$$\#(\mathcal{H} \cap \mathcal{K}) = 6$$

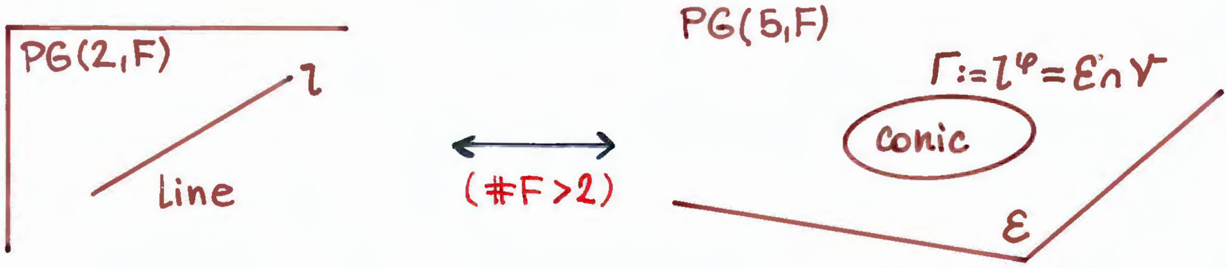
H.S.M. Coxeter, G. Pellegrino, J.A. Todd

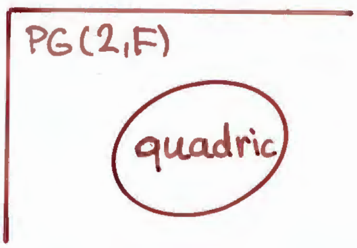
Veronese surface:

Veronese mapping:

$$\underbrace{F(x_0,x_1,x_2)}_{PG(2,F)} \stackrel{\varphi}{\longmapsto} \underbrace{F(x_0^2,\ x_0x_1,\ x_0x_2,\ x_1^2,\ x_1x_2,\ x_2^2)}_{PG(5,F)}$$

$\mathrm{im}\,\varphi =: \mathcal{V}$ ... Veronese surface





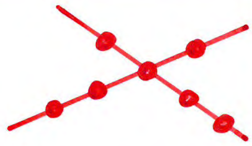
PG(5,F)

hyperplane section of $\mathcal{V}$

$$\sum_{i\le j} a_{ij}\, x_i x_j = 0 \qquad \longleftrightarrow \qquad \sum_{i\le j} a_{ij}\, y_{ij} = 0$$

Zanella - H.

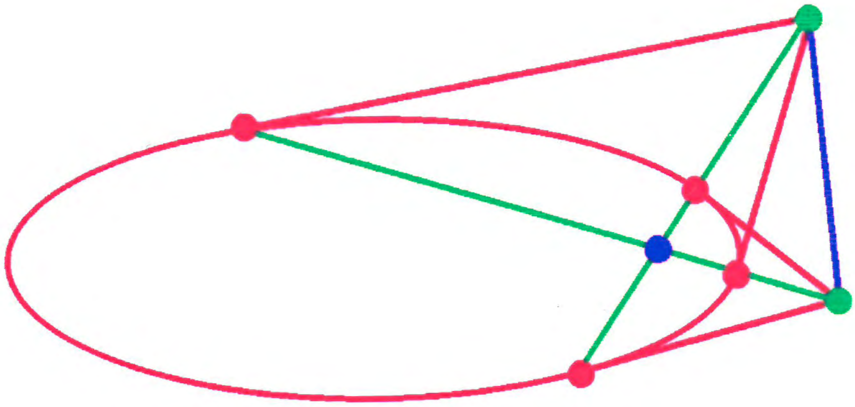
Quadrics in PG(2,3):

Equation	Name	Picture	#points
$x_0^2 + x_1^2 + x_2^2 = 0$	conic		4
$x_0^2 + x_1^2 = 0$	one point		1
$x_0^2 - x_1^2 = 0$	cross of lines		7
$x_0^2 = 0$	repeated line		4

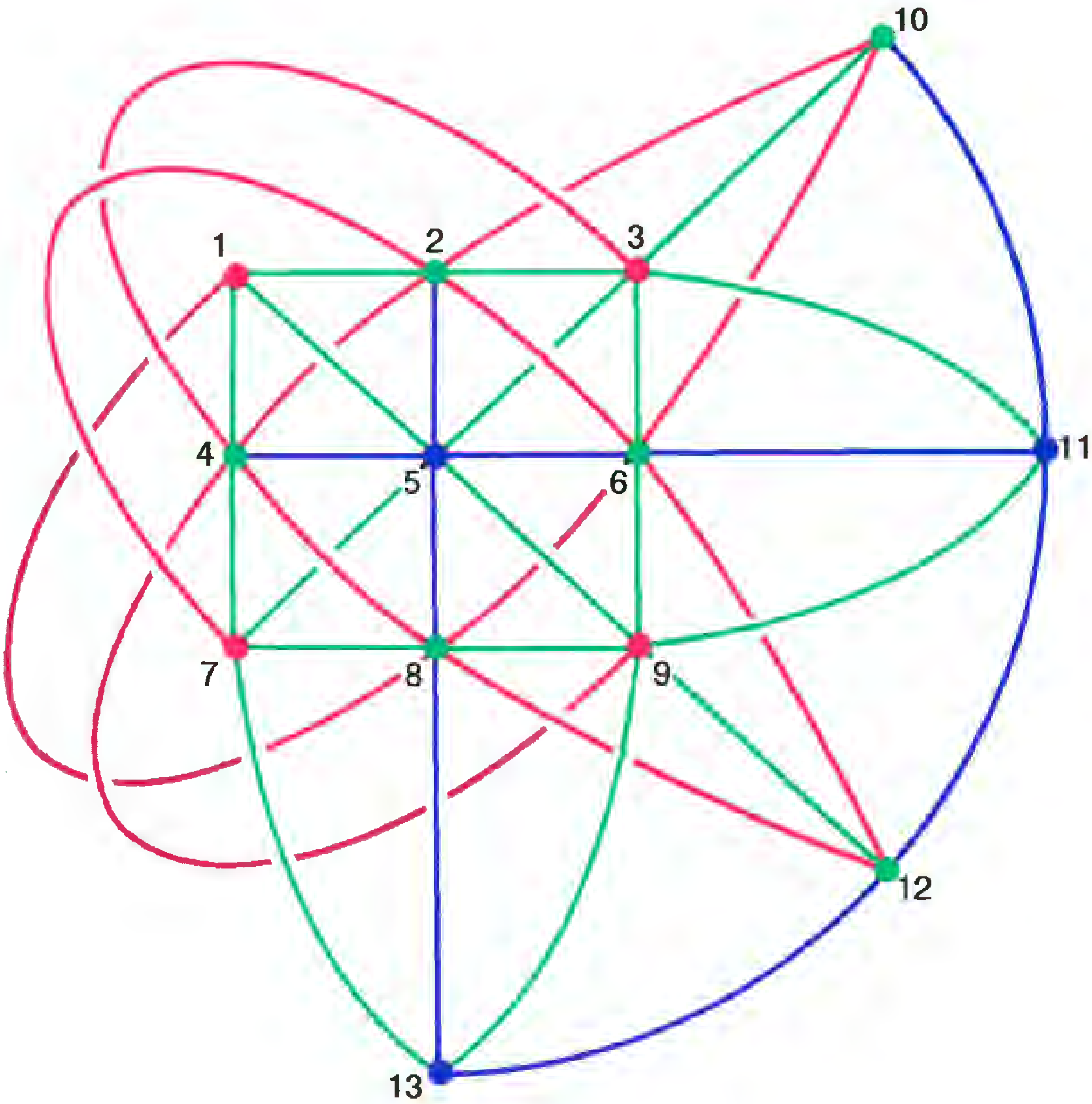
$\mathcal{H}$.. hyperplane of PG(5,3)

$$\underline{\underline{c := \#(\mathcal{H} \cap \mathcal{V}) \in \{1, 4, 7\}}}$$

Thas-Hirschfeld



Conic in the real projective plane:
Red: Points (of the conic) and tangents.
Blue: An internal point and an exterior line.
Green: External points and bisecant lines.



Conic in PG(2,3):
Red: 4 points (of the conic) and 4 tangents.
Blue: 3 internal points and 3 exterior lines.
Green: 6 external points and 6 bisecant lines.

Replacement :

l_∞ ... a line in $PG(2,3)$

$\Gamma_\infty := l_\infty^\varphi$ ... a conic ($\subset \mathcal{V}$)

Δ_∞ ... diagonal triangle of the quadrangle Γ_∞

$\mathcal{E}_\infty$ ... plane of Γ_∞

$$\mathcal{K} := (\mathcal{V} \setminus \Gamma_\infty) \cup \Delta_\infty$$

$\# \mathcal{K} = 12$

Theorem: $d := \#(\mathcal{H} \cap \mathcal{K}) \in \{0,3,6\}$ for all hyperplanes $\mathcal{H}$ of $PG(5,3)$.

Proof.: $\mathcal{H} \cap \mathcal{V} \xrightarrow{\varphi^{-1}} Q$ .. quadric

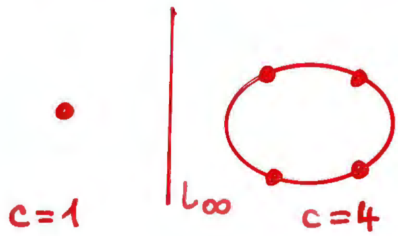
$$c := \#(\mathcal{H} \cap \mathcal{V}) \in \{1,4,7\}.$$

1. $\mathcal{E}_\infty \subset \mathcal{H} \Rightarrow d = c - 4 + 3 = c - 1 \in \{0,3,6\}$

2. $\mathcal{E}_\infty \cap \mathcal{H}$ external line of Γ_∞

   $\Rightarrow d = c - 0 + 2 = c + 2$

   Q has no points at infinity



$\Rightarrow d \in \{3,6\}$

3. $\mathcal{E}_\infty \cap \mathcal{H}$ is a tangent of $\Gamma_\infty \Rightarrow d = c - 1 + 0 = c - 1 \in \{0,3,6\}$

4. $\mathcal{E}_\infty \cap \mathcal{H}$ is a bisecant of $\Gamma_\infty \Rightarrow d = c - 2 + 1 = c - 1 \in \{0,3,6\}$

Model of W_{12} in $PG(5,3)$

$P := \mathcal{K}$

$B := \{ \mathcal{H} \cap \mathcal{K} \mid \#(\mathcal{H} \cap \mathcal{K}) \geq 4 \,,\ \mathcal{H} \text{ a hyperplane of } PG(5,3) \}$

$$\Downarrow$$

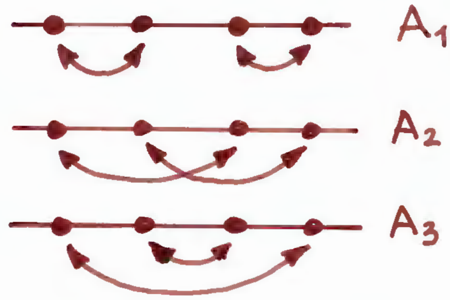
$$\#(\mathcal{H} \cap \mathcal{K}) = 6$$

Model of W_{12} in $PG(5,3) = (\mathcal{P}, \mathcal{L})$

3 elliptic involutions on l_∞

$$\downarrow$$

3 „new" points A_1, A_2, A_3



$P := (\mathcal{P} \setminus l_\infty) \cup \{A_1, A_2, A_3\}$

B ...

- affine line + all elliptic involutions (3)
- ellipse + those elliptic involutions that are not the involution of conjugate points on l_∞ (2)
- union of two disjoint affine lines (0)
- union of two non-parallel affine lines + that involution which interchanges the points at infinity. (1)

$$\underbrace{\text{affine quadric}}_{\text{Veronese}} + \underbrace{\text{some „new points"}}_{\text{replacement}}$$

Relations to Coding Theory

$F := GF(3)$ $\quad W := F^6$

$w_1, \dots, w_{12} \in W$ representing the points of $\mathcal{K}$

$f \in W^* = L(W, K)$

$\underbrace{(w_1^f, \dots, w_{12}^f) \in F^{12}}$

gives ... a linear code $G_{12} = G_{12}^{\perp}$

extended ternery Golay code

$v_1, \dots, v_{13} \in W$ representing Υ

$\underbrace{(v_1^f, \dots, v_{13}^f) \in F^{13}}$

gives ... a linear code C_{13}

$C_{13}^{\perp}$ ... generated by the lines of $PG(2,3)$

C_{13} ... — " — .... complements of lines ...

... — " — .... differences of lines ...