Matrix Spaces vs. Projective Lines over Rings

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Grassmannians

Let F be a (not necessarily commutative) field and $m, n \geq 1$.

- $\mathcal{G}_{n+m,m}(F)$ denotes the Grassmannian of all m-subspaces of the left vector space F^{n+m}.
- Two m-subspaces W_1 and W_2 are called *adjacent* if $\dim W_1 \cap W_2 = m - 1$.
- We consider $\mathcal{G}_{n+m,m}(F)$ as the set of vertices of an undirected graph, called the *Grassmann graph*. Its edges are the (unordered) pairs of adjacent m-subspaces.
- We shall frequently assume $m, n \geq 2$ in order to avoid a complete graph.

Theorem (W. L. Chow (1949) [11])

Let $m, n \geq 2$. A mapping $\varphi : \mathcal{G}_{n+m,m}(F) \to \mathcal{G}_{n+m,m}(F) : X \mapsto X^{\varphi}$ is an automorphism of the Grassmann graph if, and only if, it has the following form:

- *For arbitrary m, n:*

$$X \mapsto \{y \in F^{n+m} \mid y = x^{\sigma} P \text{ with } x \in X\},$$

  where $P \in \mathrm{GL}_{n+m}(F)$ and σ is an automorphism of F.

- *For $n = m$ and fields admitting an antiautomorphism only:*

$$X \mapsto \{y \in F^{n+m} \mid yP(x^{\sigma})^{\mathsf{T}} = 0 \text{ for all } x \in X\},$$

  where P is as above, σ is an antiautomorphism of F, and T denotes transposition.

The Matrix Approach

Each element of the Grassmannian $\mathcal{G}_{n+m,m}(F)$ can be viewed as the left row space of a matrix $A|B$ with rank m, where $A \in F^{m\times n}$ and $B \in F^{m\times m}$, and vice versa.

- Let $\mathrm{rk}(A|B) = m$. Then $A|B$ and $A'|B'$ have the same row space, if and only if, there is a $T \in \mathrm{GL}_m(F)$ with

$$A' = TA \text{ and } B' = TB.$$

- One may consider a matrix pair $(A, B) \in F^{m\times n} \times F^{m\times m}$ with $\mathrm{rk}(A|B) = m$ as left homogeneous coordinates of an element of $\mathcal{G}_{n+m,m}(F)$.
- Some authors call $\mathcal{G}_{n+m,m}(F)$ the point set of the *projective space of $m \times n$ matrices over F*.

An Embedding

We have an injective mapping:

$$\begin{array}{ccccc} F^{m\times n} & \to & F^{m\times(n+m)} & \to & \mathcal{G}_{n+m,m}(F) \\ A & \mapsto & A|I_m & \mapsto & \text{left rowspace of } A|I_m \end{array}$$

Here I_m denotes the $m \times m$ identity matrix over F.

- Two matrices $A_1, A_2 \in F^{m\times n}$ are *adjacent*, i. e., $\mathrm{rk}(A_1 - A_2) = 1$, precisely when their images in $\mathcal{G}_{n+m,m}(F)$ are adjacent.

Related Work

A series of results in the spirit of Chow's theorem have been established for various (projective) matrix spaces.
Also, the assumptions in Chow's original theorem can be relaxed.

- Original work by L. K. Hua and others (1945 and later).
- Z.-X. Wan: *Geometry of Matrices* [39].
- L.-P. Huang: *Geometry of Matrices over Ring* [17].
- M. Pankov: *Grassmannians of Classical Buildings* [36].
- See also: Y. Y. Cai, L.-P. Huang, W.-I. Huang, P. Šemrl, R. Westwick, S.-W. Zou [18], [19], [20], [21], [22], [23], [24], [28], [40].

Towards Ring Geometry

- The set $F^{m\times m}$ of $m \times m$ matrices over F is a ring with unit element I_m.
- The case $m \neq n$ will not be covered by our ring geometric approach.

All our rings are associative, with a unit element $1 \neq 0$ which is preserved by homomorphisms, inherited by subrings, and acts unitally on modules. The group of units (invertible elements) of a ring R is denoted by R^*.

The Projective Line over a Ring

Let R be a ring. We consider the free left R-module R^2.

- A pair $(a, b) \in R^2$ is called *admissible* if (a, b) is the first row of a matrix in $\mathrm{GL}_2(R)$.
  This is equivalent to saying that there exists $(c, d) \in R^2$ such that $(a, b), (c, d)$ is a basis of R^2.
- *Projective line* over R:

$$\mathbb{P}(R) := \{R(a, b) \mid (a, b) \text{ admissible}\}$$

  The elements of $\mathbb{P}(R)$ are called *points*.
- Two admissible pairs generate the same point if, and only if, they are left proportional by a unit in R.

Remarks

- Our approach is due to X. Hubaut [29].
- $\mathbb{P}(R)$ may also be described as the orbit of the "starter point" $R(1,0)$ under the natural right action of $\mathrm{GL}_2(R)$ on R^2.
- Note that R^2 may also have bases with cardinality $\neq 2$.

The Distant Graph

- *Distant* points of $\mathbb{P}(R)$:

$$R(a,b) \vartriangle R(c,d) \quad :\Leftrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

- $(\mathbb{P}(R), \vartriangle)$ is called the *distant graph* of $\mathbb{P}(R)$.
- Non-distant points are also called *neighbouring*.
- The relation $\vartriangle$ is invariant under the action of $\mathrm{GL}_2(R)$ on $\mathbb{P}(R)$.

Remark

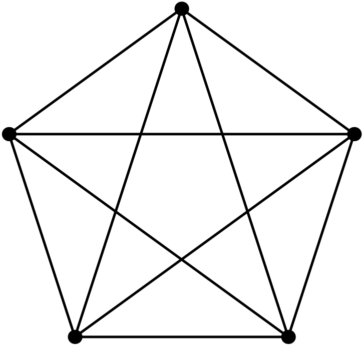
For $R = F^{m\times m}$ distant points correspond to complementary subspaces of $\mathcal{G}_{2m,m}$ due to $\mathrm{GL}_2(R) = \mathrm{GL}_{2m}(F)$.

Examples: Rings with Four Elements

Ring

- $R = \mathrm{GF}(4)$ (Galois field).
- $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.
- $R = \mathbb{Z}_4$.
- $R = \mathbb{Z}_2[\varepsilon]$, $\varepsilon^2 = 0$ (dual numbers over $\mathbb{Z}_2$).

Distant graph



$\#\mathbb{P}(R) = 5$

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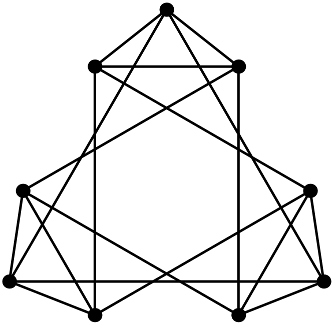

$\#\mathbb{P}(R) = 9$

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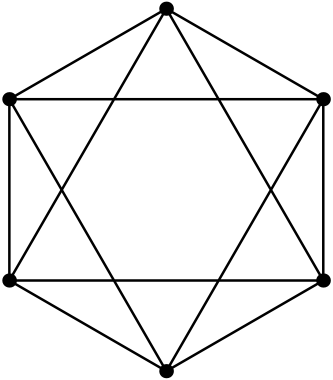

$\#\mathbb{P}(R) = 6$

Examples: Rings with Four Elements

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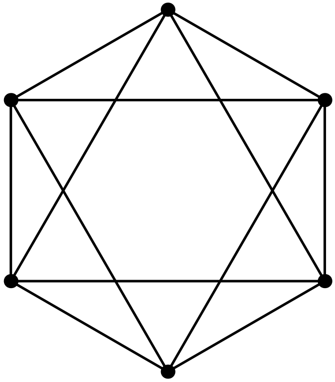
Distant graph



$\#\mathbb{P}(R) = 6$

Properties of the Distant Relation

- $(\mathbb{P}(R), \triangle)$ is a complete graph $\Leftrightarrow$ $\not\triangle$ equals the identity relation $\Leftrightarrow$ R is a field.
- The relation $\not\triangle$ is an equivalence relation $\Leftrightarrow$ R is a *local ring*, i.e., $R \setminus R^*$ is an ideal of R.

A. Herzer (survey) [16].

A. Blunck, A. Herzer: *Kettengeometrien* [9].

The Elementary Linear Group $\mathrm{E}_2(R)$

All elementary 2×2 matrices over R, i. e., matrices of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \text{ with } t \in R,$$

generate the *elementary linear group* $\mathrm{E}_2(R)$. The group $\mathrm{GE}_2(R)$ is the subgroup of $\mathrm{GL}_2(R)$ generated by $E_2(R)$ and all invertible diagonal matrices.

Lemma (P. M. Cohn [12])

A 2×2 matrix over R is in $\mathrm{E}_2(R)$ if, and only if, it can be written as a finite product of matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \text{ with } t \in R.$$

Connectedness

Theorem (A. Blunck, H. H. [4])

Let R be any ring.

- $(\mathbb{P}(R), \triangle)$ *is connected precisely when* $\mathrm{GL}_2(R) = \mathrm{GE}_2(R)$.
- *A point* $p \in \mathbb{P}(R)$ *is in the connected component of* $R(1,0)$ *if, and only if, it can be written as* $R(a,b)$ *with*

$$(a,b) = (1,0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1).$$

for some $n \in \mathbb{N}$ *and some* $t_1, t_2, \ldots, t_n \in R$.

Connectedness (cont.)

The formula

$$(a, b) = (1, 0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1)$$

reads explicitly as follows:

$$\begin{aligned}
n = 0: \quad (a, b) &= (1, 0) \\
n = 1: \quad (a, b) &= (t_1, 1) \\
n = 2: \quad (a, b) &= (t_2 t_1 - 1, t_2) \\
n = 3: \quad (a, b) &= (t_3 t_2 t_1 - t_3 - t_1, t_3 t_2 - 1) \\
&\vdots
\end{aligned}$$

Stable Rank 2

A ring has *stable rank 2* (or: stable range 1) if for any unimodular pair $(a, b) \in R^2$, i.e., there exist u, v with $au + bv \in R^*$, there is a $c \in R$ with

$$ac + b \in R^*.$$

- Surveys by F. Veldkamp [37] and [38].
- H. Chen: *Rings Related to Stable Range Conditions* [10].

Examples

Rings of stable rank 2 are ubiquitous:

- local rings;
- matrix rings over fields;
- finite-dimensional algebras over commutative fields.
- direct products of rings of stable rank 2.

$\mathbb{Z}$ is not of stable rank 2: Indeed, $(5, 7)$ is unimodular, but no number $5c + 7$ is invertible in $\mathbb{Z}$.

Examples

$\mathbb{P}(R)$ is connected if ...

- R is a ring of stable rank 2. Diameter ≤ 2
- R is the endomorphism ring of an infinite-dimensional vector space. Diameter 3.
- R is a polynomial ring $F[X]$ over a field F in a central indeterminate X. Diameter ∞.

However, in $R = F[X_1, X_2, \ldots, X_n]$ with $n \geq 2$ central indeterminates there holds

$$\begin{pmatrix} 1 + X_1X_2 & X_1^2 \\ -X_2^2 & 1 - X_1X_2 \end{pmatrix} \in \mathrm{GL}_2(R) \setminus \mathrm{GE}_2(R).$$

A Parallelism

Let $\triangle(p)$ be the set of all points distant to $p \in \mathbb{P}(R)$.

- Points with $\triangle(p) \subset \triangle(q)$ are called *(Jacobson) parallel*, in symbols $p \parallel q$.
- Despite its asymmetric definition, $\parallel$ is an equivalence relation on $\mathbb{P}(R)$. Hence

$$p \parallel q \quad \Leftrightarrow \quad \triangle(p) = \triangle(q).$$

- The relation $\parallel$ is invariant under the action of $\mathrm{GL}_2(R)$ on $\mathbb{P}(R)$.

A Parallelism (cont.)

- For all $p \in \mathbb{P}(R)$ holds:

$$p \parallel R(1,0) \quad \Leftrightarrow \quad p = R(1,b) \ \text{ with } \ b \in \operatorname{rad} R,$$

  i. e. the *Jacobson radical* of R. Indeed,

$$b \in \operatorname{rad} R \quad \Leftrightarrow \quad \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \in \mathrm{GL}_2(R) \text{ for all } a \in R.$$

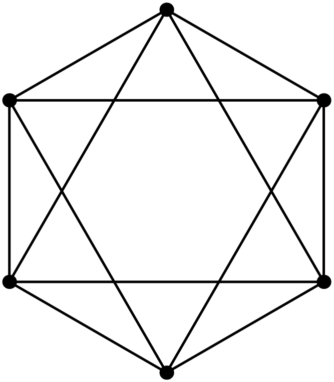
- All parallel classes of $\mathbb{P}(R)$ have cardinality $\#\operatorname{rad} R$.
- Parallel points of $\mathbb{P}(R)$ are non-distant.
- The relations $\parallel$ and $\not\triangle$ coincide precisely when R is local.

Example: Local Rings with Four Elements

Ring

- $R = \mathbb{Z}_4$.
- $R = \mathbb{Z}_2[\varepsilon]$, $\varepsilon^2 = 0$ (dual numbers over $\mathbb{Z}_2$).

Distant graph



$\#\mathbb{P}(R) = 6$, $\#\,\mathrm{rad}\,R = 2$.

Distant Homomorphisms

Given rings R and R' a mapping

$$\varphi : \mathbb{P}(R) \to \mathbb{P}(R')$$

is said to be a *distant homomorphism* if

$$p \vartriangle q \Rightarrow p^{\varphi} \vartriangle' q^{\varphi} \quad \text{for all} \quad p, q \in \mathbb{P}(R).$$

Examples: The Easy Ones

- Let $\sigma : R \to R'$ be a ring homomorphism. Then

$$\varphi : \mathbb{P}(R) \to \mathbb{P}(R') : R(a,b) \mapsto R'(a^\sigma, b^\sigma)$$

  is a distant homomorphism.

- Let $\sigma : R \to R'$ be a ring antihomomorphism. Then the mapping $\varphi : \mathbb{P}(R) \to \mathbb{P}(R')$ given by

$$R(a,b)^\varphi := \{(x', y') \in R'^2 \mid -x' b^\sigma + y' a^\sigma = 0\}$$

  is a distant homomorphism.

- Let $\alpha \in \mathrm{GL}_2(R)$. Then

$$\varphi : \mathbb{P}(R) \to \mathbb{P}(R) : R(a,b) \mapsto R\big((a,b)\cdot\alpha\big) =: R(a,b)^\alpha$$

  is a distant automorphism.

Examples: Some Ugly Ones

The following mappings $\varphi : \mathbb{P}(R) \to \mathbb{P}(R)$ are distant automorphisms:

- Let R be a field, and let $\varphi : \mathbb{P}(R) \to \mathbb{P}(R)$ be any bijection.
- Let $\mathrm{GE}_2(R) \neq \mathrm{GL}_2(R)$. With $\alpha := E(0) \in \mathrm{E}_2(R)$ define:

$$p^\varphi := \begin{cases} p^\alpha & \text{if } p \text{ is in the conneced component of } R(1,0) \\ p & \text{otherwise} \end{cases}$$

- Let $\operatorname{rad} R \neq 0$. With any bijection $\sigma : \operatorname{rad} R \to \operatorname{rad} R$ define:

$$p^\varphi := \begin{cases} R(1, b^\sigma) & \text{if } p = R(1,b) \parallel R(1,0) \\ p & \text{otherwise} \end{cases}$$

- Let $R = F[X]$ with F commutative ...
  C. Bartolone, F. Bartolozzi [2].

Jordan Homomorphisms

A mapping $\sigma : R \to R'$ is called a *Jordan homomorphism* if it satisfies the following conditions for all $x, y \in R$:

$$\begin{aligned} (x+y)^\sigma &= x^\sigma + y^\sigma, \\ (xyx)^\sigma &= x^\sigma y^\sigma x^\sigma, \\ 1^\sigma &= 1'. \end{aligned}$$

- Homomorphisms and antihomomorphisms are Jordan homomorphisms.
- Example: Let R be the direct product $\mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2}$ and define

$$\sigma : R \to R : (A, B) \mapsto (A, B^{\mathsf{T}}).$$

Theorem (C. Bartolone [1], A. Blunck, H. H. [6])

Each Jordan homomorphism $\sigma : R \to R'$ gives rise to a distant preserving mapping which is defined on the connected component of $R(1,0)$ as follows:

$$R(1,0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1)$$

is mapped to

$$R'(1',0') \cdot E(t_n^\sigma) \cdot E(t_{n-1}^\sigma) \cdots E(t_1^\sigma).$$

So, if $(\mathbb{P}(R), \triangle)$ is connected, we obtain a distant homomorphism.

Two Characterisations

Let $R = F^{m\times m}$, $m \geq 1$. Below we do not distinguish between the projective line $\mathbb{P}(R)$ and the Grassmannian $\mathcal{G}_{2m,m}(F)$.

Theorem (A. Blunck, H. H. [7])

For all $p, q \in \mathbb{P}(R)$ the following assertions hold:

1. $p \bigtriangleup q \Leftrightarrow$ *The distance of p and q in the Grassmann graph equals the diameter of this graph.*
2. *p and q are adjacent $\Leftrightarrow$ There exists a point $r \in \mathbb{P}(R)$ other than p and q such that $\bigtriangleup(r) \subset (\bigtriangleup(p) \cup \bigtriangleup(q))$.*

Consequently, the Grassmann graph and the distant graph on $\mathbb{P}(R)$ have the same group of automorphisms.

Chow's Theorem for $m = n$

Corollary

Let $m \geq 2$. A mapping

$$\varphi : \mathcal{G}_{2m,m}(F) \to \mathcal{G}_{2m,m}(F)$$

is an automorphism of the Grassmann graph if, and only if, it is the product of a linear bijection acting on $\mathcal{G}_{2m,m}(F)$ and a mapping which in terms of homogeneous coordinates has the form

$$(BA - I_m, B) \mapsto (B^\sigma A^\sigma - I_m, B^\sigma),$$

with σ being an automorphism or an antiautomorphism of F.

Related Work

- All distant automorphisms of projective lines over semisimple rings (Segre products of Grassmannians) can be described "algebraically" provided that no simple component is a field.
- Similar characterisations have been established for other spaces of matrices and spaces of linear operators.
- Characterisations of mappings preserving a bounded distance.
- See the papers by A. Blunck, H. H., L.-P. Huang, W.-l. Huang, J. Kosiorek, M. Kwiatkowski, M. H. Lim, A. Matraś, A. Naumowicz, M. Pankov, K. Prażmowski, P. Šemrl, J. J.-H. Tan: [5], [8], [14], [15], [21], [25], [26], [27], [30], [31], [32], [33], [34], [35].

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The books and surveys [3], [9], [13], [16], [17], [38], [39] contain a wealth of further references.

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