Pencilled regular parallelisms

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To the memory of Rolf Riesinger (1946–2018), teacher, colleague, and friend.

Basic notions

- Let $\mathrm{PG}(n, \mathbb{K})$ be the n-dimensional projective space over a (commutative) field $\mathbb{K}$; its point set is written as $\mathcal{P}_n$ and the line set is denoted by $\mathcal{L}_n$.
- A *spread* of $\mathrm{PG}(3, \mathbb{K})$ is a partition of $\mathcal{P}_3$ by (disjoint) lines.
- A *parallelism* on $\mathrm{PG}(3, \mathbb{K})$ is a partition of $\mathcal{L}_3$ by (disjoint) spreads.
- The spreads of a parallelism are called *parallel classes*.

See, among others, Hirschfeld [14], Johnson [15], [16], Karzel and Kroll [17], and Knarr [18].

Additional properties

In $\mathrm{PG}(3, \mathbb{K})$, we shall only be concerned with spreads and parallelisms that satisfy some additional properties.

- A spread is are called *regular* if it is closed under reguli.
- Regular spreads are precisely the elliptic linear congruences of lines.
- A parallelism is called *regular* if all its parallel classes are regular spreads.

Regular spreads of $\mathrm{PG}(3,\mathbb{K})$ and external lines to H_5

Let $\mathcal{C}$ be a regular spread of $\mathrm{PG}(3,\mathbb{K})$.

- The **Klein correspondence** λ sends $\mathcal{C}$ to an elliptic subquadric $\lambda(\mathcal{C})$ of the Klein quadric H_5.
- The **polarity** π_5 of the Klein quadric sends the solid spanned by $\lambda(\mathcal{C})$ to

$$\gamma(\mathcal{C}) := \pi_5\big(\mathrm{span}\,\lambda(\mathcal{C})\big),$$

which is an external line to the Klein quadric.

In this way one obtains a **bijection** γ from the set of regular spreads of $\mathrm{PG}(3,\mathbb{K})$ to the set of lines that are external to the Klein quadric H_5.

Hfd line sets (Betten and Riesinger [3])

Let ***P*** be a regular parallelism on $\mathrm{PG}(3, \mathbb{K})$.

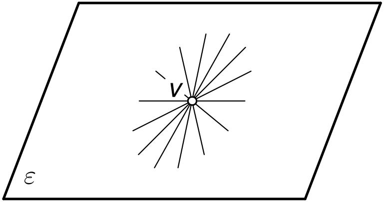
- The λ-image of ***P*** is a *hyperflock* of the Klein quadric H_5, that is, a partition of H_5 by (disjoint) elliptic subquadrics.
- The γ-image of ***P*** is a *hyperflock determining line set* (shortly: an *hfd line set*) with respect to the Klein quadric, that is, a set $\mathcal{H}$ with the following properties:

1. $\mathcal{H} \subset \mathcal{L}_5$ consists of lines that are external to the Klein quadric.
2. Each tangent hyperplane of the Klein quadric contains exactly one line of $\mathcal{H}$.

In this way one obtains a **bijection** from the set of regular parallelisms of $\mathrm{PG}(3, \mathbb{K})$ to the set of hfd line sets w. r. t. H_5.

Pencilled regular parallelisms (H. and Riesinger [13])

Notation.
For any incident point-plane pair (v, ε) we denote by $\mathcal{L}[v, \varepsilon]$ the pencil of lines with vertex v and plane ε.



Definition 1.

An hfd line set $\mathcal{H}$ is said to be *pencilled* if each element of $\mathcal{H}$ belongs to at least one pencil of lines contained in $\mathcal{H}$.

Definition 2.

A regular parallelism $\boldsymbol{P}$ on $\mathrm{PG}(3, \mathbb{K})$ is called *pencilled* if the hfd line set $\gamma(\boldsymbol{P})$ is pencilled.

Construction of pencilled hfd line sets

Theorem 1.

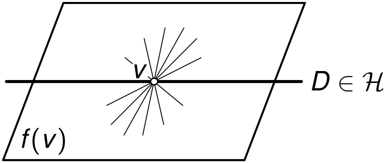
In PG$(5, \mathbb{K})$, *let* D *be a line such that*

$$\mathcal{E}_D := \{\varepsilon \subset \mathcal{P}_5 \mid D \subset \varepsilon \text{ and } \varepsilon \text{ is an external plane to } H_5\}$$

is non-empty. Then, upon choosing any mapping $f\colon D \to \mathcal{E}_D$, *the union*

$$\bigcup_{v \in D} \mathcal{L}[v, f(v)] =: \mathcal{H}$$

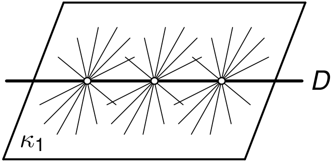
is a pencilled hfd line set.



Example

Suppose that the mapping $f\colon D \to \mathcal{E}_D$ in Theorem 1 is constant.

- The image of f contains a single plane, say κ_1.
- $\mathcal{H}$ is the plane of lines in κ_1.



- $\gamma^{-1}(\mathcal{H})$ is a *Clifford parallelism*.
  These parallelisms are commonly defined in various ways; see Betten and Riesinger [4], Blunck, Pianta and Pasotti [6], Giering [8], Karzel and Kroll [17], or H. [9], [10], [11], [12].
  Cf. also Blunck, Knarr, Stroppel and Stroppel [5].

Example

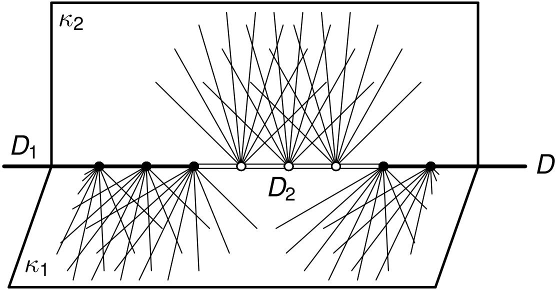
Let the image of the mapping f in Theorem 1 consist of two distinct planes κ_1 and κ_2.

- The mapping f decomposes the line D into two non-empty subsets D_1 and D_2, namely the pre-images of κ_1 and κ_2, respectively.
- The corresponding hfd line set can be written in the form

$$\Big(\bigcup_{v \in D_1} \mathcal{L}[v, \kappa_1] \Big) \cup \Big(\bigcup_{v \in D_2} \mathcal{L}[v, \kappa_2] \Big).$$

Example (cont.)

- Over the real numbers, f can be chosen in such a way that D_1 is a connected component of D with respect to the standard topology in $\mathrm{PG}(5, \mathbb{R})$. Then D_2 is also connected.



Main theorem on pencilled hfd line sets

Theorem 2.

In $\mathrm{PG}(5, \mathbb{K})$*, any hfd line set admits at least one construction as in Theorem 1.*

Existence of pencilled regular parallelisms

Theorem 3.

Given any field $\mathbb{K}$ the following assertions are equivalent.

1. *In* $\mathrm{PG}(3, \mathbb{K})$ *there exists a Clifford parallelism.*
2. *There exists an algebra $\mathbb{H}$ over the field $\mathbb{K}$ such that one of the following conditions,* (A) *or* (B)*, is satisfied:*
   - (A) $\mathbb{H}$ *is a quaternion skew field with centre $\mathbb{K}$.*
   - (B) $\mathbb{H}$ *is an extension field of $\mathbb{K}$ with degree* $[\mathbb{H} : \mathbb{K}] = 4$ *and such that $a^2 \in \mathbb{K}$ for all $a \in \mathbb{H}$.*
3. *In* $\mathrm{PG}(3, \mathbb{K})$ *there exists a pencilled regular parallelism that is not Clifford.*

Conclusion

- Hirschfeld [14, p. 69], who follows Conwell [7], uses hfd line sets to construct regular parallelisms of $\mathrm{PG}(3, 2)$. (Hirschfeld's terminology is different from ours.) These parallelisms give rise to solutions of Kirkman's Fifteen Schoolgirls problem (1850).
- Further examples of hfd line sets (pencilled or not) in $\mathrm{PG}(5, \mathbb{R})$ can be found in Betten and Riesinger [1], [2, Ex. 16 and 22], [3], and Löwen [20]. Many of these hfd line sets satisfy additional topological conditions.
- See Kroll and Vincenti [19] for results on partitions of the Klein quadric.

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