Divisible Designs from Twisted Dual Numbers

Joint work with
Andrea Blunck (Hamburg) and Corrado Zanella (Vicenza)

TECHNISCHE UNIVERSITÄT WIEN
VIENNA UNIVERSITY OF TECHNOLOGY
DIFFERENTIALGEOMETRIE UND GEOMETRISCHE STRUKTUREN

HANS HAVLICEK

FORSCHUNGSGRUPPE
DIFFERENTIALGEOMETRIE UND
GEOMETRISCHE STRUKTUREN
INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE
TECHNISCHE UNIVERSITÄT WIEN
havlicek@geometrie.tuwien.ac.at

Divisible Designs

Assume that X is a finite set of *points*, endowed with an equivalence relation $\mathcal{R}$; its equivalence classes are called *point classes*. A subset Y of X is called *$\mathcal{R}$-transversal* if for each point class C we have

$$\#(C \cap Y) \leq 1.$$

Definition. A triple $\mathcal{D} = (X, \mathcal{B}, \mathcal{R})$ is called a *t-(s, k, λ_t)-divisible design* (DD) if there exist positive integers t, s, k, λ_t such that the following axioms hold:

(A) $\mathcal{B}$ is a set of $\mathcal{R}$-transversal subsets of X, called *blocks*, with $\#B = k$ for all $B \in \mathcal{B}$.

(B) Each point class has size s.

(C) For each $\mathcal{R}$-transversal t-subset $Y \subset X$ there exist exactly λ_t blocks containing Y.

(D) $t \leq \frac{v}{s}$, where $v := \#X$.

Spera's Construction

Theorem (A. G. Spera 1992). *Let X be a finite set with v elements and $\mathcal{R}$ an equivalence relation on X. Suppose, moreover, that G is a group acting on X, and assume that the following properties hold:*

- *The equivalence relation $\mathcal{R}$ is G-invariant.*
- *All equivalence classes of $\mathcal{R}$ have the same cardinality, say s.*
- *The group G acts transitively on the set of $\mathcal{R}$-transversal t-subsets of X for some positive integer $t \leq \frac{v}{s}$.*

Finally, let B_0 be an $\mathcal{R}$-transversal k-subset of X with $t \leq k$. Then

$$(X, \mathcal{B}, \mathcal{R}) \text{ with } \mathcal{B} := B_0^G = \{B_0^g \mid g \in G\}$$

is a t-(s, k, λ_t)-divisible design, where

$$\lambda_t := \frac{\#G}{\#G_{B_0}} \frac{\binom{k}{t}}{\binom{vs^{-1}}{t} s^t}, \tag{1}$$

and where $G_{B_0} \subset G$ denotes the setwise stabilizer of B_0.

Examples

A. G. Spera, C. Cerroni, S. Giese, and R.-H. Schulz obtained many 2-DDs and 3-DDs in this way using various geometric structures, like

- finite translation planes,
- finite analogues of Minkowski space-time,
- projective spaces over finite local algebras,

together with appropriate groups.

Cf. also D. R. Hughes (1965) for a similar construction of designs.

Projective Lines

Let R be a finite local ring with unity $1 \neq 0$, and denote by $I := R \setminus R^*$ its unique maximal ideal. The *projective line* $\mathbb{P}(R)$ over R is the set of all submodules

$$R(a,b) \in R^2$$

such that $a \notin I$ or $b \notin I$. Hence

$$\mathbb{P}(R) = \{R(a,1) \mid a \in R\} \cup \{R(1,b) \mid b \in I\}.$$

Two points $p = R(a,b)$ and $q = R(c,d)$ are called *parallel* (in symbols: $p \parallel q$) if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin \mathrm{GL}_2(R).$$

DDs are Ubiquitous

Theorem. *Spera's construction can be carried out for*

$$(X, \mathcal{R}, G) := \big(\mathbb{P}(R), \parallel, \mathrm{GL}_2(R)\big)$$

and ***any*** *transversal k-subset B_0. Provided that $k \geq 3$, this gives a divisible design with parameters*

$$t = 3,\ s = \#I,\ k = \#B_0,\ v = \#R + \#I,$$

and λ_3 as given by (1).

Proof. The equivalence relation $\parallel$ is invariant under the natural action (from the right hand side) of $\mathrm{GL}_2(R)$ on $\mathbb{P}(R)$. $\mathrm{GL}_2(R)$ acts transitively on the set of $\parallel$-transversal triads (ordered triples) of $\mathbb{P}(R)$. Thus all point classes have the same size. $\square$

But, in order to calculate λ_3 one has to know the order of the stabilizer $\mathrm{GL}_2(R)_{B_0}$.

Chain Geometries

Suppose that $K \subset R$ is a subfield of R. As $\mathbb{P}(K) \subset \mathbb{P}(R)$ is $\parallel$-transversal, it can be chosen as base block B_0.

- Let K be in the centre of R, i. e., R is an algebra over K. Then a chain geometry $\Sigma(K, R)$ is obtained by Spera's construction. Cf. W. Benz (1972), A. Herzer (1995). Hence

$$\lambda_3 = 1.$$

- Let K be arbitrary. Then a generalized chain geometry $\Sigma(K, R)$ is obtained by Spera's construction. Cf. C. Bartolone (1989), A. Blunck and H. H. (2000). Let

$$N = \{n \in R^* \mid n^{-1} K^* n = K^*\}$$

be the normalizer of K^* in R^*. After some calculations, one obtains

$$\lambda_3 = \frac{\# R^*}{\# N}.$$

Twisted Dual Numbers

Let R be a finite local ring and K a subfield such that $\dim_K R = 2$. Assume that R is not a field. Then there exists an element $\varepsilon \in R \setminus R^*$ such that

$$R = \{x + y\varepsilon \mid x, y \in K\} \text{ and } \varepsilon^2 = 0.$$

Furthermore, there is an automorphism $\sigma : K \to K$ satisfying

$$\varepsilon x = x^\sigma \varepsilon \text{ for all } x \in K.$$

Conversely, each automorphism σ of K gives rise to such a ring $K(\varepsilon; \sigma)$ of *twisted dual numbers*.

General assumption. $R = K(\varepsilon; \sigma)$ is given as follows:

$$K = \mathrm{GF}(q) \text{ and } x^\sigma = x^m \text{ for all } x \in K.$$

Hence q is a power of m, $F := \mathrm{Fix}(\sigma) = \mathrm{GF}(m)$, and $\#R = q^2$.

The Normalizer

Lemma. *Let N be the normalizer of K^* in R^*. Then*

$$N = \begin{cases} R^* & \textit{if} \quad \sigma = \mathrm{id}, \\ K^* & \textit{if} \quad \sigma \neq \mathrm{id}\,. \end{cases}$$

Proof. For $\sigma = \mathrm{id}$ the assertion is clear. So let $\sigma \neq \mathrm{id}$ and $n = a + b\varepsilon \in N$ with $a, b \in K$. Take an element $x \in K$ with $x \neq x^\sigma$. Using

$$n^{-1} = a^{-1} - a^{-1}b(a^\sigma)^{-1}\varepsilon$$

we get $n^{-1}xn = x + a^{-1}b(x - x^\sigma)\varepsilon$, which must belong to K since $n \in N$. Because of our choice of x we have $x - x^\sigma \neq 0$, whence $b = 0$, as desired. □

Main Result

Theorem. *The chain geometry* $\Sigma(K,R) = (\mathbb{P}, \mathcal{B}, \parallel)$ *is a transversal* 3*-divisible design with parameters* $v = q^2 + q$*,* $s = q$*,* $k = q + 1$*, and*

$$\lambda_3 = \begin{cases} 1 & \text{\textit{if}} \quad \sigma = \mathrm{id}, \\ q & \text{\textit{if}} \quad \sigma \neq \mathrm{id}\,. \end{cases}$$

Remarks.

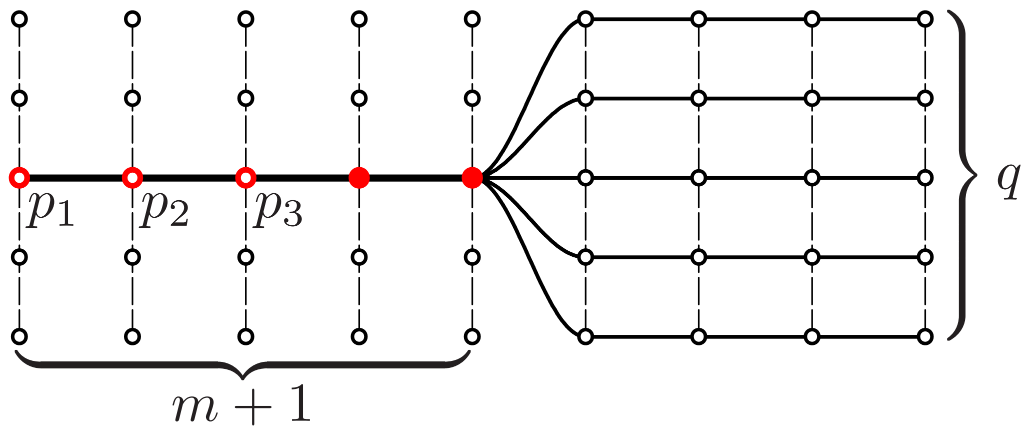
- Blocks are called chains or, more precisely, K-chains.
- For $\sigma = \mathrm{id}$ the well known Miquelian Laguerre plane over the K-algebra of dual numbers is obtained.
- For $\sigma \neq \mathrm{id}$ the parameter λ_3 does not depend on m.

Intersection of Blocks

Proposition. Let $p_1, p_2, p_3 \in \mathbb{P}(R)$ be mutually non-parallel, let T be the intersection of all blocks through p_1, p_2, p_3, and let $x \nparallel p_1, p_2, p_3$. Then the number of blocks through p_1, p_2, p_3, x is

- q, if $x \in T$,
- 0, if $x \notin T$, but $x \parallel x'$ for some $x' \in T$,
- 1, otherwise.

Furthermore, the subset T is an F-chain, i. e. the image of $\mathbb{P}(F)$ under the action of $\mathrm{GL}_2(R)$.



Final Remarks

- Let q be even and $m = 2$, i.e., $x^\sigma = x^2$ for all $x \in K$. By the previous Proposition, the 3-DD $\Sigma(K, R)$ is even a 4-DD with

$$\lambda_4 = 1.$$

- Let $\sigma \neq \mathrm{id}$. The point set of the DD (chain geometry) $\Sigma(K, R)$ can be identified with a cone in $\mathrm{PG}(4, q)$, but without its (one-point) vertex. The base of this cone depends on the automorphism σ.

References

[1] W. Benz. *Vorlesungen über Geometrie der Algebren*. Springer, Berlin, 1973.

[2] A. Blunck and H. H. Extending the concept of chain geometry. *Geom. Dedicata*, 83:119–130, 2000.

[3] A. Blunck and H. H. Projective representations I. Projective lines over rings. *Abh. Math. Sem. Univ. Hamburg*, 70:287–299, 2000.

[4] A. Blunck and A. Herzer. *Kettengeometrien*. Shaker, Aachen, 2005.

[5] A. Herzer. Chain geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 781–842. Elsevier, Amsterdam, 1995.

[6] A.G. Spera. t-divisible designs from imprimitive permutation groups. *Europ. J. Combin.*, 13:409–417, 1992.

[7] A.G. Spera. On divisible designs and local algebras. *J. Comb. Designs*, 3:203–212, 1995.