# Linear sets in the projective line over the endomorphism ring of a finite field

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# Basic assumptions

- Let $q$ be a prime power and let $t \geq 2$ be an integer.
- The field with $q^t$ elements is denoted by $\mathbb{F}_{q^t}$ and its unique subfield of order $q$ is written as $\mathbb{F}_q$.
- The vector space $\mathbb{F}_{q^t}^2$ over $\mathbb{F}_{q^t}$ determines the projective line $\mathrm{PG}(1, q^t)$. Its points have the form

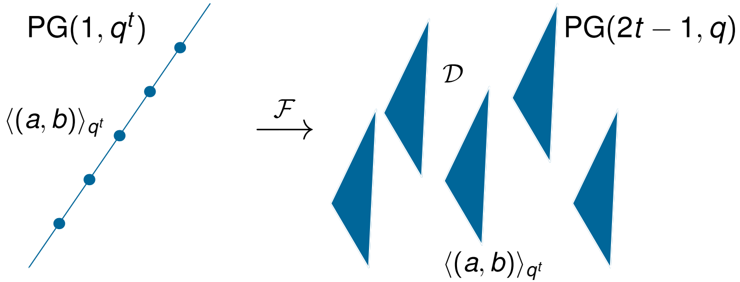
$$\langle (u, v) \rangle_{q^t} \text{ with } (0,0) \neq (u, v) \in \mathbb{F}_{q^t}^2.$$

- The vector space $\mathbb{F}_{q^t}^2$ over $\mathbb{F}_q$ determines the projective space $\mathrm{PG}(2t-1, q)$. Its points have the form

$$\langle (u, v) \rangle_{q} \text{ with } (0,0) \neq (u, v) \in \mathbb{F}_{q^t}^2.$$

- $\mathcal{G}$ denotes the Grassmannian of $(t-1)$-dimensional subspaces of $\mathrm{PG}(2t-1, q)$.

# Field reduction map $\mathcal{F}$: $\mathrm{PG}(1, q^t) \to \mathcal{G}$

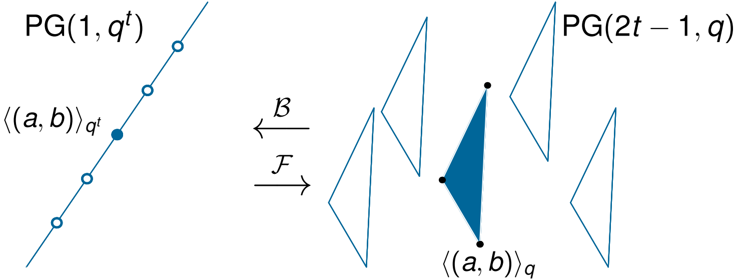



The *field reduction map* $\mathcal{F}$ assigns to each point $\langle(a,b)\rangle_{q^t}$ that element of the Grassmannian $\mathcal{G}$ which is given by $\langle(a,b)\rangle_{q^t}$ (considered as subspace of the vector space $\mathbb{F}_{q^t}^2$ over $\mathbb{F}_q$).

The image of $\mathcal{F}$ is a Desarguesian spread, say $\mathcal{D}$.

The map $\mathcal{F}$ is injective.

# Blow up map $\mathcal{B}$: PG$(2t-1,q) \to$ PG$(1,q^t)$

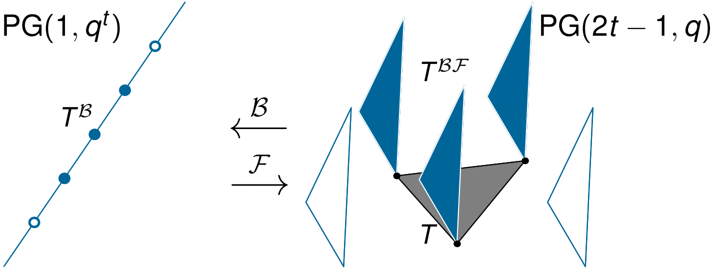



The *blow up map* $\mathcal{B}$ assigns to each point $\langle(a,b)\rangle_q$ the point $\langle(a,b)\rangle_{q^t}$.

The product $\mathcal{BF}$: PG$(2t-1,q) \to \mathcal{G}$ takes $\langle(a,b)\rangle_q$ to the only element of the spread $\mathcal{D}$ containing $\langle(a,b)\rangle_q$.

The map $\mathcal{B}$ is not injective (due to $t \geq 2$).

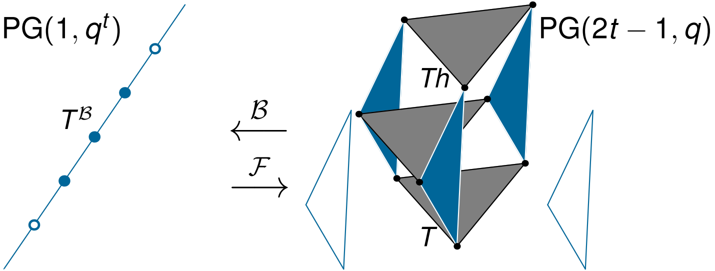
# Linear sets




By blowing up all points of an element $T \in \mathcal{G}$ we obtain a subset $T^{\mathcal{B}}$ of $\mathrm{PG}(1, q^t)$, which is called an $\mathbb{F}_q$-linear set of rank $t$.

The set $T^{\mathcal{B}\mathcal{F}}$ comprises those elements of the spread $\mathcal{D}$ which intersect $T$ non-trivially.
An element $T \in \mathcal{G}$ and its corresponding linear set $T^{\mathcal{B}}$ are said to be scattered if the restriction of $\mathcal{B}$ to $T$ is injective.

# Scattered linear sets – Two families




Let $T$ be scattered and write $Th := \{\langle(ah, bh)\rangle_q \mid \langle(a, b)\rangle_q \in T\}$, where $h \in \mathbb{F}_{q^t} \setminus \{0\} =: \mathbb{F}^*_{q^t}$. Then the families

$$\mathcal{U}(T) := T^{\mathcal{BF}} \text{ and } \mathcal{U}'(T) := \{Th \mid h \in \mathbb{F}^*_{q^t}\},$$

constitute two partitions (by elements of $\mathcal{G}$) of the same hypersurface of degree $t$ in PG($2t - 1, q$).

See M. Lavrauw, J. Sheekey, C. Zanella [15, Prop. 2].

# The projective line over $E$

- We consider the endomorphism ring

$$E := \mathrm{End}_q(\mathbb{F}_{q^t}).$$

- An element $(\alpha, \beta) \in E^2$ is called *admissible* if it can be extended to a basis of the left $E$-module $E^2$.
- The *projective line over $E$* is the set $\mathrm{PG}(1, E)$ of all cyclic submodules $E(\alpha, \beta)$ of $E^2$, where $(\alpha, \beta) \in E^2$ is admissible. The elements of $\mathrm{PG}(1, E)$ are called *points*.
- The map

$$\Psi : \mathrm{PG}(1, E) \to \mathcal{G} : E(\alpha, \beta) \mapsto \left\{ \langle (u^\alpha, u^\beta) \rangle_q \mid u \in \mathbb{F}^*_{q^t} \right\}$$

is a bijection (X. Hubaut [11], Z.-X. Wan [24], and others).

# The distant relation

Let $P = E(\alpha, \beta)$ and $Q = E(\gamma, \delta)$ be points of $\mathrm{PG}(1, E)$.

- $P$ and $Q$ are called *distant*, in symbols $P \vartriangle Q$, if $((\alpha, \beta), (\gamma, \delta))$ is a basis of $E^2$.
- $P \vartriangle Q$ if, and only if, the subspaces $P^\Psi$ and $Q^\Psi$ are skew (see, among others, A. Blunck [1, Thm. 2.4]).

# Embedding of $\mathrm{PG}(1, q^t)$ in $\mathrm{PG}(1, E)$

- The mapping

$$\mathbb{F}_{q^t} \to E \colon a \mapsto (\rho_a \colon x \mapsto xa)$$

  is a monomorphism of rings taking $1 \in \mathbb{F}_{q^t}$ to the identity $\mathbb{1} \in E$.

- This allows us to define an embedding

$$\iota : \mathrm{PG}(1, q^t) \to \mathrm{PG}(1, E) : \langle (a, b) \rangle_{q^t} \mapsto E(\rho_a, \rho_b).$$

# Projectivities

- Given a matrix
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(E)$$
we obtain a projectivity of $\mathrm{PG}(1, E)$ by letting
$$E(\xi, \eta) \mapsto E\left((\xi, \eta) \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)$$
and a projectivity of $\mathrm{PG}(2t-1, q)$ by letting
$$\langle (u, v) \rangle_q \mapsto \langle (u^\alpha + v^\gamma, u^\beta + v^\delta) \rangle_q.$$
- All projectivities of $\mathrm{PG}(1, E)$ and $\mathrm{PG}(2t-1, q)$ can be obtained in this way (S. Lang [13, 642–643]).
- The actions of $\mathrm{GL}_2(E)$ on $\mathrm{PG}(1, E)$ and $\mathcal{G}$ are isomorphic.

# Dictionary

| $\mathrm{PG}(1, E)$ | Grassmannian $\mathcal{G}$ |
|---|---|
| point $T$ | subspace $T^{\Psi} \in \mathcal{G}$ |
| subline $\mathrm{PG}(1, q^t)^{\iota}$ | spread $\mathcal{D}$ |
| $L_T := \left\{ X \in \mathrm{PG}(1, q^t)^{\iota} \mid X \not\triangle T \right\}$ | $\mathcal{U}(T^{\Psi}) = (T^{\Psi})^{\mathcal{BF}}$ |
| $L'_T = \left\{ T \cdot \mathrm{diag}(\rho_h, \rho_h) \mid h \in \mathbb{F}^*_{q^t} \right\}$ | $\mathcal{U}'(T^{\Psi})$ |

The sets $L_T$, with $T$ varying in $\mathrm{PG}(1, E)$, are precisely the images under $\iota$ of the $\mathbb{F}_q$-linear sets of rank $t$ in $\mathrm{PG}(1, q^t)$.

# Linear sets of pseudoregulus type

Let $\tau$ be a generator of the Galois group $\mathrm{Gal}(\mathbb{F}_{q^t}/\mathbb{F}_q)$ and write $T_0 := E(\mathbb{1}, \tau)$. Then $L_{T_0}$ corresponds to a scattered linear set.

A linear set of $\mathrm{PG}(1, q^t)$ is said to be of *pseudoregulus type* if it is projectively equivalent to the linear set corresponding to $T_0$.

Cf. B. Czajbók, C. Zanella [4],
G. Donati, N. Durante [6],
M. Lavrauw, J. Sheekey, C. Zanella [15],
G. Lunardon, G. Marino, O. Polverino, R. Trombetti [20].

# Main result

**Theorem (H. H., C. Zanella [9])**

*A scattered linear set of* $\mathrm{PG}(1, q^t)$*,* $t \geq 3$*, arising from* $T \in \mathrm{PG}(1, E)$ *is of pseudoregulus type if, and only if, there exists a projectivity* $\varphi$ *of* $\mathrm{PG}(1, E)$ *such that* $L_T^{\varphi} = L'_T$*.*

Proof.

"$\Leftarrow$" See M. Lavrauw, J. Sheekey, C. Zanella [15, Cor. 18] or [9].

"$\Rightarrow$" For the most part, the proof can be done neatly in $\mathrm{PG}(1, E)$ using the representation of projectivities in terms of $\mathrm{GL}_2(E)$ ...

Essence: We establish the existence of a cyclic group of projectivities of $\mathrm{PG}(1, E)$ acting regularly on $L_T$ and fixing $L'_T$ pointwise.

# References

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