Von Staudt's Theorem Revisited

Hans Havlicek

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Research Group
Differential Geometry and Geometric Structures
Institute of Discrete Mathematics and Geometry

Conference on Geometry: Theory and Applications,
Kefermarkt, June 11, 2015

Von Staudt, Geometrie der Lage (1847)

Zwei einförmige Grundgebilde heissen zu einander projektivisch ($\barwedge$), wenn sie so auf einander bezogen sind, dass jedem harmonischen Gebilde in dem einen ein harmonisches Gebilde im andern entspricht.

Next, after defining perspectivities, the following theorem is established:

Any projectivity is a finite composition of perspectivities and vice versa.

It was noticed later that there is a small gap in von Staudt's reasoning.

Any result in this spirit now is called a *von Staudt's theorem*.

The projective line over a ring

- Let R be a ring with unity $1 \neq 0$.
- Let M be a free left R-module of rank 2, i. e., M has a basis with two elements.
- We say that $a \in M$ is *admissible* if there exists $b \in M$ such that (a, b) is a basis of M (with two elements).
  (We do not require that all bases of M have the same number of elements.)

Definition

The *projective line* over M is the set $\mathbb{P}(M)$ of all cyclic submodules Ra, where $a \in M$ is admissible. The elements of $\mathbb{P}(M)$ are called *points*.

The distant relation

Definition

Two points p and q of $\mathbb{P}(M)$ are called *distant*, in symbols $p \triangle q$, if $M = p \oplus q$.

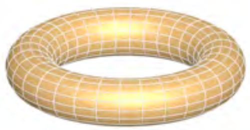
Examples

The projective line over some rings can be modelled as surfaces with a system of distinguished curves that illustrate the non-distant relation.

Cylinder:

Real dual numbers $\mathbb{R}(\varepsilon)$.

Torus:



Real double numbers $\mathbb{R} \times \mathbb{R}$.

Harmonic quadruples

Definition

A quadruple $(p_0, p_1, p_2, p_3) \in \mathbb{P}(M)^4$ is *harmonic* if there exists a basis (g_0, g_1) of M such that

$$p_0 = Rg_0, \quad p_1 = Rg_1, \quad p_2 = R(g_0 + g_1), \quad p_3 = R(g_0 - g_1).$$

Given four harmonic points as above we obtain:

- $p_0 \triangle p_1$ and $\{p_0, p_1\} \triangle \{p_2, p_3\}$.
- $p_2 \neq p_3$ if, and only if, $2 \neq 0$ in R.
- $p_2 \triangle p_3$ if, and only if, 2 is a unit in R.

Harmonicity preservers

Let M' be a free left module of rank 2 over a ring R'.

Definition

A mapping $\mu : \mathbb{P}(M) \to \mathbb{P}(M')$ is said to be a *harmonicity preserver* if it takes all harmonic quadruples of $\mathbb{P}(M)$ to harmonic quadruples of $\mathbb{P}(M')$.

No further assumptions, like injectivity or surjectivity of μ will be made.

Main problem

Give an algebraic description of all harmonicity preservers between projective lines over rings R and R'.

Solutions and Contributions

Many authors addressed our main problem :

- **(Skew) Fields** with characteristic $\neq 2$:
  O. Schreier and E. Sperner [19],
  G. Ancochea [1], [2], [3],
  L.-K. Hua [10], [11].
- **(Non) Commutative Rings** subject to varying extra assumptions:
  W. Benz [6], [7],
  H. Schaeffer [18],
  B. V. Limaye and N. B. Limaye [12], [13], [14],
  N. B. Limaye [15], [16],
  B. R. McDonald [17],
  C. Bartolone and F. Di Franco [5].

A wealth of articles is concerned with generalisations.

Jordan homomorphisms of rings

Definition

A mapping $\alpha : R \to R'$ is a *Jordan homomorphism* if for all $x, y \in R$ the following conditions are satisfied:

1. $(x + y)^\alpha = x^\alpha + y^\alpha$,
2. $1^\alpha = 1'$,
3. $(xyx)^\alpha = x^\alpha y^\alpha x^\alpha$.

Examples

- All homomorphisms of rings, in particular $\mathrm{id}_R : R \to R$.
- All antihomomorphisms of rings; e. g. the conjugation of real quaternions: $\mathbb{H} \to \mathbb{H}$ with $z \mapsto \bar{z}$.
- The mapping $\mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H} : (z, w) \mapsto (\bar{z}, w)$ which is neither homomorphic nor antihomomorphic.

Beware of Jordan homomorphisms

Let $\alpha : R \to R'$ be a Jordan homomorphism. Given bases (e_0, e_1) of M and (e'_0, e'_1) of M' the mapping $M \to M'$ defined by

$$x_0 e_0 + x_1 e_1 \mapsto x_0^\alpha e'_0 + x_1^\alpha e'_1 \text{ for all } x_0, x_1 \in R$$

need not take submodules to submodules (let alone points to points).

Assumption

Let $\mu : \mathbb{P}(M) \to \mathbb{P}(M')$ be a harmonicity preserver. Furthermore, we assume that R contains “sufficiently many” units; in particular 2 has to be a unit in R.

Step 1: A local coordinate representation of μ

There are bases (e_0, e_1) of M and (e'_0, e'_1) of M' such that

$$(Re_0)^\mu = R'e'_0, \quad (Re_1)^\mu = R'e'_1, \quad \big(R(e_0 \pm e_1)\big)^\mu = R'(e'_0 \pm e'_1).$$

Then there exists a unique mapping $\beta : R \to R'$ with the property

$$\big(R(xe_0 + e_1)\big)^\mu = R'(x^\beta e'_0 + e'_1) \quad \text{for all} \quad x \in R.$$

This β is additive and satisfies $1^\beta = 1'$.

Step 2: Change of coordinates

We may repeat Step 1 for the new bases

$$(f_0, f_1) := (te_0 + e_1, -e_0) \quad \text{and} \quad (f'_0, f'_1) := (t^\beta e'_0 + e'_1, -e'_0),$$

where $t \in R$ is arbitrary. So the transition matrices are

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad E(t^\beta) := \begin{pmatrix} t^\beta & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the new local representation of μ yields the same mapping β as in Step 1.

Step 3: β is a Jordan homomorphism

By combining Step 1 and Step 2 (for $t = 0$) one obtains:

The mapping β from Step 1 is a Jordan homomorphism.

Part of the proof relies on previous work.

Step 4: Induction

Suppose that a point $p \in \mathbb{P}(M)$ can be written as

$$p = R(x_0 e_0 + x_1 e_1)$$

with

$$(x_0, x_1) = (1, 0) \cdot E(t_1) \cdot E(t_2) \cdots E(t_n) \quad \text{for some} \quad t_1, t_2, \ldots, t_n \in R,$$

where n is variable.

Then the image point of p under μ is

$$R'(x_0' e_0' + x_1' e_1')$$

with

$$(x_0', x_1') = (1', 0') \cdot E(t_1^\beta) \cdot E(t_2^\beta) \cdots E(t_n^\beta).$$

Concluding remarks

- For a wide class of rings in order to reach all points of $\mathbb{P}(M)$ it suffices to let $n \leq 2$ in Step 4.
- There are rings where the the description from Step 4 will not cover the entire line $\mathbb{P}(M)$. Here μ can be described in terms of several Jordan homomorphisms.
- Any Jordan homomorphism $R \to R'$ gives rise to a harmonicity preserver. This follows from previous work of C. Bartolone [4] and A. Blunck, H. H. [8].
- For precise statements and further references see [9].

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