The isometries of Cayley's surface

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Differentialgeometrie und Geometrische Strukturen

Basic notions

Let $\mathbb{P}_3(K)$ be the *3-dimensional projective space* over a commutative field K.

Given a homogeneous polynomial $g(\boldsymbol{X}) \in K[\boldsymbol{X}] = K[X_0, X_1, X_2, X_3]$ then

$$\mathcal{V}\big(g(\boldsymbol{X})\big) := \big\{K\boldsymbol{p} \in \mathbb{P}_3(K) \mid g(\boldsymbol{p}) = 0\big\}$$

denotes the *set of K-rational points* of the variety given by this form.

We regard $\omega := \mathcal{V}(X_0)$ as the *plane at infinity*.

Cayley's ruled cubic surface

The *Cayley surface* is given by $F := \mathcal{V}\big(f(\boldsymbol{X})\big)$, where

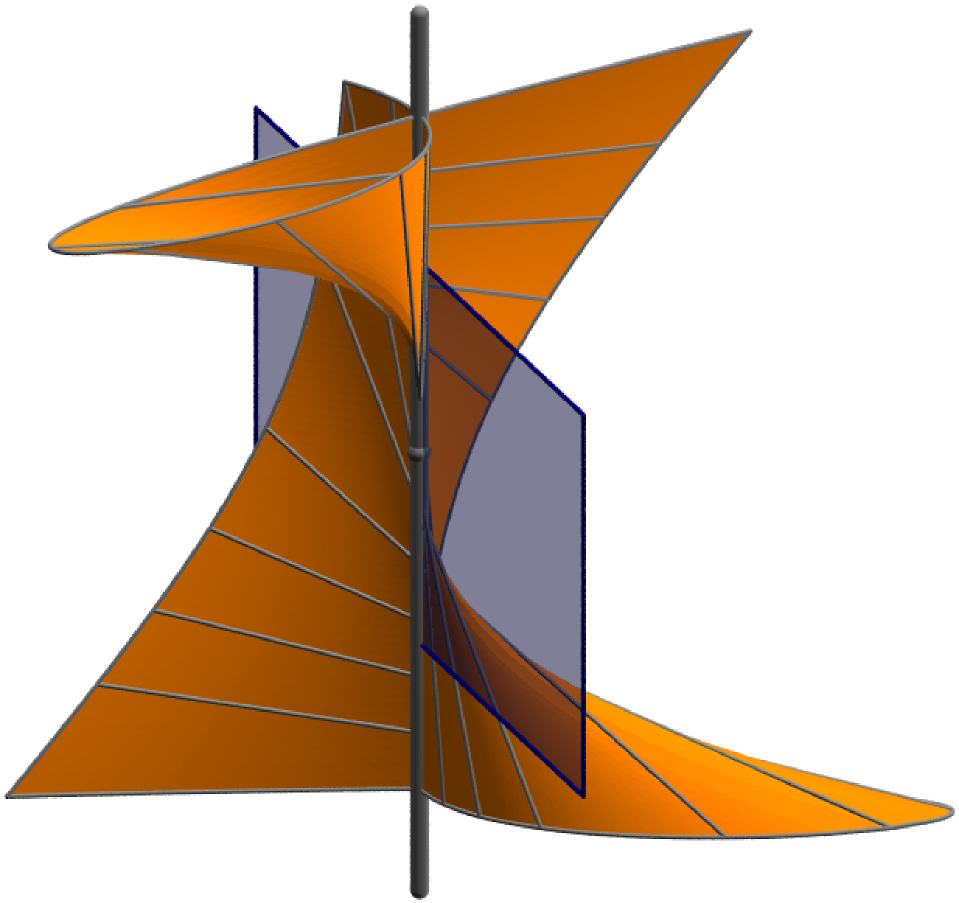
$$f(\boldsymbol{X}) := X_0X_1X_2 - X_1^3 - X_0^2X_3. \tag{1}$$

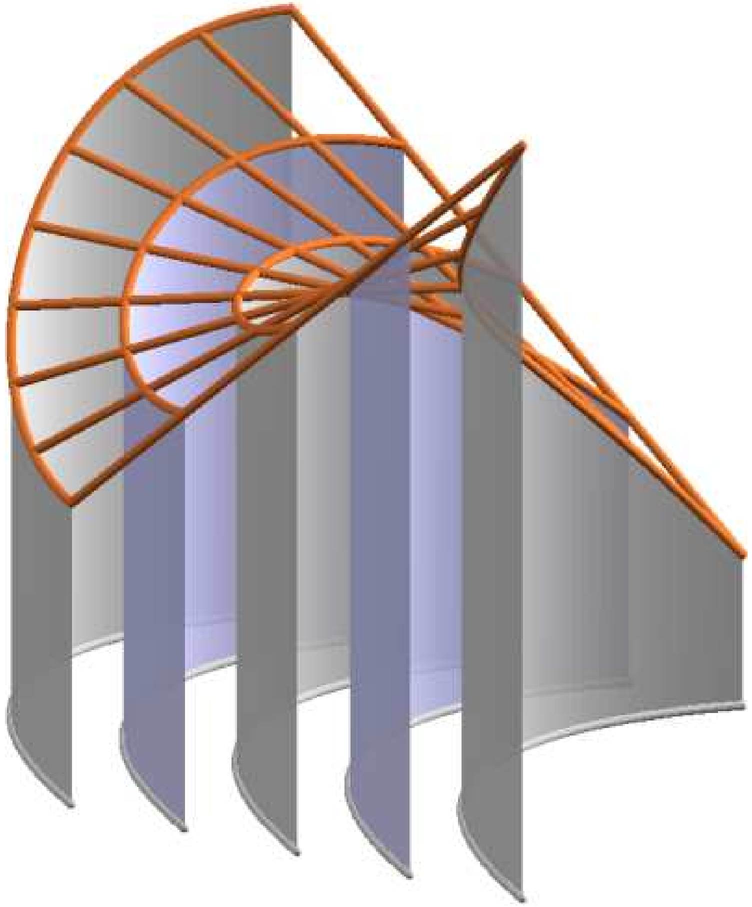
The parametrization

$$K^2 \to \mathbb{P}_3(K) : (u_1, u_2) \mapsto K(1, u_1, u_2, u_1u_2 - u_1^3)^{\mathrm{T}} =: P(u_1, u_2) \tag{2}$$

is injective, and its image coincides with $F \setminus \omega$ (the affine part of F).

Two pictures





Automorphic collineations

The set of all matrices

$$M_{a,b,c} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & c & 0 & 0 \\ b & 3\,ac & c^2 & 0 \\ ab - a^3 & bc & ac^2 & c^3 \end{pmatrix} \tag{3}$$

where $a, b \in K$ and $c \in K \setminus \{0\}$ is a group, say $G(F)$, under multiplication.

Each matrix in $G(F)$ leaves invariant the cubic form $f(\boldsymbol{X})$ to within the factor c^3. Consequently, the group $G(F)$ acts on F as a group of projective collineations.

Theorem 1. *There are no automorphic projective collineations of the Cayley surface F other than the ones given by (3) if, and only if, $|K| \geq 4$.*

A distance function on $F \setminus \omega$

From now on we shall assume $|K| \geq 4$.

We define a function

$$\delta : (F \setminus \omega) \times (F \setminus \omega) \to K \cup \{\infty\}$$

as follows. Let $A = P(u_1, u_2)$ and $B = P(v_1, v_2)$, where $u_1, u_2, v_1, v_2 \in K$.

- $u_1 = v_1 \Leftrightarrow A, B$ are on a common generator of F: $\delta(A, B) := \infty$ ($A \parallel B$ . . . *parallel points*)

- $u_1 \neq v_1$, $AB \cap F =: \{A, B, C\}$, $AB \cap \omega := \{I\}$:

$$\delta(A, B) := \mathrm{CR}(C, B, A, I) = \frac{2u_1^2 - u_2 - u_1 v_1 + v_2 - v_1^2}{(u_1 - v_1)^2}$$

Properties of the distance function

The following properties hold for all $A, B \in F \setminus \omega$:

- $\delta(A, A) = \infty$.
- There exists a point $C \in F \setminus \omega$ with $C \neq A$ and $\delta(A, C) = \infty$.
- $\delta(A, B) = 1 - \delta(B, A)$ (with $1 - \infty := \infty$).
- $\delta(A, B) \in \{0, 1\} \Leftrightarrow AB$ is a tangent of F.

H. Brauner (1964), $K = \mathbb{R}$ using differential geometry and Lie groups:

$\widehat{\delta}(A, B) := \frac{3}{2}\left(\frac{1}{2} - \delta(A, B)\right)^{-1}$, $\widehat{\delta}(A, A) = 0$, and $\widehat{\delta}(A, B) = -\widehat{\delta}(B, A)$.

Circles

Given a point $A \in F \setminus \omega$ and an element $\rho \in K \cup \{\infty\}$ we define the *circle with midpoint A and radius ρ* in the obvious way as

$$\mathcal{C}(A, \rho) := \{Y \in F \setminus \omega \mid \delta(A, Y) = \rho\}.$$

By the *extended circle* $\mathcal{E}(A, \rho)$ we mean the circle $\mathcal{C}(A, \rho)$ together with its midpoint A.

A family of curves

For all $\alpha, \beta, \gamma \in K$ the rationally parameterized curve

$$\mathcal{R}_{\alpha,\beta,\gamma} := \left\{ K(1, t, \alpha + \beta t + (\gamma+1)t^2, \alpha t + \beta t^2 + \gamma t^3)^{\mathrm{T}} \mid t \in K \cup \{\infty\} \right\} \quad (4)$$

is lying on F. It is

- a *parabola* for $\gamma = 0$,
- a *planar cubic* for $\gamma = -1$,
- a *twisted cubic parabola* (i.e. a twisted cubic having the plane at infinity as an osculating plane) otherwise.

Remark. $F \setminus \omega$ together with the affine traces of the curves (4) is isomorphic to the affine chain geometry on the ring $K[\varepsilon]$ of dual numbers over K. An isomorphism is given by $P(u_1, u_2) \mapsto u_1 + \varepsilon u_2$.

Description of extended circles

Proposition 2. *Suppose that a point $A = P(a_1, a_2)$, $a_1, a_2 \in K$, and an element $\rho \in K \cup \{\infty\}$ are given.*

- *If $\rho \in K$ then the extended circle $\mathcal{E}(A, \rho)$ equals the set of affine points of $\mathcal{R}_{\alpha,\beta,\gamma}$, where*

$$\alpha := (\rho - 2)a_1^2 + a_2,\ \beta := (1 - 2\rho)a_1,\ \gamma := \rho.$$

- *If $\rho = \infty$ then $\mathcal{C}(A, \rho) = \mathcal{E}(A, \rho)$ is the unique generator of F through A, but without its point at infinity.*

Proposition 3. *Given a curve $\mathcal{R}_{\alpha,\beta,\gamma}$, with $\alpha, \beta, \gamma \in K$, there are three possibilities.*

(a) $1 - 2\gamma \neq 0 : \mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$ *coincides with the extended circle* $\mathcal{E}(A, \rho)$, *where*

$$A := P\left(\frac{\beta}{1-2\gamma}, \alpha - \frac{(\gamma-2)\beta^2}{(1-2\gamma)^2}\right) \text{ and } \rho := \gamma.$$

(b) $1 - 2\gamma = 0 \neq \beta : \mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$ *is not an extended circle.*

(c) $1 - 2\gamma = 0 = \beta : \mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$ *is an extended circle* $\mathcal{E}(A, \frac{1}{2})$ *for all points* $A \in \mathcal{R}_{\alpha,\beta,\gamma} \setminus \omega$.

$\mathrm{Char}\, K \neq 2$: All cases occur.

$\mathrm{Char}\, K = 2$: $1 - 2\gamma = 1 \neq 0$. There are no circles with more than one midpoint.

Transitivity of $G(F)$

Theorem 4. *The matrix group $G(F)$ has the following properties:*

(a) *$G(F)$ acts on $F \setminus \omega$ as a group of isometries.*

(b) *$G(F)$ acts regularly on the set of antiflags of $F \setminus \omega$.*

(c) *For each $d \in K$ the group $G(F)$ acts regularly on the set*

$$\Delta_d := \{(A, B) \in (F \setminus \omega)^2 \mid \delta(A, B) = d\}.$$

(d) *Given $A = P(u_1, u_2) \parallel B = P(u_1, v_2)$ and $A' = P(u_1', u_2') \parallel B' = P(u_1', v_2')$, with $u_1, u_2, \ldots, v_2' \in K$, the number of matrices in $G(F)$ mapping (A, B) to (A', B') equals the number of distinct elements $c \in K \setminus \{0\}$ such that*

$$c^2(v_2 - u_2) = (v_2' - u_2').$$

All isometries

Following W. Benz an *isometry* of $F \setminus \omega$ is just a mapping $\mu : F \setminus \omega \to F \setminus \omega$ such that

$$\delta(A, B) = \delta\big(\mu(A), \mu(B)\big) \text{ for all } A, B \in F \setminus \omega.$$

Theorem 5. *Each isometry $\mu : F \setminus \omega \to F \setminus \omega$ is induced by a unique matrix in $G(F)$. Consequently, μ is bijective and it can be extended in a unique way to a projective collineation of $\mathbb{P}_3(K)$ fixing the Cayley surface F.*