On bijections that preserve complementary subspaces

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The distant graph

$V \ldots$ vector space over a (skew) field $K$.
$\mathcal{G} \ldots$ all subspaces $X \leq V$ such that $X \cong V/X$.
We assume that $\mathcal{G} \neq \emptyset$.

The distant relation

$$X \triangle Y :\iff X \oplus Y = V,$$

where $X, Y \in \mathcal{G}$, yields the distant graph on $\mathcal{G}$.

The distant graph is connected. Its diameter is as follows:

<table>
<thead>
<tr>
<th>dim $V$</th>
<th>0</th>
<th>2</th>
<th>4, 6, …</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>diameter</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

The Grassmann graph

The adjacency relation

\[ X \sim Y \iff 1 = \dim((X + Y)/X) = \dim((X + Y)/Y), \]

where \( X, Y \in \mathcal{G} \) yields the Grassmann graph on \( \mathcal{G} \).

Distance of \( X, Y \in \mathcal{G} \) in the Grassmann graph:

\[ d = \dim((X + Y)/X) = \dim((X + Y)/Y) \]

which is equivalent to

\[ d = \dim(X/(X \cap Y)) = \dim(Y/(X \cap Y)) \]

The Grassmann graph is connected iff \( \dim V < \infty \).
A characterization of adjacency

**Theorem.** For all $P, Q \in \mathcal{G}$ the following statements are equivalent:

- $P$ and $Q$ are adjacent.
- There is an element $R \in \mathcal{G}$ satisfying the following conditions:
  
  \[
  R \neq P, Q, \\
  \forall X \in \mathcal{G} : X \triangle R \Rightarrow X \triangle P \text{ or } X \triangle Q.
  \]
Comparing Automorphisms

Theorem. The following statements hold:

• Every automorphism of the distant graph is an automorphism of the Grassmann graph.

• If \( \dim V < \infty \) then every automorphism of the Grassmann graph is an automorphism of the distant graph.
Theorem. Let $4 \leq \dim V < \infty$ and let $\varphi : G \to G$ be an automorphism of the distant graph. Then there is either a semilinear bijection

$$f : V \to V \text{ such that } X^\varphi = X^f$$

or a semilinear bijection

$$f : V^* \to V \text{ such that } X^\varphi = (X^\perp)^f;$$

here $V^*$ denotes the dual of $V$.
