1. Introduction

$\mathfrak{V}_K$	$\{\mathfrak{v}_0, \ldots\ldots, \mathfrak{v}_n\}$	$\mathfrak{a}$
$\vdots$	$\vdots$	$\vdots$
(n+1)-dimensional right K-vector space	basis	$\in \mathfrak{V}_K \setminus \{\mathfrak{o}\}$

$\mathcal{P}(\mathfrak{V}_K)$ ... projective space on $\mathfrak{V}_K$

$\mathfrak{F} = \{\mathfrak{v}_0 K, \ldots, \mathfrak{v}_n K, (\sum_{j=0}^{n} \mathfrak{v}_j)K\}$ .... frame

$P = \mathfrak{a} K$

Φ ... group of all projective collineations which leave $\mathfrak{F}$ elementwise invariant

P^Φ ... orbit of P under the group Φ

Every $\varphi \in \Phi$ is induced by a linear automorphism of $\mathfrak{V}_K$ such that

(*) $\mathfrak{v}_j \mapsto \mathfrak{v}_j x$, $x \in K^\times$

and conversely every $x \in K^\times$ yields a map $\in \Phi$.

$\Phi \cong K^\times / Z(K)^\times$ $\qquad$ Z(K)... centre of K.

$\Phi = \{id\} \iff \mathcal{P}(\mathfrak{V}_K)$ pappian.

Representation of (*) in coordinates

$$\begin{tikzcd}
\mathfrak{v} \arrow[r, dash, dotted] & (a_0,\ldots,a_n) \arrow[d, mapsto, "\mathrm{diag}(x,\ldots,x)"] \\
x\mathfrak{v} \arrow[r, dash, dotted] & (xa_0,\ldots,xa_n)
\end{tikzcd}$$

⚠ This left multiplication depends on the basis $\{\mathbb{v}_0,\ldots,\mathbb{v}_n\}$.

2. A dimension formula

$$\underset{\mathcal{P}(\mathcal{W}_K)}{\dim}\,[P\Phi] \;?$$

Equivalent problem

$$\underset{\mathcal{W}_K}{\dim}\,[K^{\times}\mathfrak{a}] = \underset{\mathcal{W}_K}{\dim}\,[\{x\mathfrak{a} \mid x \in K^{\times}\}]$$

LEMMA 1. If U is a sub-semigroup of $(K^{\times};\cdot)$ and $Z(U)$ denotes the centralizer of U in K, then

$$\underset{\mathcal{W}_K}{\dim}\,[U\mathfrak{a}] = \underset{{}_{Z(U)}K}{\dim}\,[\{a_0,\dots,a_n\}]$$

THEOREM 1. The subspace of $\mathcal{P}(\mathcal{W}_K)$ spanned by the orbit of $P=\mathfrak{a}K$ under Φ has dimension equal to

$$\underset{{}_{Z(K)}K}{\dim}\,[\{a_0,\dots,a_n\}]-1\,.$$

3. The "internal structure" of $P\Phi$

$\mathfrak{a} \ldots\ldots (a_0, \ldots, a_n)$

Without loss of generality $a_0 = 1$

$A = \{a_0 = 1, a_1, \ldots, a_n\}$ $\quad$ $Z(A)$... centralizer of A in K

If we fix the basis $\{\mathfrak{n}_0, \ldots, \mathfrak{n}_n\}$ then $Z(A)$ is determined by $P\Phi$ up to transformation under inner automorphisms of K.

This result remains true if we restrict ourselves to those bases of $\mathfrak{W}_K$ which determine a frame of Φ-invariant points.

$K\mathfrak{a}$:

$$x_0\mathfrak{a} + x_1\mathfrak{a} = (x_0 + x_1)\mathfrak{a}$$
$$(x\mathfrak{a})s = (xs)\mathfrak{a} \qquad \forall\, s \in Z(A)$$

Hence $K\mathfrak{a}$ is a $Z(A)$-right vector space

The map $x \mapsto x\mathfrak{a}$ is an isomorphism

$$K_{Z(A)} \rightarrow (K\mathfrak{a})_{Z(A)}$$

In projective terms we get a surjective map

$$\begin{aligned} \iota : \mathcal{P}(K_{Z(A)}) &\longrightarrow P\Phi \\ xZ(A) &\longmapsto (x\mathfrak{r})K \end{aligned}$$

Is ι an injective map ?

How "good" is the embedding ι ?

Every line of $\mathcal{P}(K_{Z(A)})$ is mapped into a line of $\mathcal{P}(\mathfrak{W}_K)$, but different lines of $\mathcal{P}(K_{Z(A)})$ may have images within the same line of $\mathcal{P}(\mathfrak{W}_K)$.

If $\{x_i\mathfrak{v} \mid x_i \in K^\times, i=0,\dots,m\}$ is linearly independent in $\mathfrak{W}_K$, then this family is also linerarly independent in $(K\mathfrak{v})_{Z(A)}$ and hence $\{x_i \mid i=0,\dots,m\}$ is linearly independent in $K_{Z(A)}$.

Conversely, if $\dim[K\mathfrak{v}] = d+1$ then there are linearly independent elements $\bar{x}_i \in K_{Z(A)}$ such that $\{\bar{x}_i\mathfrak{v} \mid i=0,\dots,d\}$ is a basis of $[K\mathfrak{v}]$. But not every family of $d+1$ (or even less) linearly independent elements of $K_{Z(A)}$ yields necessarily an independent family in $\mathfrak{W}_K$.

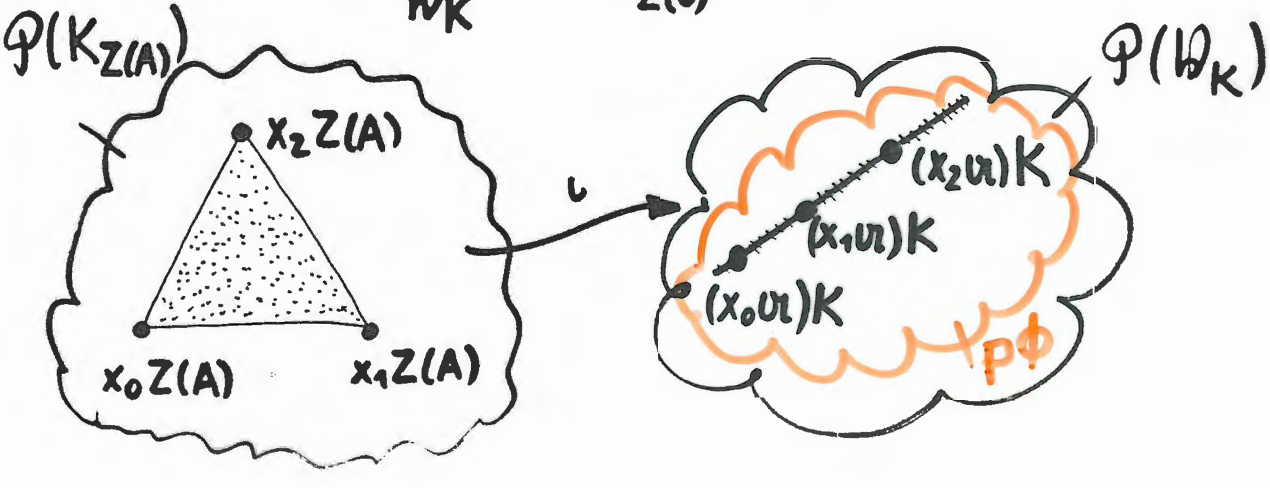
Example: K ... skew field , $|K:Z(K)| = 9$

$a_1, a_2 \in K$ such that $a_1a_2 \neq a_2a_1$

(a_0, a_1, a_2) with $a_0 = 1$ $\Rightarrow$ $Z(A) = Z(K)$

$x_0 = 1$, $x_1 = a_1$, $x_2 = a_1^2$; $U = \langle a_1 \rangle \Rightarrow Z(U) \neq Z(K)$

By Lemma 1 : $\underset{\mathfrak{W}_K}{\dim}[U\mathfrak{v}] = \underset{Z(U)^K}{\dim}[\{1, a_1, a_2\}] = 2 \neq 3$



4. Lifting

Let $B = \{b_0 = 1, \ldots\ldots, b_\iota\} \subset K^\times$ such that $Z(B) \subset Z(A)$.

$$\mathcal{W}_K \xrightarrow{B} (\mathcal{W}_1)_K = \underbrace{\mathcal{W} \oplus \cdots\cdots \oplus \mathcal{W}}_{(\iota+1)\text{times}}$$

$$\{\mathfrak{v}_j \mid j = 0, \ldots n\} \xrightarrow{B} \{\mathfrak{v}_j \delta_{0r} \oplus \cdots\cdots \oplus \mathfrak{v}_j \delta_{r\iota} \mid j = 0, \ldots, n;\ r = 0, \ldots, \iota\}$$

$$\mathfrak{v} \xrightarrow{B} \mathfrak{v}_1 = \mathfrak{v} b_0 \oplus \cdots\cdots \oplus \mathfrak{v} b_\iota$$

$$\mathcal{P}(\mathcal{W}_K), P, \phi, \ldots \xrightarrow{B} \mathcal{P}((\mathcal{W}_1)_K),\ P_1,\ \phi_1, \ldots\ldots$$

$$Z(A) \xrightarrow{B} Z(B)$$

„B-lifting of $(\mathcal{W}_K, \{\mathfrak{v}_j \mid j = 0, \ldots, n\}, \mathfrak{v})$"

Lemma 2: If $\{x_i \mathfrak{v} \mid i = 0, \ldots, m-1\}$ is lin. ind. in $\mathcal{W}_K$ and $\{x_i \mid i = 0, \ldots\ldots, m\}$ is lin. ind. in $K_{Z(B)}$ then

$$\{x_i \mathfrak{v}_1 \mid i = 0, \ldots, m\}$$

is linearly independent in $(\mathcal{W}_1)_K$.

Application:

$$K_K \xrightarrow{A} K_K^{n+1}$$

$$\{1\} \xrightarrow{A} \text{canonical basis}$$

$$1 \xrightarrow{A} (a_0, \ldots, a_n)$$

$\{x_0, x_1\}$ lin. ind. in $K_{Z(A)} \implies x_0 \mathfrak{v}, x_1 \mathfrak{v}$ lin. ind. in $\mathcal{W}_K$

Corollary: The map $\iota : \mathcal{P}(K_{Z(A)}) \to P\Phi$ is injective.

Theorem 2. If $\mathbb{W}_K, \{w_j \mid j=0,\dots,n\}, \iota$ can be written as an h-fold lifting ($h \geq 1$) then any $h+1$ independent points of $\mathcal{P}(K_{Z(A)})$ are mapped under ι into $h+1$ independent points.

If $h \geq 2$ then $\iota : \mathcal{P}(K_{Z(A)}) \to P\Phi$ is a collineation of the projective space on $K_{Z(A)}$ onto the trace space of $\mathcal{P}(\mathbb{W}_K)$ determined by $P\Phi$.

5. Final remarks

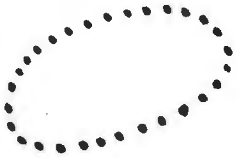
$\mathfrak{A}$ .... $(1, a, a^2, \dots, a^n)$ $\Rightarrow$ n-fold lifting of $K_K, \{1\}, 1$

P^Φ ....subset of a normal curve of $\mathcal{P}(\mathfrak{W}_K)$

↳ conjugacy class (H.H. 1984)

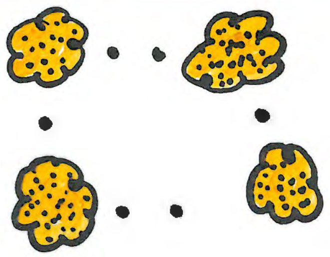
e.g. $n=2$

K commutative



a conic is a union of points ($P^\Phi = \{P\}$)

K non-commutative



a conic is a union of projective spaces some of which are single points.

(L. Vogt, 1986)

$\varphi \in \Phi$ and suppose $|P^\Phi| > 1$. If $\varphi | P^\Phi = \mathrm{id}_{P^\Phi}$, then $\varphi = \mathrm{id}_{\mathcal{P}(\mathfrak{W}_K)}$.