Mutually Inscribed and Circumscribed Simplices— Where Möbius Meets Pauli

Joint work with
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Part 1

Introduction

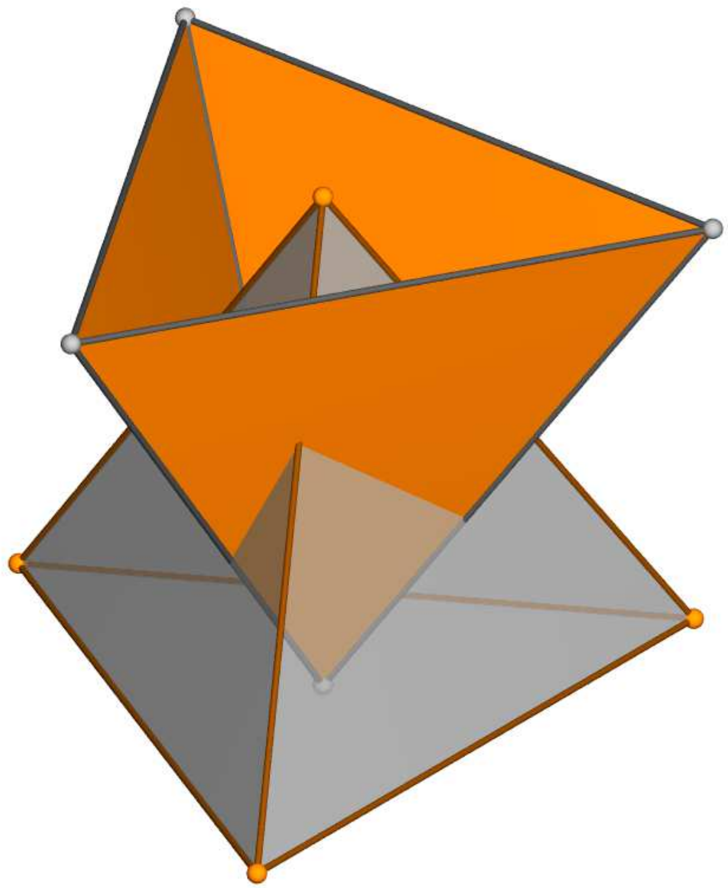
F. A. Möbius gave an affirmative answer to the following question in 1828:

Do there exist two tetrahedra each of which has all its vertices lying in planes of the other?

F. A. Möbius. Kann von zwei dreiseitigen Pyramiden eine jede in Bezug auf die andere um- und eingeschrieben zugleich heissen? *J. reine angew. Math.*, 3:273–278, 1828.

Example

Here is an example in the three-dimensional Euclidean space.



The two (regular) tetrahedra are mutually inscribed and circumscribed.

We call them a *Möbius pair of tetrahedra* or shortly a *Möbius pair*.

The Three-Dimensional Case

The result of Möbius involves only incidence properties, so it is a theorem of three-dimensional projective geometry over the real numbers.

There is a wealth of older and newer papers on Möbius pairs (H. S. M. Coxeter, A. P. Guinand, K. Witczyński, . . .).

It turns out that Möbius pairs exist in the three-dimensional projective space over any field F. (All our fields are understood to be commutative.)

Möbius Pairs

In what follows we consider the n-dimensional projective space $\mathrm{PG}(n, F)$ over any field F, where $n \geq 1$.

Two n-simplices of $\mathrm{PG}(n, F)$ are *mutually inscribed and circumscribed* if each point of the first simplex is in a hyperplane of the second simplex, and *vice versa* for the points of the second simplex.

Two such n-simplices will be called a *Möbius pair of simplices* in $\mathrm{PG}(n, F)$ or shortly a *Möbius pair*.

Existence

A systematic account of the n-dimensional case seems to be missing. We could find just a few results:

- In $\mathrm{PG}(n, F)$, with n odd, choose any null polarity and any n-simplex, say $\mathcal{P}$. Then the poles of the hyperplanes of $\mathcal{P}$ comprise a simplex $\mathcal{Q}$, say. The simplices $\mathcal{P}$ and $\mathcal{Q}$ form a Möbius pair (folklore, mentioned in a book by H. Brauner).

- The Klein image of a double six of lines in $\mathrm{PG}(3, F)$ gives a Möbius pair in $\mathrm{PG}(5, F)$ (folklore, mentioned in a book by J. W. P. Hirschfeld).

- Other examples are due to L. Berzolari and H. S. M. Coxeter.

Non-Degeneracy

A Möbius pair is said to be *non-degenerate* if each point of either simplex is incident with one and only one hyperplane of the other simplex.

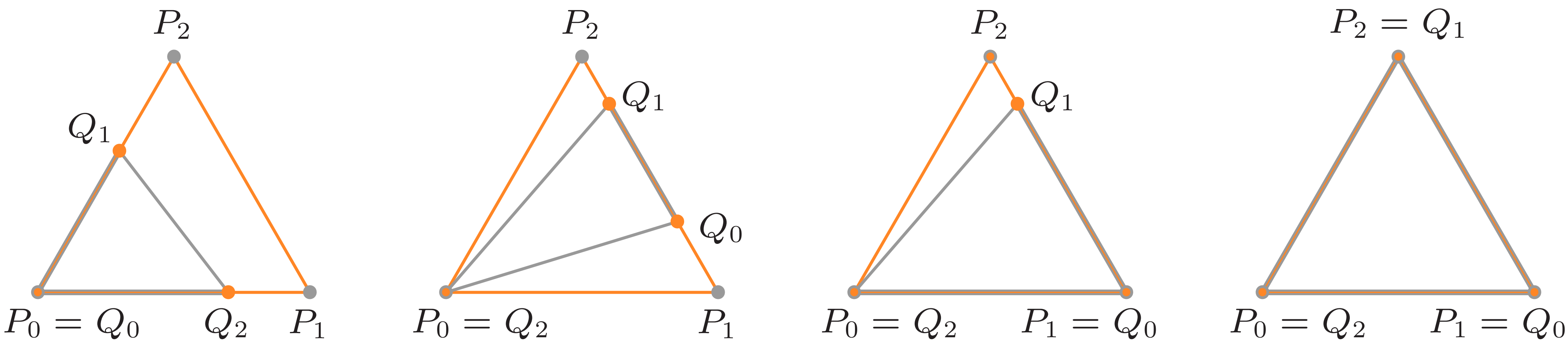
Question:

Do non-degenerate Möbius pairs exist in $\mathrm{PG}(n, F)$ for all $n \geq 1$ and all fields F?

A Negative Answer

Any triangle P_0, P_1, P_2 in the projective plane $\mathrm{PG}(2, F)$ can be extended to a Möbius pair.

However, all solutions are degenerate.



Part 2

Non-degenerate Möbius Pairs

In the second part the existence of non-degenerate Möbius pairs will be established for projective spaces $\mathrm{PG}(n, F)$ of odd dimension $n \geq 1$.

The problem of finding all non-degenerate Möbius pairs is not within the scope of this lecture.

Basic Assumptions

We define an alternating $(n+1) \times (n+1)$ matrix

$$A := \begin{pmatrix} 0 & -1 & \dots & -1 \\ 1 & 0 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{pmatrix}. \tag{1}$$

It is easily verified that A is an invertible matrix. Thus A defines a null polarity π of $\mathrm{PG}(n, F)$.

Basic Assumptions (cont.)

Let

$$\mathcal{P} := \{P_0, P_1, \dots, P_n\}$$

be the n-simplex which is determined by the vectors $e_0, e_1, \dots, e_n$ of the standard basis of F^{n+1}, i. e.,

$$P_j = Fe_j \text{ for all } j \in \{0, 1, \dots, n\}. \tag{2}$$

The elements of F^{n+1} are understood as column vectors.

Towards an Affirmative Answer

Lemma 1. *Let S be a subspace of $\mathrm{PG}(n, F)$ which is generated by $k+1 \geq 1$ distinct points of the simplex $\mathcal{P}$, say $P_{j_0}, P_{j_1}, \ldots, P_{j_k}$ with $0 \leq j_0 < j_1 < \cdots < j_k \leq n$. Then the following assertions hold:*

- *k odd: $S \cap \pi(S) = \emptyset$.*
- *k even: $S \cap \pi(S)$ is a single point, namely*

$$P_{j_0,j_1,\ldots,j_k} := F\Big(\sum_{i=0}^{k}(-1)^{i+1}e_{j_i}\Big).$$

Hence $P_{j_0,j_1,\ldots,j_k}$ is in general position to the chosen points of $\mathcal{P}$.

The proof is an elementary calculation.

2^n Distinguished Points

The null polarity π and the simplex $\mathcal{P}$ give rise to the following points:

- $P_0, P_1 \dots, P_n$ (the points of $\mathcal{P}$).

- $P_{012}, P_{013}, \dots, P_{n-2,n-1,n}$ (one point in each plane of $\mathcal{P}$)

$\dots$

- $P_{0,1,\dots,n-1}, \dots, P_{1,2,\dots,n}$ (one point in each hyperplane of $\mathcal{P}$).

All together these are

$$\binom{n+1}{1} + \binom{n+1}{3} + \cdots + \binom{n+1}{n} = \sum_{i=0}^{n} \binom{n}{i} = 2^n \tag{3}$$

mutually distinct points.

Main Result

Given $P_{j_0,j_1,\dots,j_k}$ let $0 \leq m_0 < m_1 < \cdots < m_{n-k} \leq n$ be those indices which do not appear in $(j_0, j_1, \dots, j_k)$. Then we define

$$P_{j_0,j_1,\dots,j_k} =: Q_{m_0,m_1,\dots,m_{n-k}}. \tag{4}$$

Theorem 1. *In* $\mathrm{PG}(n, F)$, n *odd, let the null polarity* π *and the* n*-simplex*

$$\mathcal{P} = \{P_0, P_1, \dots, P_n\}$$

be given according to (1) *and* (2)*, respectively. Then* $\mathcal{P}$ *and*

$$\mathcal{Q} := \{Q_0, Q_1, \dots, Q_n\},$$

where the points Q_m *are defined by* (4)*, is a non-degenerate Möbius pair of* n*-simplices.*

Further Results

Under the assumptions of Theorem 1 the following assertions hold:

- For $n = 1$ holds $P_0 = Q_1$ and $P_1 = Q_0$, otherwise n-simplices $\mathcal{P}$ and $\mathcal{Q}$ have no points in common.
- For $n \geq 3$ the n-simplices $\mathcal{P}$ and $\mathcal{Q}$ are in perspective from a point if, and only if, F is a field of characteristic two.
- Any choice of an even number of points from $\mathcal{P}$ gives rise to a nested Möbius pair. It shares, *mutatis mutandis* the properties of $\mathcal{P}$ and $\mathcal{Q}$. This gives an interpretation for all the 2^n points from (3).

Part 3

Pauli Operators

In the third part it will be sketched—in terms of one example only—how to apply our geometric results to get rather peculiar systems of commuting / non-commuting Pauli operators.

The Pauli Group

We consider the complex matrices

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

The sixteen matrices

$$i^\alpha \sigma_\beta \qquad \text{with } i := \sqrt{-1}, \; \alpha \in \{0,1,2,3\}, \; \text{and } \beta \in \{0,x,y,z\}$$

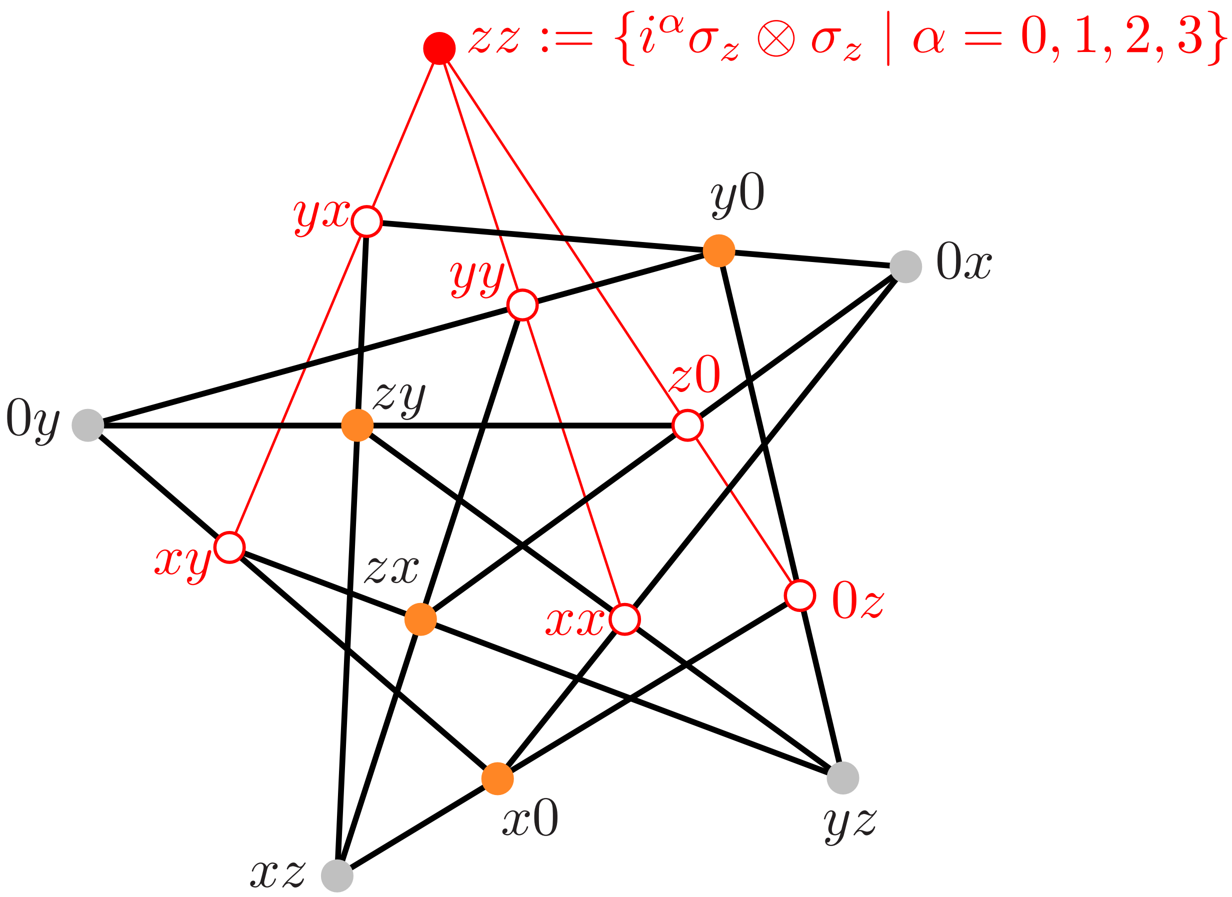
constitute the *Pauli group* P. It acts on the two-dimensional Hilbert space of a single quantum bit (qubit).

Symplectic Geometry

Let $G = P \otimes_{\mathbb{C}} P$ be the the Kronecker product of the Pauli group with itself. This group acts on the four-dimensional Hilbert space of two qubits.

- $\#G = 64$.
- Centre of G: $Z(G) = \{i^\alpha \sigma_0 \otimes \sigma_0 \mid \alpha = 0, 1, 2, 3\}$, $\#Z(G) = 4$.
- $G/Z(G)$ can be viewed as a 4-dimensional vector space V over $\mathrm{GF}(2)$ which is equipped with a symplectic bilinear form.
- Commutation in G is equivalent to (symplectic) perpendicularity in V.

The Projective Point of View



Here the Cremona-Richmond configuration is used to depict the three-dimensional symplectic polar space over $\mathrm{GF}(2)$ (points and null lines only), a Möbius pair, all centres of perspectivity, and the corresponding cosets of operators from P (using shorthand notation).

References

For further details and references see:

H. Havlicek, B. Odehnal, and M. Saniga. Möbius pairs of simplices and commuting Pauli operators. *Math. Pannonica* **21** (2010), 115–128.