A note on Segre varieties in characteristic two

Hans Havlicek

TECHNISCHE UNIVERSITÄT WIEN
Vienna University of Technology

Research Group
Differential Geometry and Geometric Structures
Institute of Discrete Mathematics and Geometry

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Boris Odehnal (Vienna) and Metod Saniga (Tatranská Lomnica)

Our Segre varieties

Let $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_m$ be $m \geq 1$ two-dimensional vector spaces over a commutative field F.

$\mathbb{P}(\boldsymbol{V}_k) = \mathrm{PG}(1, F)$ are projective lines over F for $k \in \{1, 2, \ldots, m\}$.

The non-zero decomposable tensors of $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ determine the *Segre variety*

$$\mathcal{S}_{\underbrace{1,1,\ldots,1}_{m}}(F) = \mathcal{S}_{(m)}(F) = \left\{ F\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \cdots \otimes \boldsymbol{a}_m \mid \boldsymbol{a}_k \in \boldsymbol{V}_k \setminus \{0\} \right\}$$

with ambient projective space $\mathbb{P}\left(\bigotimes_{k=1}^{m} \boldsymbol{V}_k\right) = \mathrm{PG}(2^m - 1, F)$.

Bases

Given a basis $(\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)})$ for each vector space $\boldsymbol{V}_k$, $k \in \{1, 2, \ldots, m\}$, the tensors

$$\boldsymbol{E}_{i_1, i_2, \ldots, i_m} := \boldsymbol{e}_{i_1}^{(1)} \otimes \boldsymbol{e}_{i_2}^{(2)} \otimes \cdots \otimes \boldsymbol{e}_{i_m}^{(m)} \quad \text{with} \quad (i_1, i_2, \ldots, i_m) \in I_m := \{0, 1\}^m \tag{1}$$

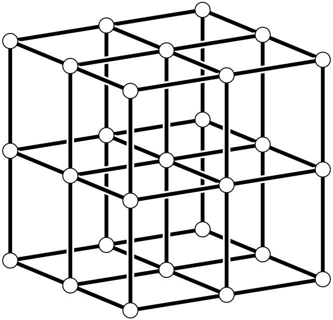
constitute a basis of $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$.

For any multi-index $\boldsymbol{i} = (i_1, i_2, \ldots, i_m) \in I_m$ the *opposite* multi-index $\boldsymbol{i}' \in I_m$ is characterised by

$$i_k \neq i'_k \quad \text{for all} \quad k \in \{1, 2, \ldots, m\}.$$

Examples

- $\mathcal{S}_1(F) = \mathrm{PG}(1, F)$.
- $\mathcal{S}_{1,1}(F)$ is a hyperbolic quadric of $\mathrm{PG}(3, F)$.
- $\mathcal{S}_{1,1,1}(2)$ has 27 points and contains precisely 27 lines (three through each point). The ambient $\mathrm{PG}(7, 2)$ has 255 points.



Collineations

The subgroup of $\mathrm{GL}\left(\bigotimes_{k=1}^{m} \boldsymbol{V}_k\right)$ preserving decomposable tensors is generated by the following transformations:

$$f_1 \otimes f_2 \otimes \cdots \otimes f_m \quad \text{with} \quad f_k \in \mathrm{GL}(\boldsymbol{V}_k) \text{ for } k \in \{1, 2, \ldots, m\}. \qquad (2)$$

$$f_\sigma \quad \text{with} \quad \boldsymbol{E}_{(i_1, i_2, \ldots, i_m)} \mapsto \boldsymbol{E}_{(i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \ldots, i_{\sigma^{-1}(m)})} \quad \text{for all} \quad \boldsymbol{i} \in I_m, \qquad (3)$$

where $\sigma \in S_m$ is arbitrary.

This subgroup induces the stabiliser $G_{\mathcal{S}_{(m)}(F)}$ of the Segre $\mathcal{S}_{(m)}(F)$ within the projective group $\mathrm{PGL}\left(\bigotimes_{k=1}^{m} \boldsymbol{V}_k\right)$.

Bilinear forms

Each of the vector spaces $\boldsymbol{V}_k$ admits a symplectic bilinear form

$$[\cdot,\cdot] : \boldsymbol{V}_k \times \boldsymbol{V}_k \to F.$$

Consequently, $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ is equipped with a bilinear form which is given by

$$\left[\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \cdots \otimes \boldsymbol{a}_m, \boldsymbol{b}_1 \otimes \boldsymbol{b}_2 \otimes \cdots \otimes \boldsymbol{b}_m\right] := \prod_{k=1}^{m} [\boldsymbol{a}_k, \boldsymbol{b}_k] \quad \text{for } \boldsymbol{a}_k, \boldsymbol{b}_k \in \boldsymbol{V}_k, \tag{4}$$

and extending bilinearly.

All these bilinear forms are unique up to a non-zero factor in F.

Bilinear forms (cont.)

Given $\boldsymbol{i}, \boldsymbol{j} \in I_m$ we have

$$
\begin{aligned}
[\boldsymbol{E_i}, \boldsymbol{E_{i'}}] &= \prod_{k=1}^{m} [\boldsymbol{e}_{i_k}^{(k)}, \boldsymbol{e}_{i'_k}^{(k)}] = (-1)^m [\boldsymbol{E_{i'}}, \boldsymbol{E_i}] \neq 0, && (5) \\
[\boldsymbol{E_i}, \boldsymbol{E_j}] &= 0 \quad \text{for all} \quad \boldsymbol{j} \neq \boldsymbol{i}'. && (6)
\end{aligned}
$$

Hence the form $[\cdot,\cdot]$ on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ is non-degenerate. Furthermore, it is

- symmetric when m is even and $\mathrm{Char}\, F \neq 2$;
- alternating otherwise (*i. e.*, when m is odd or $\mathrm{Char}\, F = 2$).

The fundamental polarity

In projective terms the form $[\cdot,\cdot]$ on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ (or any proportional one) determines the fundamental polarity of the Segre $\mathcal{S}_{(m)}(F)$, *i. e.*, a polarity of $\mathbb{P}(\bigotimes_{k=1}^{m} \boldsymbol{V}_k)$ which sends $\mathcal{S}_{(m)}(F)$ to its dual.

This polarity is

- associated with a regular quadric when m is even and $\operatorname{Char} F \neq 2$;
- null otherwise (*i. e.*, when m is odd or $\operatorname{Char} F = 2$).

The associated quadric

Let m be even and $\mathrm{Char}\, F \neq 2$.

The mapping

$$Q : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F : \boldsymbol{X} \mapsto [\boldsymbol{X}, \boldsymbol{X}]$$

is a quadratic form with Witt index 2^{m-1} and rank 2^m.

The fundamental polarity of the Segre $\mathcal{S}_{(m)}(F)$ is the polarity of the regular quadric given by Q.

The Segre coincides with this quadric precisely when $m = 2$.

Characteristic two

Let $\operatorname{Char} F = 2$.

Here $[\cdot,\cdot]$ is a symplectic bilinear form on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ for all $m \geq 1$, whence the fundamental polarity of the Segre $\mathcal{S}_{(m)}(F)$ is always null.

Furthermore, (5) simplifies to

$$[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] = \prod_{k=1}^{m} [\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] = [\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{E}_{\boldsymbol{i}}] \neq 0. \tag{7}$$

A quadratic form

Proposition

Let $m \geq 2$ and $\mathrm{Char}\, F = 2$. Then there is a unique quadratic form

$$Q : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F$$

satisfying the following two properties:

1. *Q vanishes for all decomposable tensors.*
2. *The symplectic bilinear form*

$$[\cdot,\cdot] : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \times \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F$$

   is the polar form of Q.

Proof

We denote by $I_{m,0}$ the set of all multi-indices $(i_1, i_2, \ldots, i_m) \in I_m$ with $i_1 = 0$.

In terms of our basis (1) a quadratic form is given by

$$Q : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F : \boldsymbol{X} \mapsto \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{X}][\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{X}]}{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}]}. \tag{8}$$

Proof (cont.)

Given an arbitrary decomposable tensor we have

$$
\begin{aligned}
Q(\boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m) &= \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m][\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m]}{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}]} \\
&= \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\mathbf{e}_0^{(1)}, \boldsymbol{a}_1][\mathbf{e}_1^{(1)}, \boldsymbol{a}_1] \cdots [\mathbf{e}_0^{(m)}, \boldsymbol{a}_m][\mathbf{e}_1^{(m)}, \boldsymbol{a}_m]}{[\mathbf{e}_0^{(1)}, \mathbf{e}_1^{(1)}] \cdots [\mathbf{e}_0^{(m)}, \mathbf{e}_1^{(m)}]} \\
&= 2^{m-1} \frac{[\mathbf{e}_0^{(1)}, \boldsymbol{a}_1][\mathbf{e}_1^{(1)}, \boldsymbol{a}_1] \cdots [\mathbf{e}_0^{(m)}, \boldsymbol{a}_m][\mathbf{e}_1^{(m)}, \boldsymbol{a}_m]}{[\mathbf{e}_0^{(1)}, \mathbf{e}_1^{(1)}] \cdots [\mathbf{e}_0^{(m)}, \mathbf{e}_1^{(m)}]} \\
&= 0,
\end{aligned}
$$

where we used (7), $\# I_{m,0} = 2^{m-1}$, $m \geq 2$, and $\operatorname{Char} F = 2$. This verifies property 1.

Proof (cont.)

Let $\boldsymbol{j}, \boldsymbol{k} \in I$ be arbitrary multi-indices. Polarising Q gives

$$\begin{aligned} Q(\boldsymbol{E_j}+\boldsymbol{E_k})+Q(\boldsymbol{E_j})+Q(\boldsymbol{E_k}) &= Q(\boldsymbol{E_j}+\boldsymbol{E_k})+0+0 \\ &= \sum_{\boldsymbol{i}\in I_{m,0}} \frac{[\boldsymbol{E_i},\boldsymbol{E_j}+\boldsymbol{E_k}][\boldsymbol{E_{i'}},\boldsymbol{E_j}+\boldsymbol{E_k}]}{[\boldsymbol{E_i},\boldsymbol{E_{i'}}]}. \end{aligned}$$

The numerator of a summand of the above sum can only be different from zero if

$$\boldsymbol{i} \in \{\boldsymbol{j}',\boldsymbol{k}'\} \text{ and } \boldsymbol{i}' \in \{\boldsymbol{j}',\boldsymbol{k}'\}.$$

These conditions can only be met for $\boldsymbol{k} = \boldsymbol{j}'$, whence in fact at most one summand, namely the one with $\boldsymbol{i} \in \{\boldsymbol{j},\boldsymbol{j}'\} \cap I_{m,0}$ can be non-zero.

Proof (cont.)

So

$$Q(\boldsymbol{E_j} + \boldsymbol{E_k}) + Q(\boldsymbol{E_j}) + Q(\boldsymbol{E_k}) = 0 = [\boldsymbol{E_j}, \boldsymbol{E_k}] \quad \text{for} \quad \boldsymbol{k} \neq \boldsymbol{j}'.$$

Irrespective of whether $\boldsymbol{i} = \boldsymbol{j}$ or $\boldsymbol{i} = \boldsymbol{j}'$, we have

$$Q(\boldsymbol{E_j}+\boldsymbol{E_{j'}})+Q(\boldsymbol{E_j})+Q(\boldsymbol{E_{j'}}) = \frac{[\boldsymbol{E_j}, \boldsymbol{E_j} + \boldsymbol{E_{j'}}][\boldsymbol{E_{j'}}, \boldsymbol{E_j} + \boldsymbol{E_{j'}}]}{[\boldsymbol{E_j}, \boldsymbol{E_{j'}}]} = [\boldsymbol{E_j}, \boldsymbol{E_{j'}}].$$

This implies that the quadratic form Q polarises to $[\cdot, \cdot]$, *i. e.*, also the second property is satisfied.

Proof (cont.)

Let $\widetilde{Q}$ be a quadratic form satisfying properties 1 and 2. Hence the polar form of $Q - \widetilde{Q} = Q + \widetilde{Q}$ is zero.

We consider F as a vector space over its subfield $F^\square$ comprising all squares in F. So

$$(Q + \widetilde{Q}) : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F$$

is a semilinear mapping with respect to the field isomorphism $F \to F^\square : x \mapsto x^2$.

The kernel of $Q + \widetilde{Q}$ is a subspace of $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ which contains all decomposable tensors and, in particular, our basis (1). Hence $Q + \widetilde{Q}$ vanishes on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$, and $Q = \widetilde{Q}$ as required. $\square$

Explicit equation

From (8) and (7), the quadratic form Q can be written in terms of tensor coordinates $x_{\boldsymbol{j}} \in F$ as

$$Q\Big(\sum_{\boldsymbol{j}\in I_m} x_{\boldsymbol{j}}\boldsymbol{E}_{\boldsymbol{j}}\Big) = \sum_{\boldsymbol{i}\in I_{m,0}} [\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] x_{\boldsymbol{i}} x_{\boldsymbol{i}'} = \prod_{k=1}^{m} [\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] \cdot \sum_{\boldsymbol{i}\in I_{m,0}} x_{\boldsymbol{i}} x_{\boldsymbol{i}'}. \tag{9}$$

Remarks

The previous results may be slightly simplified by taking symplectic bases, *i. e.*,

$$[\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] = 1 \text{ for all } k \in \{1, 2, \ldots, m\},$$

whence also

$$[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] = 1 \text{ for all } \boldsymbol{i} \in I_m.$$

Proposition 1 fails to hold for $m = 1$: A quadratic form Q vanishing for all decomposable tensors of $\boldsymbol{V}_1$ is necessarily zero, since any element of $\boldsymbol{V}_1$ is decomposable. Hence the polar form of such a Q cannot be non-degenerate.

Main result

Theorem

Let $m \geq 2$ and $\mathrm{Char}\, F = 2$. There exists in the ambient space of the Segre $\mathcal{S}_{(m)}(F)$ a regular quadric $\mathcal{Q}(F)$ with the following properties:

1. *The projective index of $\mathcal{Q}(F)$ is $2^{m-1} - 1$.*
2. *$\mathcal{Q}(F)$ is invariant under the group of projective collineations stabilising the Segre $\mathcal{S}_{(m)}(F)$.*

Proof

Any $f_k \in \mathrm{GL}(\boldsymbol{V}_k)$, $k \in \{1, 2, \dots, m\}$, preserves the symplectic form $[\cdot, \cdot]$ on $\boldsymbol{V}_k$ up to a non-zero factor.

Any linear bijection f_σ as in (3) is a symplectic transformation of $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$.

Hence any transformation from the stabiliser group $G_{\mathcal{S}_{(m)}(F)}$ preserves the symplectic form (4) up to a non-zero factor.

By the proposition, also Q is invariant up to a non-zero factor under the action of $G_{\mathcal{S}_{(m)}(F)}$.

Proof (cont.)

From (9) the linear span of the tensors $\boldsymbol{E}_{\boldsymbol{j}}$ with $\boldsymbol{j}$ ranging in $I_{m,0}$ is a singular subspace with respect to Q.

So the Witt index of Q equals $\#I_{m,0} = 2^{m-1}$, because $[\cdot,\cdot]$ being non-degenerate implies that a greater value is impossible.

We conclude that the quadric with equation $Q(\boldsymbol{X}) = 0$ has all the required properties. $\square$

Conclusion

We call $\mathcal{Q}(F)$ the *invariant quadric* of the Segre $\mathcal{S}_{(m)}(F)$.

The case $m = 2$ deserves special mention, as the Segre $\mathcal{S}_{1,1}(F)$ coincides with its invariant quadric $\mathcal{Q}(F)$ given by

$$Q\Big(\sum_{\boldsymbol{j} \in I_2} x_{\boldsymbol{j}} \boldsymbol{E}_{\boldsymbol{j}}\Big) = x_{00}x_{11} + x_{01}x_{10} = 0.$$

This result parallels the situation for $\operatorname{Char} F \neq 2$.

Problem: Is there a "better" definition of the quadratic form Q?

References

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