Projective Ring Lines and Their Generalisations

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Our Rings

All our rings are associative, with a unit element $1 \neq 0$ which is preserved by homomorphisms, inherited by subrings, and acts unitally on modules. The group of units (invertible elements) of a ring R is denoted by R^*.

The Projective Line over a Ring

Let R be a ring. We consider the free left R-module R^2.

- A pair $(a, b) \in R^2$ is called *admissible* if (a, b) is the first row of a matrix in $\mathrm{GL}_2(R)$.
  This is equivalent to saying that there exists $(c, d) \in R^2$ such that $(a, b), (c, d)$ is a basis of R^2.
- *Projective line* over R (X. Hubaut [30]):

  $$\mathbb{P}(R) := \{R(a, b) \mid (a, b) \text{ admissible}\}$$

  The elements of $\mathbb{P}(R)$ are called *points*.
- Two admissible pairs generate the same point if, and only if, they are left proportional by a unit in R.
- Note that R^2 need not have an invariant basis number: There may also be bases with cardinality $\neq 2$.

The Distant Graph

- *Distant* points of $\mathbb{P}(R)$:

$$R(a,b) \vartriangle R(c,d) \quad :\Leftrightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

- $(\mathbb{P}(R), \vartriangle)$ is called the *distant graph* of $\mathbb{P}(R)$.
- Non-distant points are also called *neighbouring*.
- The relation $\vartriangle$ is invariant under the action of $\mathrm{GL}_2(R)$ on $\mathbb{P}(R)$.
- The group $\mathrm{GL}_2(R)$ acts transitively on the triples of mutually distant points of $\mathbb{P}(R)$.

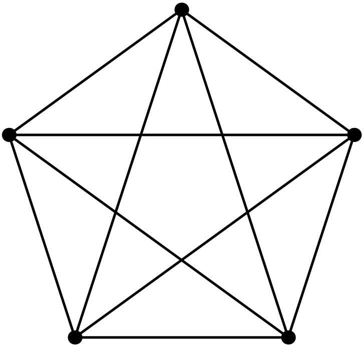
A. Blunck, A. Herzer: *Kettengeometrien* [12].
A. Herzer: *Chain Geometries* [25].

Examples: Rings with Four Elements

Ring

- $R = \mathrm{GF}(4)$ (Galois field).
- $R = \mathbb{Z}_2 \times \mathbb{Z}_2$.
- $R = \mathbb{Z}_4$.
- $R = \mathbb{Z}_2[\varepsilon]$, $\varepsilon^2 = 0$ (dual numbers over $\mathbb{Z}_2$).

Distant graph



$\#\mathbb{P}(R) = 5$

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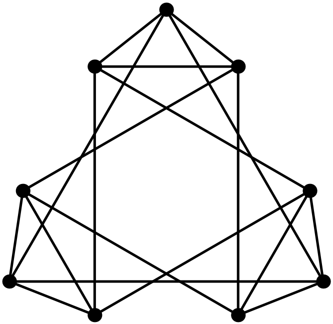
Distant graph



$\#\mathbb{P}(R) = 9$

Examples: Rings with Four Elements

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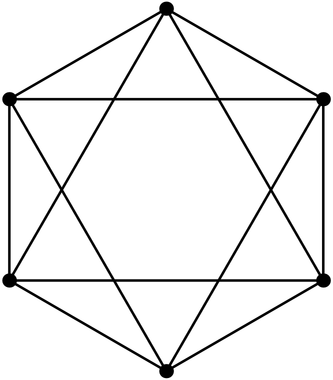
Distant graph



$\#\mathbb{P}(R) = 6$

Examples: Rings with Four Elements

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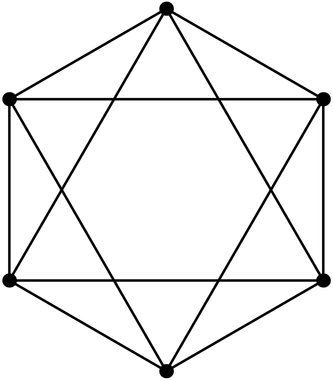
Distant graph



$\#\mathbb{P}(R) = 6$

The Elementary Linear Group $\mathrm{E}_2(R)$

All elementary 2×2 matrices over R, i. e., matrices of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \text{with} \quad t \in R,$$

generate the *elementary linear group* $\mathrm{E}_2(R)$. The group $\mathrm{GE}_2(R)$ is the subgroup of $\mathrm{GL}_2(R)$ generated by $\mathrm{E}_2(R)$ and all invertible diagonal matrices.

Lemma (P. M. Cohn [17])

A 2×2 matrix over R is in $\mathrm{E}_2(R)$ if, and only if, it can be written as a finite product of matrices

$$E(t) := \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} \quad \textit{with} \quad t \in R.$$

Connectedness

Theorem (A. Blunck, H. H. [8])

Let R be any ring.

- $(\mathbb{P}(R), \triangle)$ *is connected precisely when* $\mathrm{GL}_2(R) = \mathrm{GE}_2(R)$.
- *A point* $p \in \mathbb{P}(R)$ *is in the connected component of* $R(1,0)$ *if, and only if, it can be written as* $R(a,b)$ *with*

$$(a,b) = (1,0)\cdot E(t_n)\cdot E(t_{n-1})\cdots E(t_1).$$

for some $n \in \mathbb{N}$ *and some* $t_1, t_2, \ldots, t_n \in R$.

See A. Blunck [6] and [7] for the orbit of $R(1,0)$ under certain subgroups of $\mathrm{GL}_2(R)$.

Connectedness (cont.)

The formula

$$(a, b) = (1, 0) \cdot E(t_n) \cdot E(t_{n-1}) \cdots E(t_1)$$

reads explicitly as follows:

$$\begin{aligned} n = 0: \quad (a, b) &= (1, 0) \\ n = 1: \quad (a, b) &= (t_1, 1) \\ n = 2: \quad (a, b) &= (t_2 t_1 - 1, t_2) \quad \text{(Cf. C. Bartolone [1]).} \\ n = 3: \quad (a, b) &= (t_3 t_2 t_1 - t_3 - t_1, t_3 t_2 - 1) \\ &\ \vdots \end{aligned}$$

Recursive formulas for the entries of $E(t_n) \cdot E(t_{n-1}) \cdots E(t_1)$ can be found in A. Blunck, H. H. [9].

Stable Rank 2

A ring has *stable rank 2* (or: stable range 1) if for any unimodular pair $(a, b) \in R^2$, i.e., there exist u, v with $au + bv \in R^*$, there is a $c \in R$ with

$$ac + b \in R^*.$$

Surveys by F. Veldkamp [40] and [41].
H. Chen: *Rings Related to Stable Range Conditions* [16].

Examples

Rings of stable rank 2 are ubiquitous:

- local rings;
- matrix rings over fields;
- finite-dimensional algebras over commutative fields;
- finite rings;
- direct products of rings of stable rank 2.

$\mathbb{Z}$ is not of stable rank 2: Indeed, $(5, 7)$ is unimodular, but no number $5c + 7$ is invertible in $\mathbb{Z}$.

Examples

$\mathbb{P}(R)$ is connected if ...

- R is a ring of stable rank 2. Diameter ≤ 2 (C. Bartolone [1]).
- R is the endomorphism ring of an infinite-dimensional vector space. Diameter 3 (A. Blunck, H. H. [8]).
- R is a polynomial ring $F[X]$ over a field F in a central indeterminate X. Diameter ∞ (A. Blunck, H. H. [8]).

However, in $R = F[X_1, X_2, \ldots, X_n]$ with $n \geq 2$ central indeterminates there holds

$$\begin{pmatrix} 1 + X_1X_2 & X_1^2 \\ -X_2^2 & 1 - X_1X_2 \end{pmatrix} \in \mathrm{GL}_2(R) \setminus \mathrm{GE}_2(R)$$

(J. R. Silvester [39]).

Chain Spaces

A *chain space* $\Sigma = (\mathbb{P}, \mathcal{C})$ is an incidence structure (consisting of *points* and *chains*) such that the following axioms hold:

1. Each point is on at least one chain. Each chain contains at least one point.
2. There is a unique chain through any three mutually distant points of $\mathbb{P}$.
   Here two points $p, q \in \mathbb{P}$ are called *distant* (in symbols: $p \vartriangle q$) if they are distinct and on at least one common chain.
3. For each point $p \in \mathbb{P}$ the *residue* $\Sigma_p := (\vartriangle(p), \mathcal{C}_p)$, where

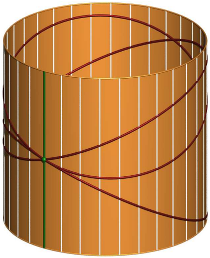
   $$\vartriangle(p) := \{q \in \mathbb{P} \mid q \vartriangle p\} \quad \text{and} \quad \mathcal{C}_p := \{C \setminus \{p\} \mid p \in C \in \mathcal{C}\},$$

   is a *partial affine space*, i.e., an incidence structure resulting from an affine space by removing some (but not all) parallel classes of lines.

Example: The Chain Space on a Cylinder

An elliptic cylinder in the three-dimensional real affine space gives rise to a chain space $\Sigma = (\mathbb{P}, \mathcal{C})$ as follows:

The set $\mathbb{P}$ is the set of points of the cylinder. The set of chains $\mathcal{C}$ is the set of ellipses on the cylinder.



- Two points are distant precisely when they are not on a common generator.
- The point set of any residue Σ_p arises by removing the generator through p from $\mathbb{P}$.
- All residues Σ_p are real affine planes from which precisely one parallel class of lines is removed.

Any projective quadric (up to some degenerate cases) determines a chain space in a similar way.

The Chain Geometry of an Algebra

Let R be an algebra over a commutative field K. By identifying $x \in K$ with $x \cdot 1_R \in R$ we may assume $K \subset R$.

- The injective mapping

$$\mathbb{P}(K) \to \mathbb{P}(R) : K(a, b) \mapsto R(a, b)$$

  is used to identify $\mathbb{P}(K)$ with a subset of $\mathbb{P}(R)$.
- The $\mathrm{GL}_2(R)$ orbit of $\mathbb{P}(K)$ is called the set of *K-chains* in $\mathbb{P}(R)$ and will be denoted by $\mathcal{C}(K, R)$.
- For $K \neq R$ the incidence structure

$$\Sigma(K, R) := (\mathbb{P}(R), \mathcal{C}(K, R))$$

  is the *chain geometry* on (K, R).

Properties of $\Sigma(K, R)$

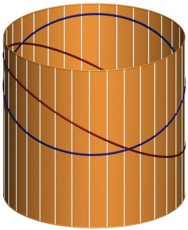
Proposition

- *The chain geometry $\Sigma(K, R)$ is a chain space.*
- *The distant relation of the chain space $\Sigma(K, R)$ coincides with the distant relation of the projective line $\mathbb{P}(R)$.*
- *All residues of $\Sigma(K, R)$ are isomorphic to the partial affine space which arises from the vector space R over K by removing all lines with a non-invertible direction vector.*

A bijective correspondence between R and the residue at $R(1, 0)$ is given by $a \mapsto R(a, 1)$.

W. Benz: *Vorlesungen über Geometrie der Algebren* [2].
A. Herzer: *Chain Geometries* [25].
A. Blunck, A. Herzer: *Kettengeometrien* [12].

Example: The Blaschke Cylinder



The chain space on the cylinder which we exhibited before is actually a model for the chain geometry

$$\Sigma(\mathbb{R}, \mathbb{R}[\varepsilon]),$$

where $\mathbb{R}[\varepsilon]$ denotes the real dual numbers (W. Blaschke [3]).

Example

Let $R = K^{n \times n}$ be the K-algebra of $n \times n$ matrices over a commutative field K. There is the a bijective correspondence:

Chain geometry $\Sigma(K, R)$	Vector space K^{2n}
Point	Subspace with dimension n
Chain	Regulus
$\triangle$	Complementarity relation

Theorem (A. Blunck and H. H. [11])

The K-chains of $\Sigma(K, K^{n \times n})$ are definable in terms of the distant relation of $\mathbb{P}(K^{n \times n})$.

Actually, in [11] a more general result is shown.

Cf. also M. Pankov [38] and Z.-X. Wan [42] for relations with Grassmann spaces and the geometry of matrices.

Subspaces of Chain Spaces

Let $(\mathbb{P}, \mathcal{C})$ be a chain space. Given any subset $\mathbb{S}$ of $\mathbb{P}$ we denote by $\mathcal{C}(\mathbb{S})$ the set of all chains which are entirely contained in $\mathbb{S}$.

The set $\mathbb{S}$ is called a *subspace* of the chain space $(\mathbb{P}, \mathcal{C})$ if it satisfies the following conditions:

1. $\mathbb{S}$ has at least three mutually distant points.
2. For any three mutually distant points of $\mathbb{S}$ the unique chain through them belongs to $\mathcal{C}(S)$.
3. $(\mathbb{S}, \mathcal{C}(\mathbb{S}))$ is a chain space.

Subspaces of $\Sigma(K, R)$

Examples:

- Any connected component of the distant graph on $\mathbb{P}(R)$ is a subspace.
- Let S is a K-subalgebra of R which is *inversion invariant*, i. e., for all $x \in S \cap R^*$ holds $x^{-1} \in S$.
  Then $\mathbb{P}(S)$ (embedded in $\mathbb{P}(R)$) is a subspace.
- There are various "sporadic" examples of subspaces.

Problem

Find all subspaces of a chain geometry $\Sigma(K, R)$ containing $R(1, 0)$, $R(0, 1)$, and $R(1, 1)$ with a neat algebraic description.

Jordan Systems of (K, R)

A *Jordan System* J of (K, R) is K-subspace of R satisfying the following conditions:

1. $1 \in J$.
2. For all $x \in J \cap R^*$ holds $x^{-1} \in J$.

A Jordan system J is called *strong* provided that the following extra condition holds:

3. For all $x \in J$ we have

$$\#(k \in K | x + k \notin R^*) < \#(k \in K | x + k \in R^*).$$

A. Herzer [24], H. J. Kroll [31].

See O. Loos [35] for relations with Jordan algebras and Jordan pairs.

Examples

- Let R be the algebra of $n \times n$ matrices over a commutative field K. Then the subset of symmetric matrices is a Jordan system. It is strong if $\# K > 2n$.
  This may be generalised to Hermitian matrices.
- For commutative algebras (K, R) with $\operatorname{Char} K \neq 2$, any strong Jordan system is necessarily a subalgebra (H. J. Kroll [32], [33]).
- Many examples, even for commutative algebras, can be found in A. Blunck, A. Herzer [12], A. Herzer [26].
- All inversion invariant additive subgroups of a field R were determined by D. Goldstein et al. [19] and A. Mattarei (R commutative) [36].

Properties

An essential tool in the investigation of strong Jordan systems is Hua's identity: Let a, b and $a - b$ be invertible elements of a ring R. Then $a^{-1} - b^{-1}$ is invertible too, and there holds

$$(a^{-1} - b^{-1})^{-1} = a - a(a-b)^{-1}a.$$

Theorem (A. Herzer [24])

Any strong Jordan-System J is closed under the Jordan triple product:

$$xyx \in J \quad \text{for all} \quad x, y \in J.$$

Easy consequences:

- $x^n \in J$ for all $x \in J$ and all $n \in \mathbb{N}$.
- $xy + yx \in J$ for all $x, y \in J$.

The Projective Line over a Strong Jordan System

Let J be a strong Jordan system in R. The *projective line* over J is defined as

$$\mathbb{P}(J) = \{R(t_2 t_1 - 1, t_2) \mid t_1, t_2 \in J\}.$$

Theorem (A. Herzer [24])

The projective line over any strong Jordan-System J in R is a connected subspace of $\Sigma(K, R)$.

Under certain technical conditions the theorem describes all connected subspaces containing $R(1,0)$, $R(0,1)$, and $R(1,1)$ (A. Herzer [24]).

See also A. Blunck [4], H.-J. Kroll [31], [32], [33].

Final Remarks

- Strong Jordan systems of the matrix algebra $R = K^{n\times n}$ (K commutative) yield subsets of Grassmannians which are closed under reguli (A. Herzer [24]).
- Chain spaces on quadrics (with quadratic form Q) can be described algebraically via strong Jordan systems of the Clifford algebra of Q (A. Blunck [5]).

Question

Is it possible to replace the strongness condition for Jordan systems by closedness under triple multiplication without affecting the known results about projective lines?

Cf. [10] for an affirmative answer concerning Hermitian matrices, using results about dual polar spaces (see P. J. Cameron [15]) and matrix spaces (see Z.-X. Wan [42]) rather than ring geometry.

References

The bibliography focusses on the presented material and recent related work.
The books and surveys [2], [12], [20], [25], [29], [41], [42] contain a wealth of further references.

Refer to [13], [14], [18], [21], [22], [23], [34] for deviating definitions of projective lines which we could not present in our lecture.

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