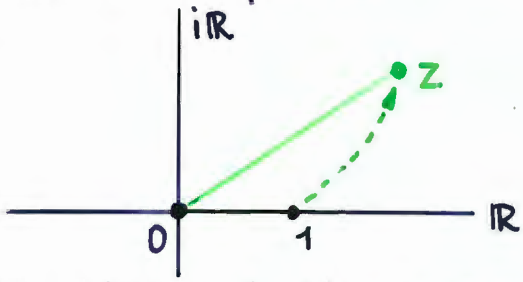
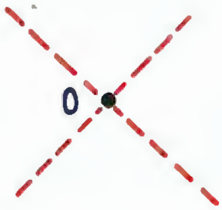
THE CLASSICAL EXAMPLE $\mathbb{R} \subset \mathbb{C}$

Euclidean plane $\xrightarrow{\text{complexification}}$ Minkowskian plane



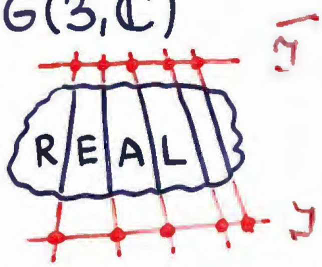
dilative rotations $\Leftrightarrow$ multiplication in $\mathbb{C}^\times$



two isotropic directions $\Leftrightarrow$ absolute circular points.

$PG(1,\mathbb{C}) \xrightarrow{\text{restriction}} PG(3,\mathbb{R}) \xrightarrow{\text{complexif.}} PG(3,\mathbb{C})$

$\mathcal{S}$... regular spread (elliptic linear congruence of lines)



$\subset$ hyperbolic lin. congruence

$\downarrow$ Klein $\qquad \downarrow$ Klein

$PG(5,\mathbb{R}) \xrightarrow{\text{complexif.}} PG(5,\mathbb{C})$

elliptic quadric $= Q \cap \mathcal{H}$

3-space

Klein quadric

$\subset$ hyperbolic quadric

GENERALIZATION

$L|K$ right quadratic field extension

How do things alter?

Case 1: general

Case 2: L is not commutative
K is commutative

Case 3: L is commutative

C. Segre (1891)
E. Study (1923/24)
B. Segre (1964)
R.H. Bruck (1969)
G. Lunardon (1984)
A. Beutelspacher –
J. Ueberberg (1991)
H.H.

QUADRATIC FIELD EXTENSIONS

(P.M. Cohn, 1961)

L/K $\{1, i\}$ .... basis of L over K (right)

$\uparrow$ ANY element of $L\setminus K$

$\forall x \in L$: $x = \alpha + i\beta$, $\alpha, \beta \in K$

$\boxed{i^2 + i\lambda + \mu = 0}$ $\lambda, \mu \in K$, $\mu \neq 0$ ①

$\boxed{\alpha i = i\alpha^S + \alpha^D}$ $\forall \alpha \in K$ ②

$S: K \to K$ injective endomorphism

$D: K \to K$ S-derivation $\left((\alpha\beta)^D = \alpha^D \beta^S + \alpha\beta^D \right)$

Case 1: $\alpha + i\beta \ (\in L) \longrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in K^2$ $(K^2, +, \circ) \cong (L, +, \cdot)$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \circ \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \alpha^D - \mu\beta^S \\ \beta & \alpha^S - \lambda\beta^S + \beta^D \end{pmatrix}}_{=: M_{\alpha\beta}} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \forall \alpha, \beta, \xi, \eta \in K \qquad ③$$

$\forall M \in K^{2\times 2}$: $M = M_{\alpha\beta} \iff \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is eigenvector of M

$\uparrow \in L^2 \supset K^2$

Eigenvalues of $\begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$ with $m_{10} \neq 0$

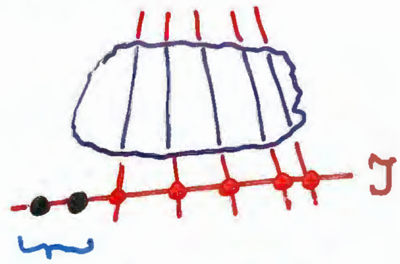
are zeros of

$$m_{00}(m_{10}^{-1} m_{11} m_{10}) - m_{01} m_{10} - (m_{00} + m_{10}^{-1} m_{11} m_{10})x + x^2$$

and vice versa. (Riesinger 1982)

Case 1 : general

$PG(1,L) \longrightarrow PG(3,K) \longrightarrow PG(3,L)$



$\mathcal{S}$... desarguesian spread

c.f. G. Kist – B. Reinmiedl (1991)

no line of the spread through certain points of $\mathcal{J}$ $\Longleftrightarrow$

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s is not surjective $\Longleftrightarrow$

$|L:K|_{LEFT} \neq 2$ $\Longleftrightarrow$

$\mathcal{S}$ is not a dual spread $\Longleftrightarrow$

$PG(3,K)$ is not a BAER-subspace of $PG(3,L)$

algebraically :

$s \in End(K)$ extends to $\hat{s} \in End(L)$

s surjective $\Longleftrightarrow$ $\hat{s}$ is surjective

Case 1: general

(a) There exist a line $\mathcal{I} \subset PG(3,L)$ such that $\mathcal{I} \cap PG(3,K) = \emptyset$.

(b) Any such line $\mathcal{I}$ is an indicator set of a desarguesian spread $\mathcal{S}(\mathcal{I})$ in $PG(3,K)$. There exists an isomorphism

$$\text{kernel of } \mathcal{S}(\mathcal{I}) \longrightarrow L$$

fixing K elementwise.

(c) Let $\mathcal{S}_1$ be a spread of $PG(3,K)$.
The following assertions are equivalent:

(I) $\mathcal{S}_1$ is indicated by a line $\mathcal{I}_1$.

(II) $\exists$ projective collineation of $PG(3,K)$ such that $\mathcal{S}_1 \to \mathcal{S}(\mathcal{I})$

(III) $\exists$ an isomorphism

$$\text{kernel of } \mathcal{S}_1 \longrightarrow \text{kernel of } \mathcal{S}(\mathcal{I})$$

fixing K elementwise

(d) L/K is Galois $\iff$ $\mathcal{S}(\mathcal{I})$ is indicated by at least two distinct lines.

Case 2: L is not commutative, K is commutative

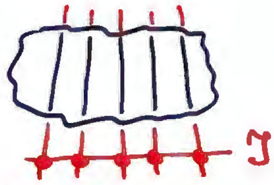
Z ... centre of L $\Rightarrow$ K is <u>maximal</u> commutative

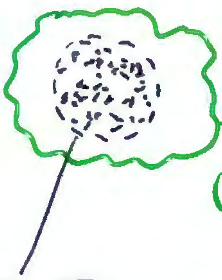
$\Rightarrow Z \subset K \Rightarrow$

$$|L:Z| = 4$$
$$|K:Z| = 2$$
$$|L:K| = 2$$

(left degrees = right degrees)

$$\begin{tikzcd} PG(1,L) \arrow[r] & PG(3,K) \arrow[r] \arrow[d, "\text{Klein}"] & PG(3,L) \\ & PG(5,K) & \end{tikzcd}$$





$\Pi \cong PG(5,Z)$

(Baer subspace)

$Q \cap \Pi$: viewed from Π:
an elliptic quadric
(„4-sphere")

cf. H. Hotje (1974)

however: $\Sigma(Z,L)$

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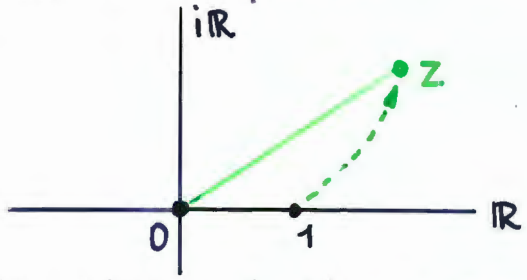
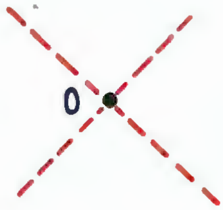
Case 3.1 : L commutative , L|K Galois

Euclidean plane $\xrightarrow{\text{complexification}}$ Minkowskian plane



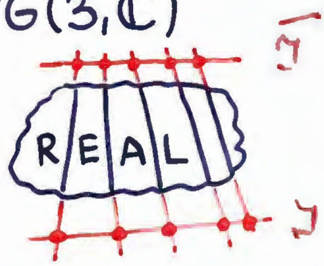
dilative rotations $\Longleftrightarrow$ multiplication in $\mathbb{C}^\times$



two isotropic directions $\Longleftrightarrow$ absolute circular points.

$\mathrm{PG}(1,\mathbb{C}) \xrightarrow{\text{restriction}} \mathrm{PG}(3,\mathbb{R}) \xrightarrow{\text{complexif.}} \mathrm{PG}(3,\mathbb{C})$

$\mathcal{S}$... regular spread (elliptic linear congruence of lines)



$\subset$ hyperbolic lin. congruence

$\downarrow$ Klein $\qquad\qquad \downarrow$ Klein

$\mathrm{PG}(5,\mathbb{R}) \xrightarrow{\text{complexif.}} \mathrm{PG}(5,\mathbb{C})$

elliptic quadric

$= \mathcal{Q} \cap \mathcal{H}$

Klein quadric

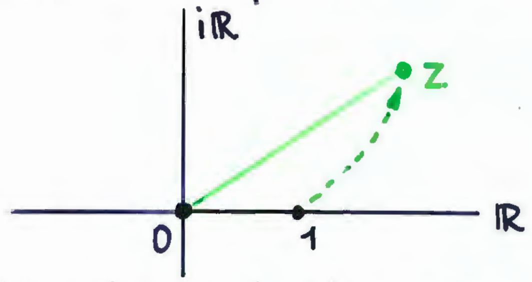
3-space

$\subset$ hyperbolic quadric

~~THE CLASSICAL EXAMPLE $\mathbb{R} \subset \mathbb{C}$~~

Case 3.2.: L commutative, $L|K$ <u>not</u> Galois

Euclidean plane $\xrightarrow{\text{complexification}}$ Galileian ~~Minkowskian~~ plane



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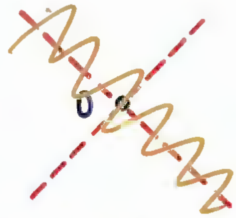
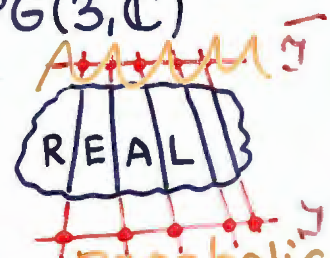


one ~~two~~ isotropic direction~~s~~ $\Longleftrightarrow$ absolute circular point~~s~~.

$PG(1,\mathbb{C}) \xrightarrow{\text{restriction}} PG(3,\mathbb{R}) \xrightarrow{\text{complexif.}} PG(3,\mathbb{C})$

$\mathcal{S}$... regular spread (elliptic linear congruence of lines)



$\subset$ parabolic ~~hyperbolic~~ lin. congruence

$\downarrow$ Klein $\qquad\qquad$ $\downarrow$ Klein

$PG(5,\mathbb{R}) \xrightarrow{\text{complexif.}} PG(5,\mathbb{C})$

elliptic quadric $= Q \cap \mathcal{H}$

Klein quadric ... Q; 3-space ... $\mathcal{H}$

degenerate symplectic polarity

$\exists$ nucleus

~~$\subset$ hyperbolic quadric~~

$\subset$ quadratic cone.

Remark : Case 2 - Case 3

K ... commutative

L ... arbitrary

$\mathcal{S}(\mathcal{I})$ permits a symplectic polarity fixing $\mathcal{S}(\mathcal{I})$ elementwise $\Longleftrightarrow$ L is commutative.