Some Remarks on the Embeddings of Grassmann, Segre, and Veronese

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SEGRE

$X_1, X_2, \dots, X_r$ vector spaces over F, $\dim X_i = n_i + 1 < \infty$

$\mathbb{P}(X_i)$ ... projective spaces

Product space $(\mathcal{P}, \mathcal{L})$

$\mathcal{P} = \mathbb{P}(X_1) \times \mathbb{P}(X_2) \times \dots \times \mathbb{P}(X_r)$ ... points

$$\mathcal{L} = \left.\begin{array}{l} \{ \ell_1 \times \{A_2\} \times \dots \times \{A_r\} \mid \dots \} \cup \\ \{ \{A_1\} \times \ell_2 \times \{A_3\} \times \dots \times \{A_r\} \mid \dots \} \cup \\ \dots \\ \{ \{A_1\} \times \dots \times \{A_{r-1}\} \times \ell_r \mid \dots \} \end{array}\right\} \text{ r types of lines}$$

Segre mapping

$$\sigma : \mathcal{P} \longrightarrow \mathbb{P}\left(\bigotimes_{i=1}^{r} X_i\right)$$

$$(Fa_1, \dots, Fa_r) \longmapsto F(a_1 \otimes \dots \otimes a_r)$$

Parametric representation (coordinates)

$$Y_{i_1, i_2, \dots, i_r} = X^{(1)}_{i_1} \cdot X^{(2)}_{i_2} \cdot \dots \cdot X^{(r)}_{i_r}$$

$\prod_{i=1}^{r} (n_i + 1)$ coordinates

$(\mathcal{P}, \mathcal{L})$ **product** space $(\mathcal{P}', \mathcal{L}')$ projective space

points lines

$\sigma : \mathcal{P} \longrightarrow \mathcal{P}'$ a mapping (**Segre mapping**)

(I) Injectivity ✓

(L) Lines

σ-images of lines are lines

(K) Klein ↯

$\alpha \in Aut(\mathcal{P}) \not\Rightarrow \sigma^{-1}\alpha\sigma$ extends to $\alpha' \in Aut(\mathcal{P}')$

(S) von Staudt ✓

$\beta : \ell_1 \to \ell_2$ projectivity $\Rightarrow \beta^\sigma : \ell_1^\sigma \to \ell_2^\sigma$ projectivity

(D) Dimension

im σ spans a space of dimension $\prod_{i=1}^{r}(n_i+1)-1$

(P) Primes ↯

$\mathcal{H} \subset \mathcal{P}$ prime $\not\Leftrightarrow \mathcal{H}^\sigma$ hyperplane section of im σ

$\ell \subset \mathcal{H}$ or $\#(\ell \cap \mathcal{H}) = 1$ } $\forall$ lines $\ell \in \mathcal{L}$

$\gamma : \mathcal{P} \to \mathcal{P}''$ a mapping , $r=2$

(I) + (L) + (D) ... regular embedding

$\Rightarrow$ $\operatorname{im}\gamma \sim \operatorname{im}\sigma$ (Segre variety)

(I) + (L) ... embedding $\Rightarrow$ decomposition

$$\mathcal{P} \xrightarrow[\text{Aut}]{(\mathrm{id},\alpha)} \mathcal{P} \xrightarrow[\text{Segre}]{\sigma} \mathcal{P}' \xrightarrow[\text{projection}]{\pi} \mathcal{P}_1' \xrightarrow[\text{coll.}]{\varkappa} \mathcal{P}_2'' \subset \mathcal{P}''$$

$\vdots$

(D) $\Leftrightarrow$ $\pi = \mathrm{id}_{\mathcal{P}'}$

Zanella

Examples: Herzer

B. Segre
Krüger
Blunck
H.

} F... skewfield , $n_1 = 1$, $n_2 \leq \infty$

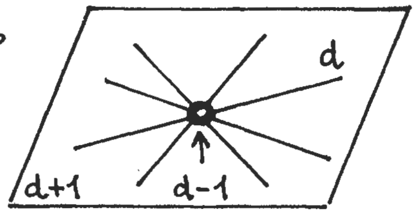
GRASSMANN

$\mathbb{P}(X)$, $\dim X = n+1 < \infty$, $1 \le d \le n-2$

Grassmann space $(\mathcal{P}, \mathcal{L})$

$\mathcal{P}$ ... d-dimensional subspaces of $\mathbb{P}(X)$

$\mathcal{L}$ ... pencils of d-subspaces



Grassmann mapping

$$\sigma : \mathcal{P} \longrightarrow \mathbb{P}(\Lambda^{d+1} X)$$

$$[Fa_0, \dots, Fa_d] \mapsto F(a_0 \wedge a_1 \wedge \dots \wedge a_d)$$

Parametric representation (Grassmann coord.)

$$y_{i_0, i_1, \dots, i_d} = \begin{vmatrix} x^{(0)}_{i_0} & \dots & x^{(0)}_{i_d} \\ \vdots & & \\ x^{(d)}_{i_0} & \dots & x^{(d)}_{i_d} \end{vmatrix} \qquad 0 \le i_0 < i_1 < \dots < i_d \le n$$

$\vdots$

$\binom{n+1}{d+1}$ Coordinates

$(\mathcal{P},\mathcal{L})$ Grassmann space, $(\mathcal{P}',\mathcal{L}')$ projective space

points · lines

$\sigma : \mathcal{P} \to \mathcal{P}'$ a mapping (Grassmann mapping)

(I) Injectivity ✓

(L) Lines

σ-images of lines are Lines

(K) Klein ✓

$\alpha \in \mathrm{Aut}(\mathcal{P}) \Rightarrow \sigma^{-1}\alpha\sigma$ extends to $\alpha' \in \mathrm{Aut}(\mathcal{P}')$

(S) von Staudt ✓

$\beta : \ell_1 \to \ell_2$ projectivity $\Rightarrow \beta^\sigma : \ell_1^\sigma \to \ell_2^\sigma$ projectivity

(D) Dimension

im σ spans a space of dimension $\binom{n+d}{d+1} - 1$

(P) Primes

$\mathcal{H} \subset \mathcal{P}$ prime $\Leftrightarrow \mathcal{H}^\sigma$ hyperplane section of im σ

$\ell \subset \mathcal{H}$ or $\#(\ell \cap \mathcal{H}) = 1$ $\forall$ lines $\ell \in \mathcal{L}$

σ^{-1}: linear complex of d-spaces

$\gamma : \mathcal{P} \to \mathcal{P}''$ a mapping

(I)+(L)+(D) ... regular embedding

$\Rightarrow$ $\operatorname{im} \gamma \sim \operatorname{im} \sigma$ (Grassmann variety)

(I)+(L) ... embedding $\Rightarrow$ decomposition

$$\mathcal{P} \xrightarrow[\text{Grassmann}]{} \mathcal{P}' \xrightarrow[\text{projection}]{\pi} \mathcal{P}_1' \xrightarrow[\text{collineation}]{} \mathcal{P}_2'' \subset \mathcal{P}''$$

H.
Zanella
Wells
Shult
H.
Shult - Hall } F skewfield

VERONESE

$\mathbb{P}(X)$, $\dim X = n+1$, $t \geq 2$

Projective space $\mathcal{P} = \mathbb{P}(X)$
$\mathcal{L}$.. lines

Veronese mapping

$$\sigma : \mathcal{P} \longrightarrow \mathbb{P}(\underbrace{X \otimes \ldots \otimes X}_{t})$$

$$Fa \longmapsto F(a \otimes \ldots \otimes a)$$ ⟸ too many dimensions: $(n+1)^t - 1$

Parametric representation

$$Y_{e_0, e_1, \ldots, e_n} = x_0^{e_0} \cdot x_1^{e_1} \cdot \ldots \cdot x_n^{e_n} \qquad \left(\sum_{i=0}^{n} e_i = t\right)$$

⋮

$\binom{n+t}{t}$ coordinates

Coordinate-free definition

Herzer, Gmainer - H.

$(\mathcal{P}, \mathcal{L})$ projective space, $(\mathcal{P}', \mathcal{L}')$ projective space

($\mathcal{P}$: points, $\mathcal{L}$: lines)

$\sigma : \mathcal{P} \longrightarrow \mathcal{P}'$ a mapping (Veronese mapping)

(I) Injectivity ✓

(L) Lines

σ-images of lines are normal rational curves

(K) Klein ✓

$\alpha \in \mathrm{Aut}(\mathcal{P}) \Rightarrow \sigma^{-1}\alpha\sigma$ extends to $\alpha' \in \mathrm{Aut}(\mathcal{P}')$

(S) von Staudt ✓

$\beta : \ell_1 \to \ell_2$ projectivity $\Rightarrow \beta^\sigma : \ell_1^\sigma \to \ell_2^\sigma$ projectivity

(D) Dimension

im σ spans a space of dimension $\binom{n+t}{t} - 1$

(P) Primes ?

$\mathcal{H} \subset \mathcal{P}$ prime $\Leftrightarrow \mathcal{H}^\sigma$ hyperplane section of im σ

$\mathcal{H} \subset \mathcal{P}$ prime ↓ ?

$\mathcal{H}^\sigma$ ↓ σ^{-1} algebraic hypersurfaces of degree t

$t=2$, $\gamma : \mathcal{P} \to \mathcal{P}''$ a mapping

prime $\Leftrightarrow$ quadric

$\mathcal{M} \subset \mathcal{P}$ ... $\overline{\overline{\mathcal{M}}}$ quadratic closure

(PC) $\overline{\overline{\mathcal{M}}} = \gamma^{-1}(\overline{\gamma(\mathcal{M})}) \quad \forall \mathcal{M} \subset \mathcal{P}$

... quadratic embedding

(PC) $\Rightarrow$ (I)

$\Rightarrow$ (D)

$\not\Rightarrow$ $\operatorname{im} \gamma \sim \operatorname{im} \sigma$

$\Rightarrow$ (L_1) ... are planar arcs.

(L_2) ... are quadratic sets

(PC) +(L_2) ... regular embedding $\Rightarrow$

decomposition

$$\mathcal{P} \xrightarrow[\text{Veronese}]{\sigma} \mathcal{P}' \xrightarrow[\text{collineation}]{} \mathcal{P}''$$

Zanella, H.

Dual Veronese mapping

$$Y^*_{e_0,e_1,\dots,e_n} = \binom{t}{e_0,e_1,\dots,e_n} x_0^{*e_0}\cdot x_1^{*e_1}\cdot\ldots\cdot x_n^{*e_n}$$

$Y^*_{e_0,e_1,\dots,e_n}$: coo. of a hyp. of $\mathcal{P}'$ → osculating hyperplane

$\binom{t}{e_0,e_1,\dots,e_n}$ → $\dfrac{t!}{e_0!\,e_1!\dots e_n!}$

$x_0^{*e_0}\cdot x_1^{*e_1}\cdot\ldots\cdot x_n^{*e_n}$: coo. of a hyperplane of $\mathcal{P}$

$\left(\sum_{i=0}^{n} e_i = t\right)$

Examples $(n=1)$

Conic

$F(x_0,x_1) \mapsto F(x_0^2,\ x_0x_1,\ x_1^2)$ ... points

$F(x_0^*,x_1^*) \mapsto F(x_0^{*2},\ \underline{2}\,x_0x_1,\ x_1^{*2})$ ... tangents

Twisted cubic

$F(x_0,x_1) \mapsto F(x_0^3,\ x_0^2x_1,\ x_0x_1^2,\ x_1^3)$ ... points

$F(x_0^*,x_1^*) \mapsto F(x_0^{*3},\ \underline{3}\,x_0^{*2}x_1^*,\ \underline{3}\,x_0^*x_1^{*2},\ x_1^{*3})$ ... osculating planes

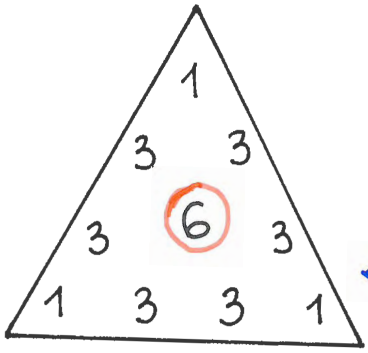
Nucleus ... $\bigcap$ osc. hyperplanes

Gmainer, Gmainer-H.

Characterization : Zanella – H.

Example

$n = 2$, $t = 3$, $\mathrm{Char}\, F = 2$



nucleus : one point

← a slice of Pascal's pyramid

$F(x_0, x_1, x_2) \overset{\sigma}{\mapsto} F(x_0^3, \dots, x_0x_1x_2, \dots, x_2^3)$ … 9-space

$\downarrow \pi$ … projection ; γ

$F(x_0^3, \dots, 0, \dots, x_2^3)$ … 8-space

(I) ✓

(K) ✓

(L) … are twisted cubics

(S) ✓

(D) ↯