The Betten-Walker Spread Revisited

Joint work with Rolf Riesinger (Wien)

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Spreads

Let $\mathbb{P}_3(K)$ be the 3-dimensional projective space over a field K, and let $\mathcal{L}$ be its set of lines.

Definition. Let $\mathcal{S} \subset \mathcal{L}$ be a set of lines satisfying some of the following conditions:

1. Any two distinct lines of $\mathcal{S}$ are skew.
2. Each point is incident with at least one line of $\mathcal{S}$.
3. Each plane is incident with at least one line of $\mathcal{S}$.

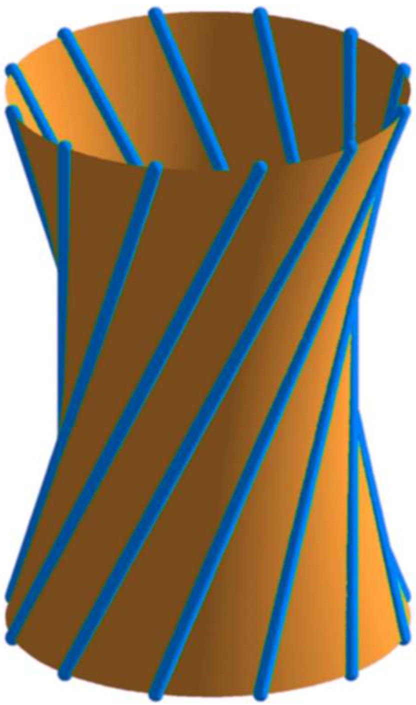
A *partial spread* is characterised by condition 1.

A *spread* is characterised by conditions 1 and 2.

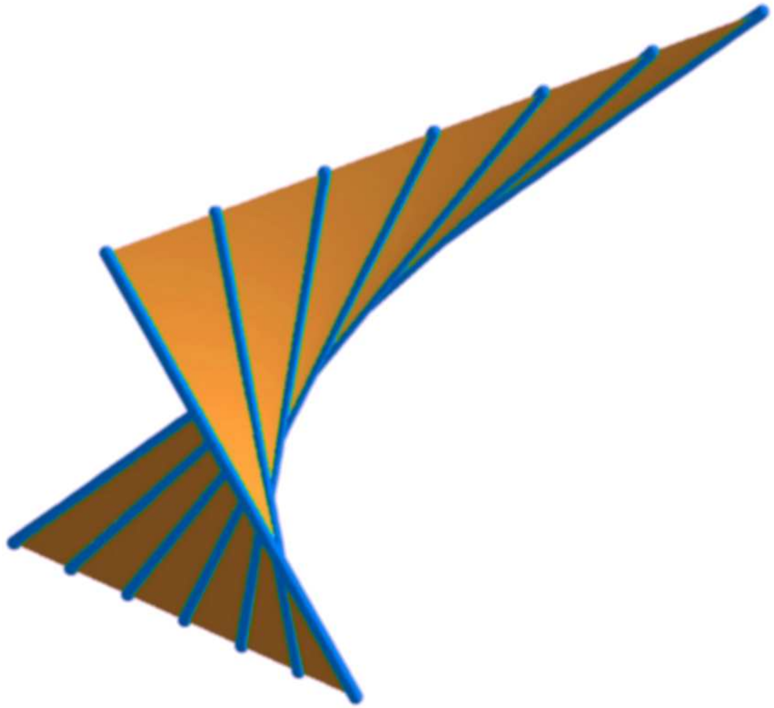
A *dual spread* is characterised by conditions 1 and 3.

Reguli

On each hyperbolic quadric there are two families of generators. Each of them forms a *regulus*. From an affine point of view there are two possibilities for a regulus $\mathcal{R}$:



Hyperboloid:
$\mathcal{R}$ has no line at infinity.



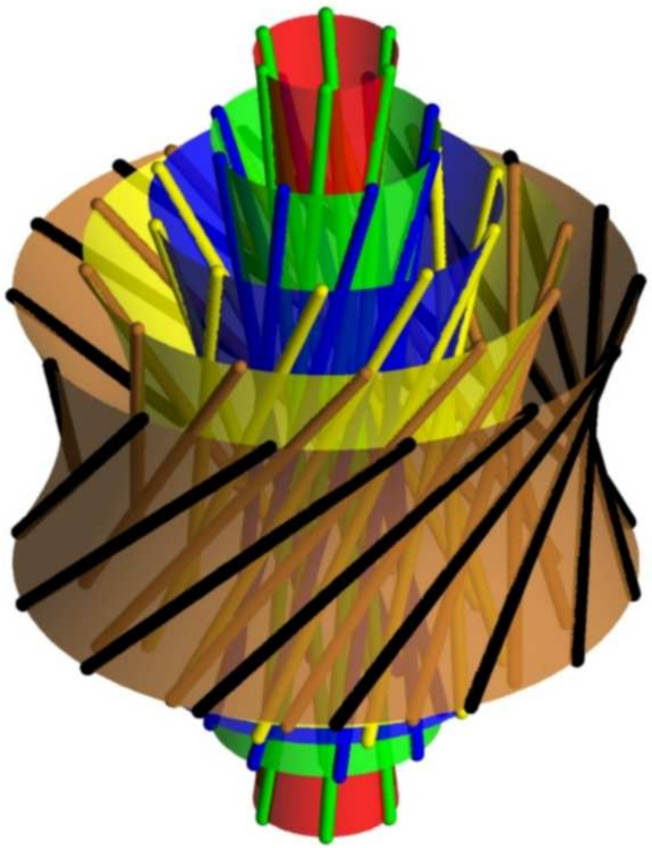
Hyperbolic paraboloid:
$\mathcal{R}$ has precisely one line at infinity.

Extra Conditions

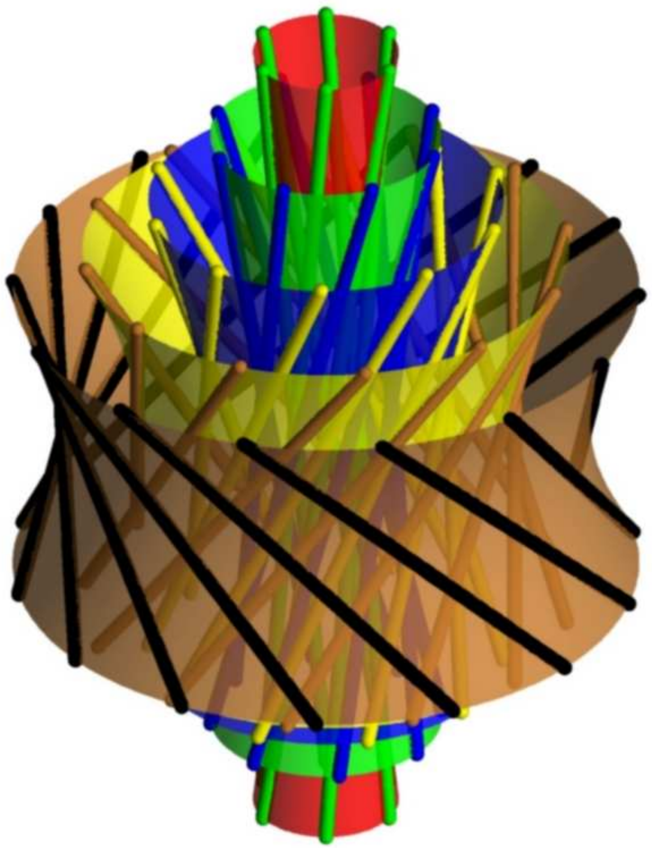
It is possible to construct very bizarre spreads, e. g. by transfinite induction, when K is infinite (M. Bernardi, 1973). Thus little can be said about spreads in general.

- A *regular spread* is closed under reguli.
- A spread is *algebraic* if its image under the Klein mapping is an algebraic variety.

Examples

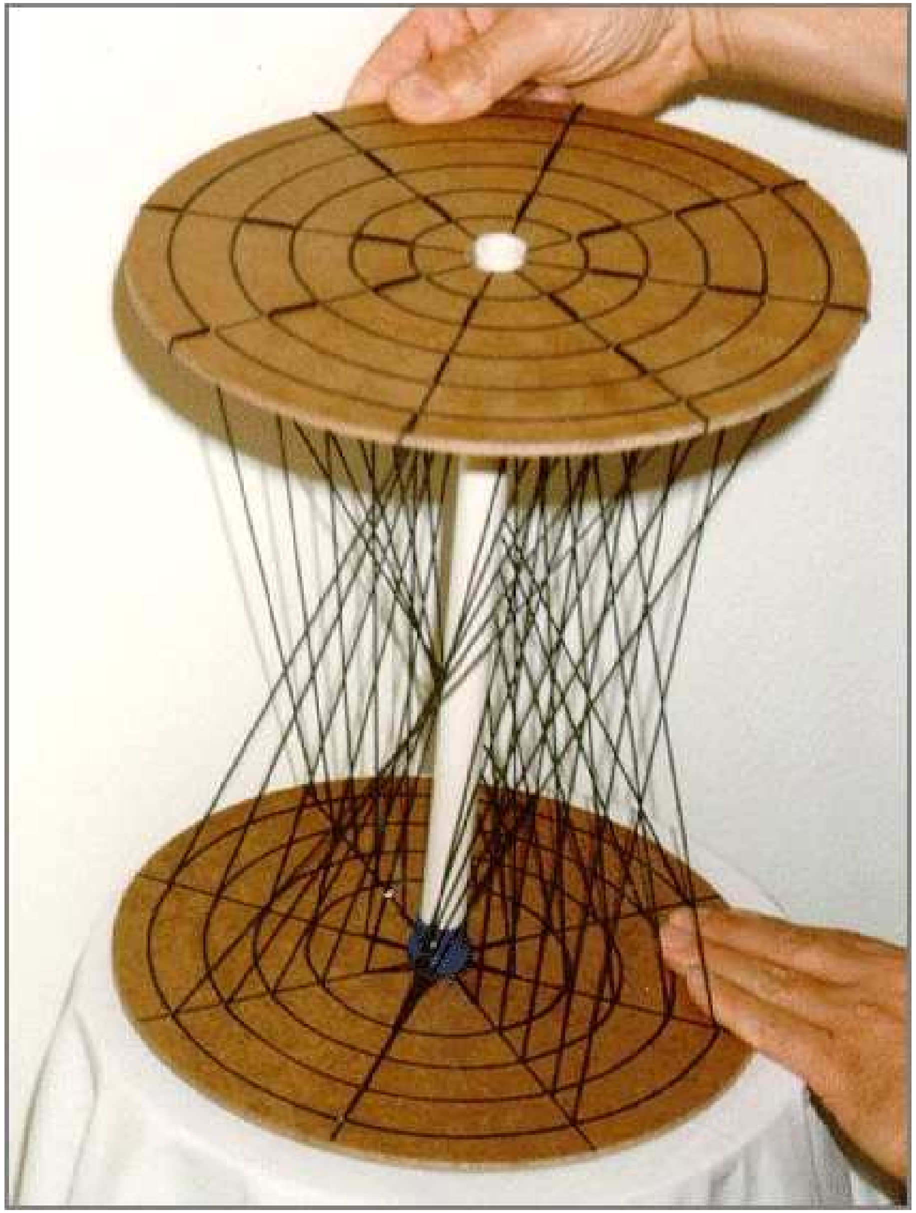


A regular spread is an elliptic linear congruence of lines.



A *subregular spread* arises from a regular spread by replacing "some" reguli with their opposite reguli.

Examples



R. Riesinger built this nice model of a regular spread some years ago.

Regular Spreads

- Regular spreads exist in $\mathbb{P}_3(K)$ if, and only if, K is not quadratically closed.
- In $\mathbb{P}_3(K)$ there is only one regular spread to within projective collineations if, and only if, K admits only one quadratic extension.
- Any regular spread of $\mathbb{P}_3(K)$ is algebraic. It is also a dual spread.
- Regular spreads do not exist in $\mathbb{P}_3(\mathbb{C})$, because $\mathbb{C}$ is quadratically closed.
  (A hyperbolic quadric in $\mathbb{P}_3(\mathbb{C})$ has no exterior lines.)

Applications

Applications of spreads to be found in the literature:

- Foundations of geometry.
  Construction of translation planes. Uses a spread in the hyperplane at infinity of a 4-dimensional affine space . . .
  J. André (1956), R. H. Bruck and R. C. Bose (1963), . . .

- Parallelisms.
  Generalizations of the Clifford-parallelism.
  W. K. Clifford (1873), . . .

- Descriptive geometry, computer vision.
  Non linear mappings on a plane. Parallel projection in 3-dimensional elliptic space. Non-central cameras, . . .
  L. Tuschel (1911), . . .

Cayley's Surface

Cayley's ruled cubic surface or, for short, the *Cayley surface* is, to within projective collineations, the point set

$$F := \mathcal{V}\big(f(\boldsymbol{X})\big) := \big\{K(p_0, p_1, p_2, p_3)^{\mathrm{T}} \in \mathbb{P}_3(K) \mid f(p_0, p_1, p_2, p_3) = 0\big\},$$

where

$$f(\boldsymbol{X}) := X_0 X_1 X_2 - X_1^3 - X_0^2 X_3 \in K[\boldsymbol{X}] = K[X_0, X_1, X_2, X_3].$$

We shall consider $\omega := \mathcal{V}(X_0)$ as plane at infinity. Hence the affine part of the Cayley surface is given by the parametrisation

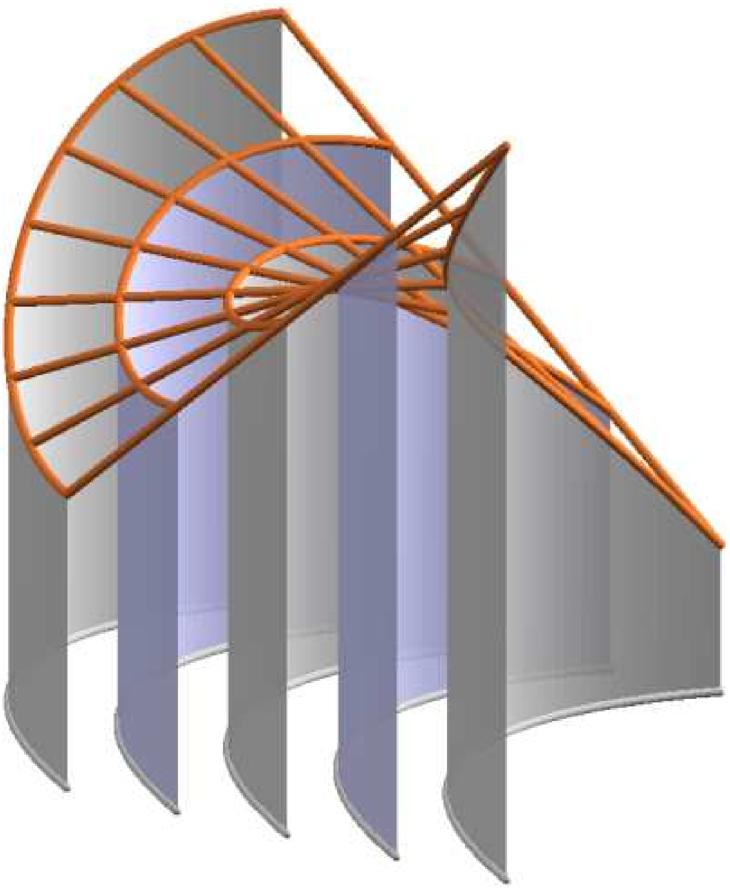
$$K^2 \to \mathbb{P}_3(K) : (u_1, u_2) \mapsto K(1, u_1, u_2, u_1 u_2 - u_1^3)^{\mathrm{T}} =: P(u_1, u_2).$$

The intersection of F with the plane ω is the line

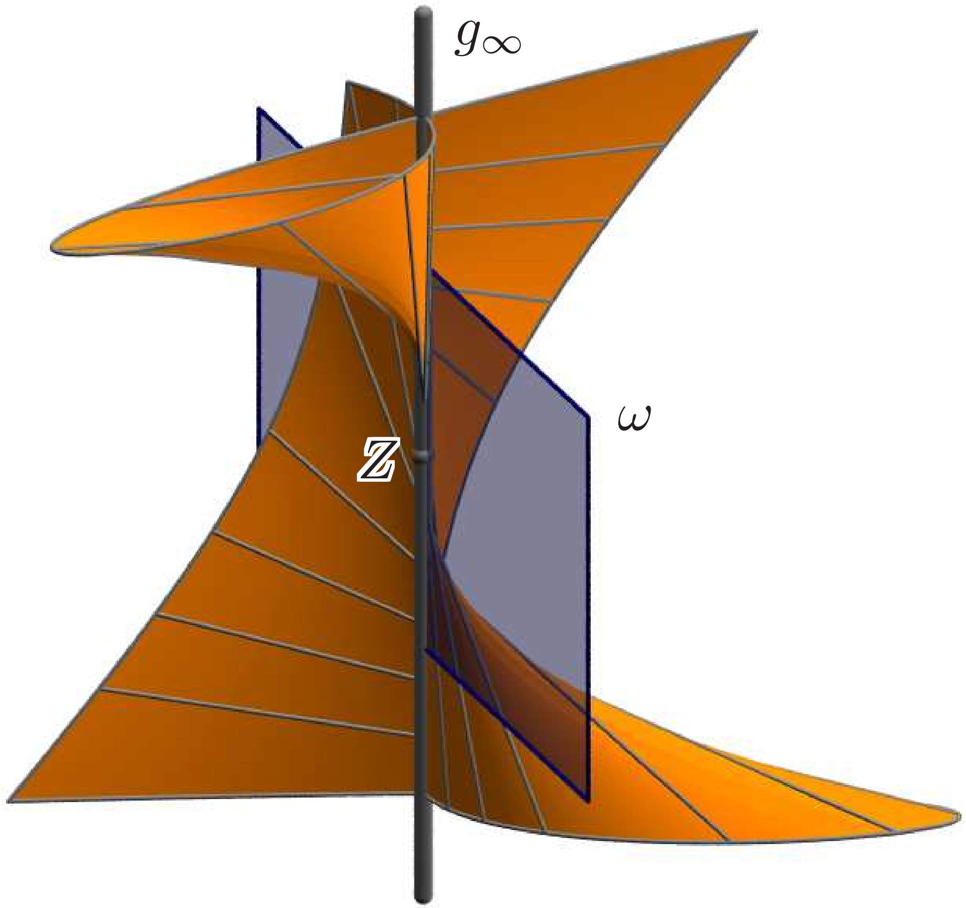
$$\mathcal{V}(X_0, X_1) =: g_\infty.$$

Pictures

All pictures illustrate the case $K = \mathbb{R}$, but in varying affine charts.



Affine point of view.
All points of $F \setminus \omega$ are simple.



Intersection with the plane at infinity.
All points of g_∞ are double points.
$Z := K(0, 0, 0, 1)^{\mathrm{T}}$ is a pinch point.

The Collineation Group

- The set of all matrices

$$M_{a,b,c} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & c & 0 & 0 \\ b & 3\,ac & c^2 & 0 \\ ab - a^3 & bc & ac^2 & c^3 \end{pmatrix}$$

  where $a, b \in K$ and $c \in K \setminus \{0\}$ is a group, say G, under multiplication.

- Each matrix in G leaves invariant the cubic form $f(\boldsymbol{X}) = X_0X_1X_2 - X_1^3 - X_0^2X_3$ to within the factor $c^3 \neq 0$.

- The group G yields all projective automorphic collineations of F unless $\#K \leq 3$.

- Under the action of the group G, the points of F fall into three orbits: $F \setminus \omega$, $g_\infty \setminus \{Z\}$, and $\{Z\}$.

References

M. Chasles

A. Cayley

The name Cayley surface is not completely appropriate.

Michel Chasles published his discovery of that surface in 1861, three years before Arthur Cayley.

There is a widespread literature on the Cayley surface:

H. Brauner (1964, 1966, 1967, 1967),
J. Gmainer and H. H. (2005),
M. Husty (1984),
R. Koch (1968),
H. Neudorfer (1925),
M. Oehler (1969),
A. Wiman (1936),
H. Wresnik (1990),
W. Wunderlich (1935),
and others.

Osculating Tangents

If a line t meets F at a simple point P with multiplicity ≥ 3 then it is called an *osculating tangent* at P. Such a tangent line is either a generator or it meets F at P only. In the latter case it will be called a *proper osculating tangent* of F.

Lemma. *At each point $P(u_1, u_2) \in F \backslash g_\infty$ there is a unique proper osculating tangent, namely the line which joins $P(u_1, u_2)$ with the point $K(0, 1, 3u_1, u_2)^{\mathrm{T}}$.*

Proof. The tangent plane of F at $P(0, 0)$ is $\mathcal{V}(X_3)$; this plane meets F along the line $\mathcal{V}(X_1, X_3)$ and the parabola given by

$$\mathcal{V}\big(X_1(X_0X_2 - X_1^2), X_3\big). \tag{1}$$

The tangent t of this parabola at $P(0, 0)$ is easily seen to be the only proper osculating tangent at $P(0, 0)$. The point at infinity of t is $K(0, 1, 0, 0)^{\mathrm{T}}$. By the action of the matrix $M_{u_1,u_2,1} \in G$ the assertion follows for any point $P(u_1, u_2) \in F \setminus g_\infty$. $\square$

Main Theorem (cont.)

Main Theorem. *The set*

$$\mathcal{O} := \{t \in \mathcal{L} \mid t \text{ is a proper osculating tangent of } F\} \cup \{g_\infty\}$$

has the following properties:

1. *$\mathcal{O}$ is a partial spread of $\mathbb{P}_3(K)$ if, and only if, $\mathrm{Char}\, K \neq 3$ and K does not contain a third root of unity other than 1.*

2. *If $\mathcal{O}$ is a partial spread then it is maximal, i.e., it is not a proper subset of any partial spread of $\mathbb{P}_3(K)$.*

3. *$\mathcal{O}$ is a covering of $\mathbb{P}_3(K)$ if, and only if, $\mathrm{Char}\, K \neq 3$ and each element of K has a third root in K.*

Proof

Ad 1. All proper osculating tangents are skew to g_∞. The osculating tangents at $P(0,0) \neq P(u_1,u_2)$ are skew if, and only if,

$$\Delta(u_1,u_2) := \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & u_1 & 1 \\ 0 & 0 & u_2 & 3u_1 \\ 0 & 0 & u_1u_2 - u_1^3 & u_2 \end{pmatrix} = u_2^2 - 3u_1^2u_2 + 3u_1^4 \neq 0.$$

For $u_1 = 0$ we have $u_2 \neq 0$ so that $\Delta(u_1,u_2) \neq 0$.

For $u_1 \neq 0$ we substitute $u_2 = (2+y)u_1^2$ with $y \in K$ and obtain the equivalent condition $u_1^4(y^2+y+1) \neq 0$. But the polynomial

$$X^2 + X + 1 \in K[X]$$

has a zero in K precisely when:

- $\mathrm{Char}\, K = 3$, since in this case $X^2+X+1 = (X-1)^2$;
- $\mathrm{Char}\, K \neq 3$ and there exists a third root of unity $w \neq 1$ in K.

Proof (cont.)

Ad 2. It suffices to show that each point at infinity is on a line of $\mathcal{O}$.

This is obviously true for any point on $g_\infty \in \mathcal{O}$.

As (u_1, u_2) varies in K^2, the osculating tangent at $P(u_1, u_2)$ contains the point

$$K(0, 1, 3u_1, u_2)^{\mathrm{T}} \in \omega.$$

By part 1, we have $\operatorname{Char} K \neq 3$. This means that each point of $\omega \setminus g_\infty$ is incident with a line of $\mathcal{O}$.

Proof (cont.)

Ad 3.

- Let $\mathrm{Char}\, K = 3$.
  By part 2, $\mathcal{O}$ is not even a covering of the plane at infinity.

- Let $\mathrm{Char}\, K \neq 3$.
  A point $K(1, p_1, p_2, p_3)$ is on a line of $\mathcal{O}$ if, and only if, there is a pair $(u_1, u_2) \in K^2$ and an $s \in K$ such that

  $$(1, p_1, p_2, p_3)^{\mathrm{T}} = (1, u_1, u_2, u_1 u_2 - u_1^3)^{\mathrm{T}} + s(0, 1, 3u_1, u_2)^{\mathrm{T}}.$$

  So we obtain the following system of equations in the unknowns $u_1, u_2, s \in K$:

  $$u_1 = p_1 - s, \quad u_2 = p_2 - 3s(p_1 - s), \quad s^3 = p_3 - (p_1 p_2 - p_1^3).$$

  This system has a solution, because $p_3 - (p_1 p_2 - p_1^3)$ has a third root in K. $\square$

Remarks

The line set $\mathcal{O}$ has the following further properties.

- Let $\operatorname{Char} K \neq 3$.
  $\mathcal{O}$ admits a projective correlation which leaves $\mathcal{O}$ invariant, as a set. Hence $\mathcal{O}$ is a spread if, and only if, it is a dual spread.

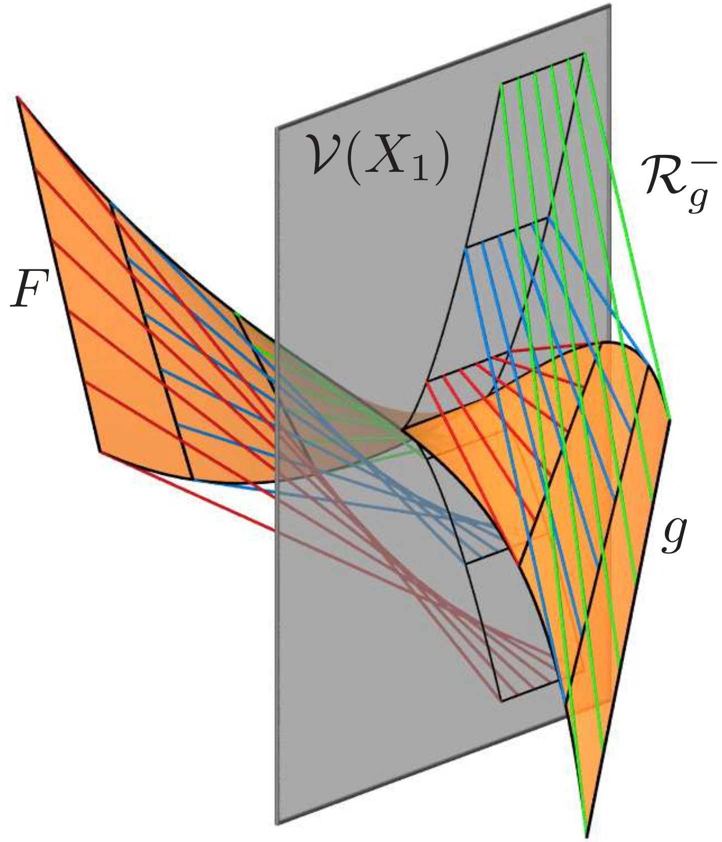
- Let $\operatorname{Char} K = 3$.
  Here $\mathcal{O}$ is part of a parabolic linear congruence. The axis of this congruence is a *line of nuclei* of the Cayley surface.

  (Cf. M. de Finis, M.-J. de Resmini, 1983.)

Covering $\mathcal{O}$ with Reguli

The following assertions can be verified by straightforward calculations:

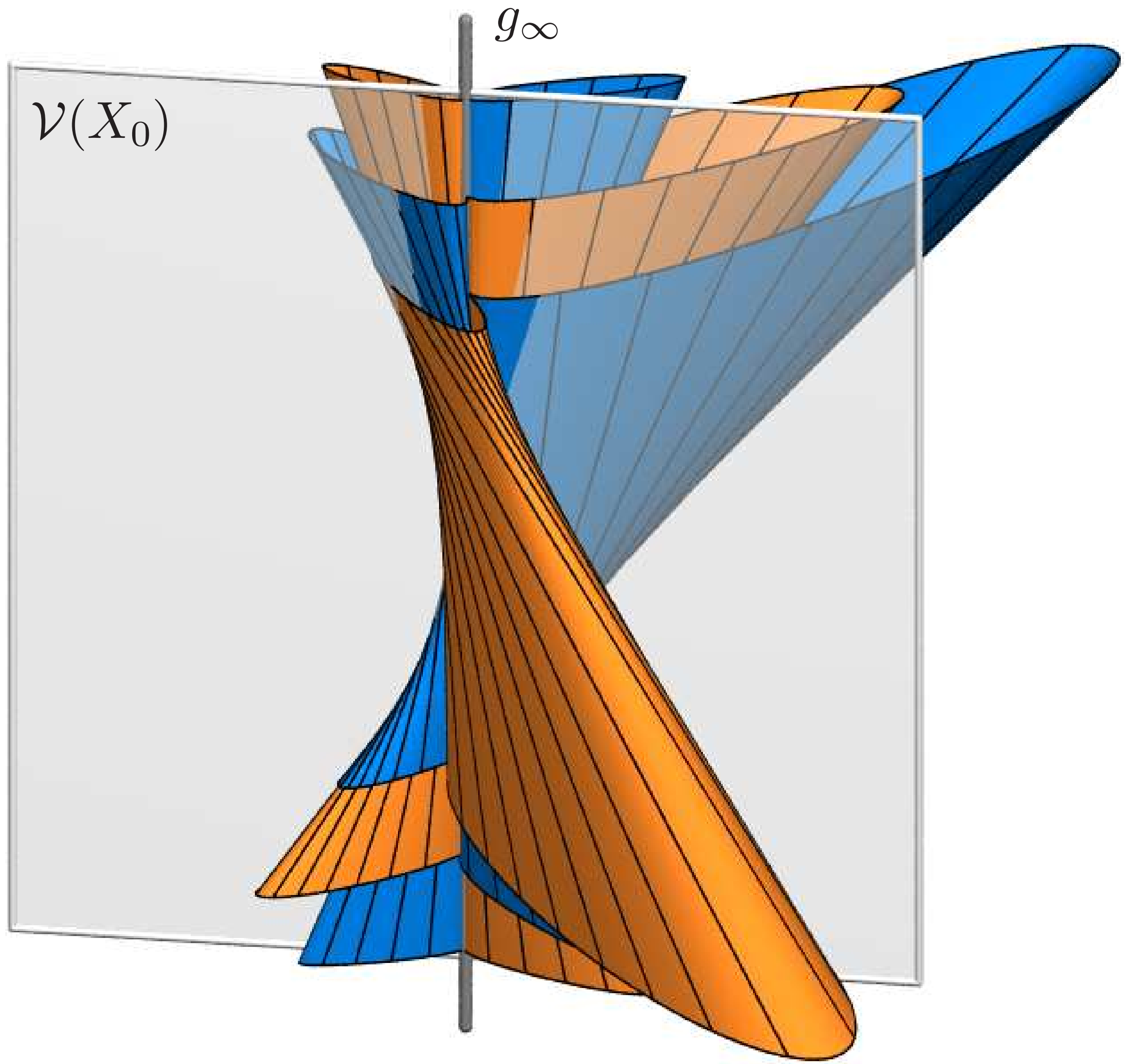
- All osculating tangents at the points of a generator $g \neq g_\infty$ together with g_∞ form a regulus $\mathcal{R}_g^-$, say. In affine terms this regulus is one family of generators on a hyperbolic paraboloid $\mathcal{H}_g$.
- The hyperbolic paraboloid $\mathcal{H}_g$ is the Lie quadric of F along the generator g.
- Given generators $g, g' \neq g_\infty$ the reguli $\mathcal{R}_g^-$ and $\mathcal{R}_{g'}^-$ have only the line g_∞ in common.
- Given generators $g, g' \neq g_\infty$ the Lie quadrics $\mathcal{H}_g$ and $\mathcal{H}_{g'}$ have the same tangent plane at each point of g_∞.



Michael Walker (1976) used the reguli $\mathcal{R}_g^-$ together with their opposite reguli $\mathcal{R}_g^+$ to construct and describe the spread $\mathcal{O}$ over certain finite fields.

Covering $\mathcal{O}$ with Reguli

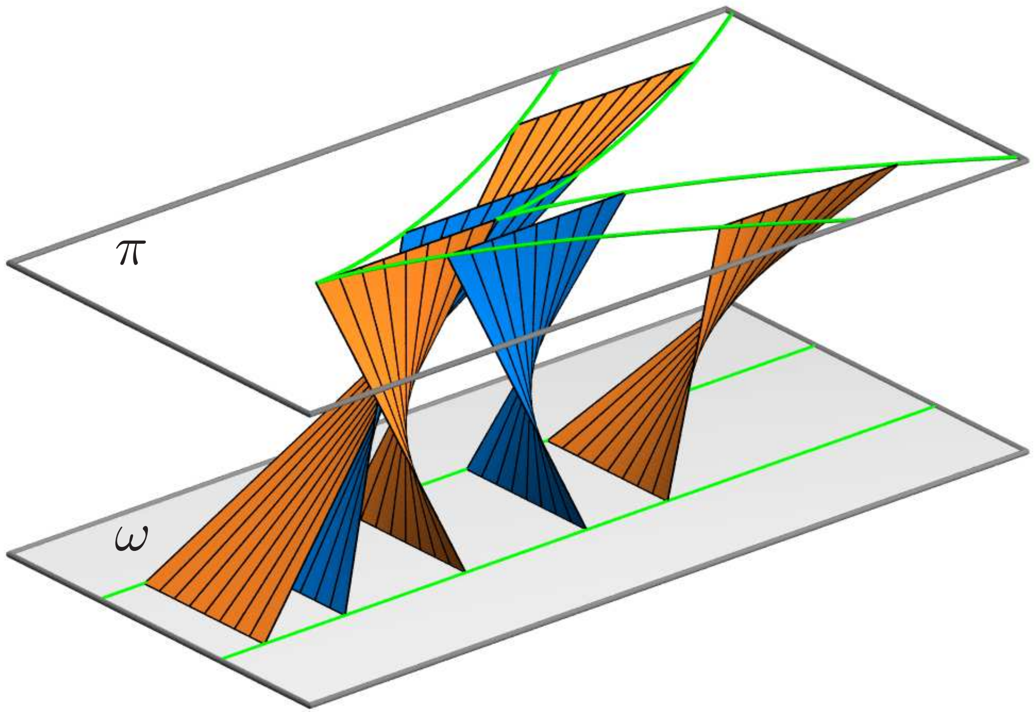
Here is another picture, where $\mathcal{V}(X_3)$ appears at infinity:



Betten's Approach

We choose the plane $\pi = \mathcal{V}(X_1)$ and the plane at infinity $\omega = \mathcal{V}(X_0)$. The lines of $\mathcal{O}$ other than g_∞ define (by intersection) a bijection

$$\tau : \omega \setminus g_\infty \to \pi \setminus g_\infty .$$



Conversely, τ can be used to generate $\mathcal{O} \setminus g_\infty$ by joining corresponding points.

Dieter Betten (1973) used a dual approach to construct the spread $\mathcal{O}$.

Final Remarks

- The *Betten-Walker spread* $\mathcal{O}$ appears in the literature under various names.

- The Betten-Walker spread in $\mathbb{P}_3(\mathbb{R})$ yields a 4-dimensional translation plane. (D. Betten).

- Let K be infinite. The union of $\mathcal{O}$ with the pencil $\mathcal{L}(Z, \omega)$ is the smallest algebraic set of lines containing $\mathcal{O}$.

  So, for example, the Betten-Walker spread in $\mathbb{P}_3(\mathbb{R})$ is not an algebraic spread, but it is very “close” to being algebraic.

- Only few algebraic spreads of $\mathbb{P}_3(\mathbb{R})$ seem to be known. Non-regular examples are due to R. Riesinger.

- For further details see: H. H. and R. Riesinger, The Betten-Walker spread and Cayley’s ruled cubic surface, *Beitr. Algebra Geometrie* **47** (2006), 527–541.