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Symplectic Plücker Transformations

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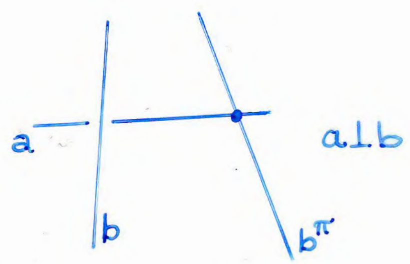
Dedicated to

Hans VOGLER

on the occasion of his 60th birthday

$(\mathcal{P},\mathcal{L},\pi)$ **symplectic space** $(3 \leq \dim(\mathcal{P},\mathcal{L}) \leq \infty)$ **with absolute quasipolarity** π.



$a,b \in \mathcal{L}$

$a \perp b \;:\Longleftrightarrow\; a \cap b^{\pi} \neq \emptyset$ (orthogonal lines)

$a \sim b \;:\Longleftrightarrow\; (a \perp b$ and $a \cap b \neq \emptyset)$ or $(a = b)$ (related lines)

$a \perp a$ (isotropic line) $\Longleftrightarrow$ $a \subset a^{\pi}$ (totally isotropic line)

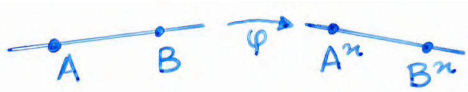
$\mathrm{P\Gamma Sp}(\mathcal{P},\pi)$ ... group of collineations commuting with π.

$(\mathcal{L},\sim)$ ... **Plücker space** (W. Benz)

$\varphi : \mathcal{L} \to \mathcal{L}$ bijective, preserving $\sim$ in both directions (**Plücker transformation**).

We say that φ is induced by a mapping $\kappa : \mathcal{P} \to \mathcal{P}$, if

$$(A \vee B)^{\varphi} = A^{\kappa} \vee B^{\kappa} \text{ for all } A,B \in \mathcal{P} \text{ with } A \neq B.$$



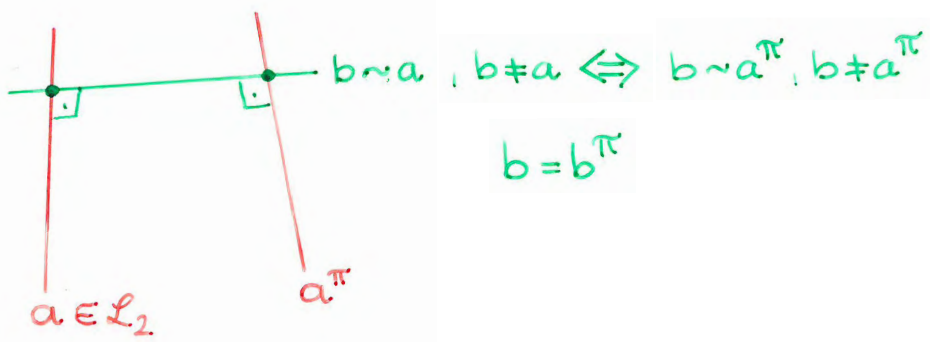
Example 1. Each $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$ is inducing a Plücker transformation.

Example 2. Let $\dim(\mathcal{P},\mathcal{L}) = 3$. For each duality τ with $\tau\pi = \pi\tau$ the restriction $\tau|\mathcal{L} : \mathcal{L} \to \mathcal{L}$ is a Plücker transformation.

Example 3. Let $\dim(\mathcal{P},\mathcal{L}) = 3$ and let $\mathcal{L}_2 \subset \mathcal{L}$ be any subset of non-isotropic lines such that $\mathcal{L}_2^{\pi} = \mathcal{L}_2$. Define

$$\delta : \mathcal{L} \to \mathcal{L}, \begin{cases} x \mapsto x & \text{if } x \in \mathcal{L}\setminus\mathcal{L}_2, \\ x \mapsto x^{\pi} & \text{if } x \in \mathcal{L}_2. \end{cases}$$

(**partial π-transformation** with respect to $\mathcal{L}_2$).



Theorem 1. *Let $(\mathcal{P},\mathcal{L},\pi)$ be a 3-dimensional symplectic space and let $\beta : \mathcal{L} \to \mathcal{L}$ be a bijection such that*

$$a \sim b \text{ implies } a^\beta \sim b^\beta \text{ for all } a,b \in \mathcal{L}.$$

Then there exists a partial π-transformation $\delta : \mathcal{L} \to \mathcal{L}$ such that $\delta\beta$ is induced by a collineation $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$.

Theorem 2. *Let $(\mathcal{P},\mathcal{L},\pi)$ be an n-dimensional symplectic space $(5 \le n \le \infty)$ and let $\beta : \mathcal{L} \to \mathcal{L}$ be a bijection such that*

$$a \sim b \text{ implies } a^\beta \sim b^\beta \text{ for all } a,b \in \mathcal{L}.$$

Then β is induced by a collineation $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$.

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Lemma 1. *If $Q \in \mathcal{P}$, then all isotropic lines through Q are given by*

$$\mathcal{I}[Q] := \{x \in \mathcal{L} \mid Q \in x \subset Q^{\pi}\}.$$

Let $a \in \mathcal{L}$ be non-isotropic. The set of isotropic lines intersecting the line a equals the set of all lines intersecting both a and a^{π}.

Lemma 2. *Distinct lines $a,b \in \mathcal{L}$ with $a \cap b \neq \emptyset$ are related if, and only if, a or b is isotropic.*

Lemma 3. *Let $\mathcal{M}$ be a set of mutually related lines. Then at most one line of $\mathcal{M}$ is non-isotropic.*

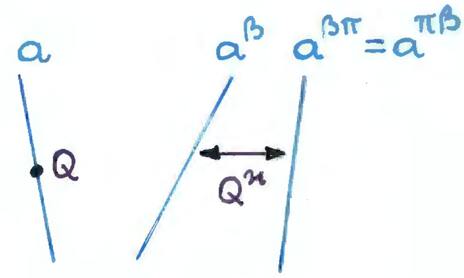
1.1. *There exists an injective mapping* $\kappa : \mathcal{P} \to \mathcal{P}$ *with*

$$\mathcal{I}[Q]^{\beta} = \mathcal{I}[Q^{\kappa}] \text{ for all } Q \in \mathcal{P}.$$

Moreover, β *is a Plücker transformation, since the set of isotropic lines is invariant under* β *and* β^{-1}.

1.2. *Let* $a \in \mathcal{L}$. *Then* $a^{\beta\pi} = a^{\pi\beta}$ *and*

$$Q^{\kappa} \in a^{\beta} \cup a^{\pi\beta} \text{ for all } Q \in a.$$



If $a \in \mathcal{L}$ *is non-isotropic, then either*

$$Q^{\kappa} \in a^{\beta} \text{ for all } Q \in a \text{ (lines of } 1^{st} \text{ kind} \ldots \mathcal{L}_1)$$

or

$$Q^{\kappa} \in a^{\pi\beta} \text{ for all } Q \in a \text{ (lines of } 2^{nd} \text{ kind} \ldots \mathcal{L}_2).$$

1.3. *The mapping*

$$\delta : \mathcal{L} \to \mathcal{L}, \begin{cases} x \mapsto x & \text{if } x \in \mathcal{L} \setminus \mathcal{L}_2, \\ x \mapsto x^{\pi} & \text{if } x \in \mathcal{L}_2, \end{cases}$$

is a partial π*-transformation. The Plücker transformation* $\delta\beta$ *takes intersecting lines to intersecting lines.*

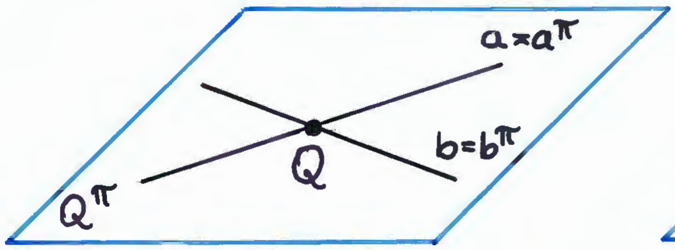
1.4. $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$. *The Plücker transformation* $\delta\beta$ *is induced by this collineation* κ.

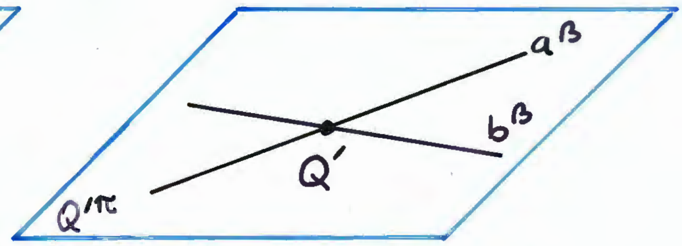
$\mathcal{J}[Q]^\beta = \mathcal{J}[Q']$

Bew.:

$x \sim y \quad \forall\, x, y \in \mathcal{J}[Q] \quad \Rightarrow \quad x^\beta \sim y^\beta \;\forall\, x^\beta, y^\beta \in \mathcal{J}[Q]^\beta$

in $\mathcal{J}[Q]^\beta$ $\exists$ höchstens eine nicht isotrope Gerade c^β

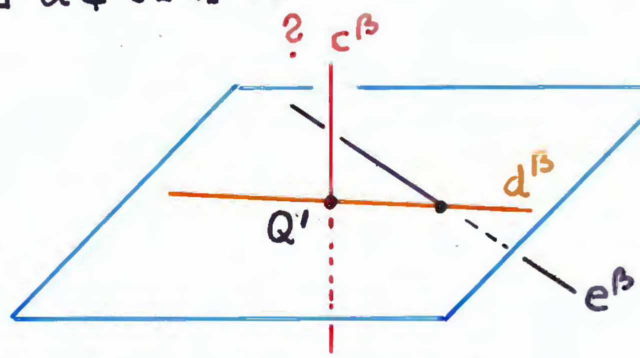




$a, b \in \mathcal{J}[Q] \setminus \{c\}, \; a \neq b \quad \Rightarrow \quad a^\beta \sim b^\beta, \; a^\beta, b^\beta \in \mathcal{J}, \; a^\beta \neq b^\beta$

$Q' := a^\beta \cap b^\beta$

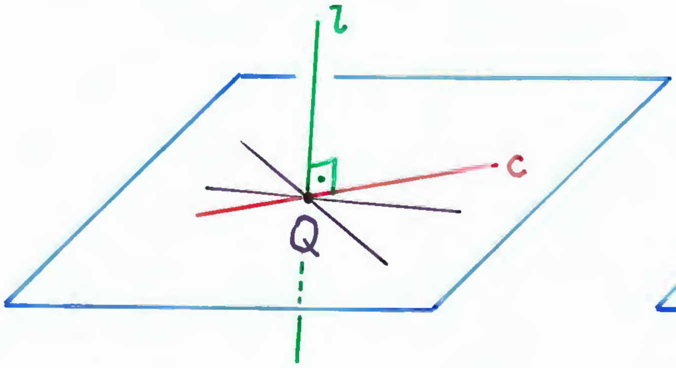
$\mathcal{J}[Q]^\beta \supset \mathcal{J}[Q']$ ind.: $\exists\, d \notin \mathcal{J}[Q]$ mit $d^\beta \in \mathcal{J}[Q']$

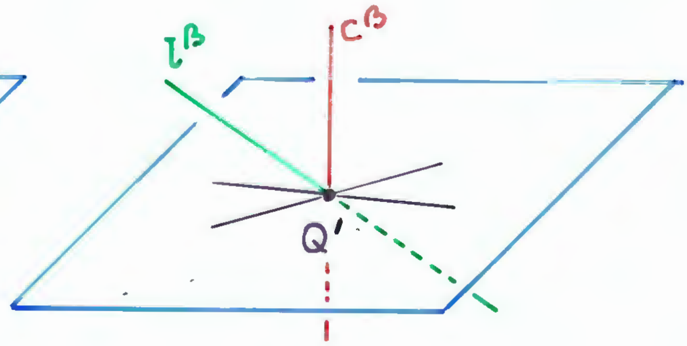


$\exists\, e : e \nsim \mathcal{J}[Q] \quad \Leftarrow \quad \exists\, e^\beta : e^\beta \nsim \mathcal{J}[Q]^\beta$

unmöglich wegen dim = 3

$\mathcal{J}[Q]^\beta \subset \mathcal{J}[Q']$, d.h. $\nexists\, c$ ind.: $\exists\, c \in \mathcal{J}[Q], \; c^\beta \notin \mathcal{J}[Q']$





$\mathcal{J}[Q] \cup \{l\}$ paarweise $\sim$

$c^\beta \nsim l^\beta$ Wid!

weil nicht isotrop

2.1. *The bijection* β *takes intersecting lines to intersecting lines. There exists an injective mapping* $\kappa : \mathcal{P} \to \mathcal{P}$ *inducing* β*. This* κ *is preserving collinearity and non-collinearity of points. Moreover*

$$\mathcal{L}[Q]^{\beta} = \mathcal{L}[Q^{\kappa}] \text{ for all } Q \in \mathcal{P}.$$

2.2. *The bijection* β *is a Plücker transformation, since the set of isotropic lines is invariant under* β *and* β^{-1}.

2.3. $\kappa \in \mathrm{P\Gamma Sp}(\mathcal{P},\pi)$.

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