K. POHLKE (1853)

$$\mathbb{E}_3 \xrightarrow{\text{parallel proj.}} \text{plane } (\subset \mathbb{E}_3) \xrightarrow{\text{affinity}} \mathbb{E}_2$$

$\Rightarrow$ decomposable into

$$\mathbb{E}_3 \xrightarrow{\text{parallel proj.}} \text{plane } (\subset \mathbb{E}_3) \xrightarrow{\text{similarity}} \mathbb{E}_2$$

Generalizations ?


Similarity
KARLSPLATZ


KARLSPLATZ

$\left.\begin{array}{l} V \ldots n\text{-dim.} \\ W \ldots m\text{-dim.} \end{array}\right\}$ euclidean vector spaces , $n > m$

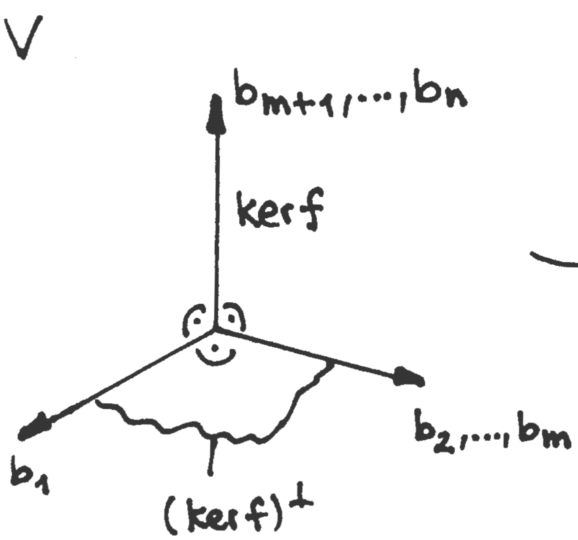
$f \in L(V,W)$, $f(V) = W$

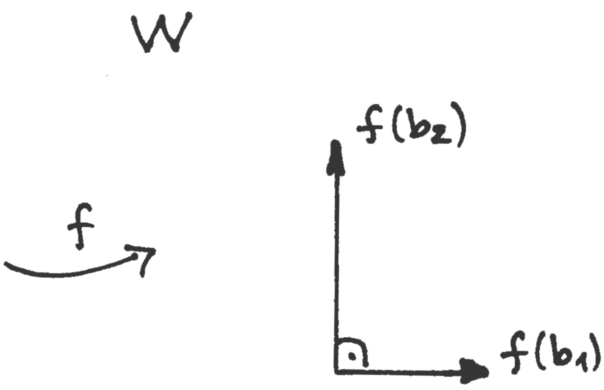
A.. coordinate matrix of f (orthonormal bases)

Singular value decomposition

$$A = \underset{\in O(m)}{C} \cdot \underbrace{\left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \end{array}\right)}_{\text{isometry}} \cdot \underbrace{\left(\begin{array}{cccc|c} \lambda_1 & & & & \\ & \lambda_2 & & & 0 \\ & & \ddots & & \\ & & & \lambda_m & \\ \hline & & 0 & & 0 \end{array}\right)}_{\substack{n \times n \\ \text{self-adjoint}}} \cdot \underset{\in O(n)}{B}$$

with $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ ... singular values





b_i ... orthonormal
$i = 1, 2, \ldots, n$

$\|f(b_j)\| = \lambda_j$
$j = 1, 2, \ldots, m$

$A^T A$ ... orthonormal eigenbasis ... b_i
eigenvalues ... λ_j^2; $0, \ldots, 0$

$f^{ad} \circ f$, $f \circ f^{ad}$ ... same eigenvalues $\neq 0$

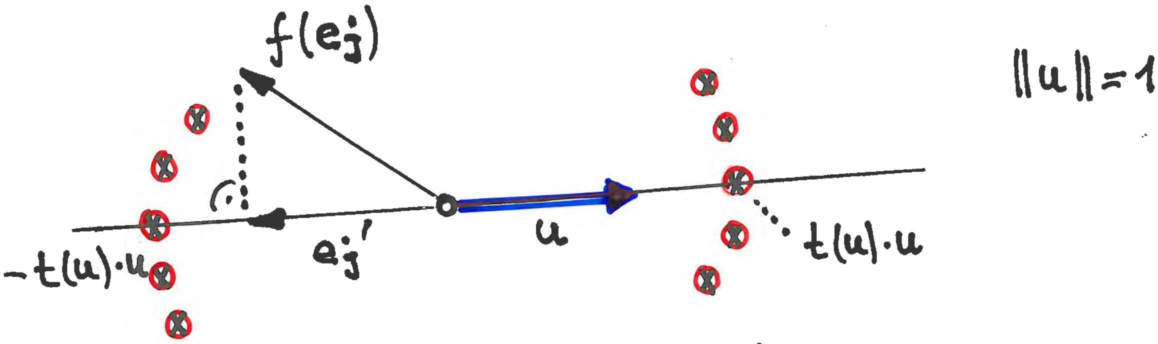
Theorem : f is decomposable into a central projection and a similarity $\iff$ the smallest singular value of g has multiplicity $\geq 2m-n+1$

(V. HAVEL 1960, H. BRAUNER 1986)

Geometric interpretation

$e_1, \dots, e_n$ … orthonormal basis of V

W:



$$t(u) := \left(\sum_{j=1}^{n} \|e_j'\|^2 \right)^{-\frac{1}{2}}$$

… ellipsoid of inertia … Λ (Naumann 1957)

$f^{ad}(\Lambda) \subset (\ker f)^{\perp}$ … unit sphere

$f \circ f^{ad}(\Lambda) \subset W$ … f-image of the unit sphere

$\mathbb{E}_n \to$ projective closure $\mathbb{P}(\mathbb{R}\times V)$

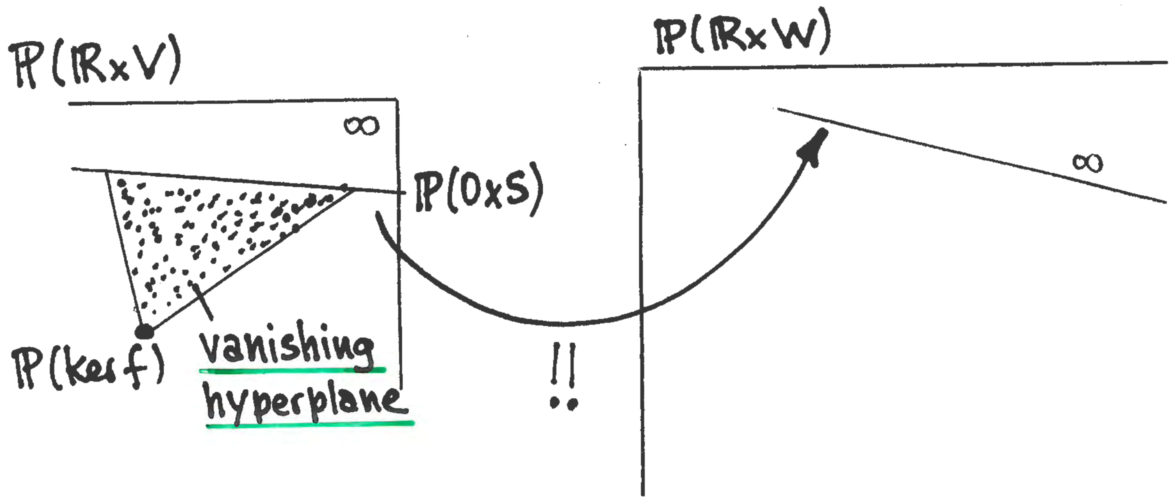
$x \mapsto \mathbb{R}(x_0, x)$, $x_0 = 1$ ... proper points

$\mathbb{R}(\underset{\neq 0}{0, x})$ ... points at infinity

$\mathbb{E}_m \to \mathbb{P}(\mathbb{R}\times W)$

$f \in L(\mathbb{R}\times V, \mathbb{R}\times W)$, surjective

$\ker f$ ... points without image , not at infinity



$0\times S := f^{-1}(0\times W) \cap (0\times V)$; $S \subset V$

$f|(0\times S) : (0, x) \mapsto (0, g(x))$; $g \in L(S, W)$

S and W are euclidean vector spaces.

A matrix characterization (H. 1996)

f ... A ... coordinate matrix (homogeneous cartesian coordinates)

$$A = \left[\begin{array}{c:ccc} a_{00} & a_{01} & \cdots & a_{0n} \\ \hdashline a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{mn} \end{array}\right] =: \left[\begin{array}{c:c} a_{00} & a_0 \\ a_{10} & a_1 \\ \vdots & \vdots \\ a_{m0} & a_m \end{array}\right]$$

vanishing hyperplane: $a_{00}x_0 + a_{01}x_1 + \cdots + a_{0n}x_n = 0$

$\ker f$ not at infinity $\Rightarrow$ $a_0 \neq 0$

New matrix:

$$\tilde{A} := \begin{bmatrix} a_1 - \dfrac{a_0 \cdot a_1}{a_0 \cdot a_0}\, a_0 \\ \vdots \\ a_m - \dfrac{a_0 \cdot a_m}{a_0 \cdot a_0}\, a_0 \end{bmatrix} \Rightarrow \tilde{g} \in L(V, W)$$

with the same singular values as g and

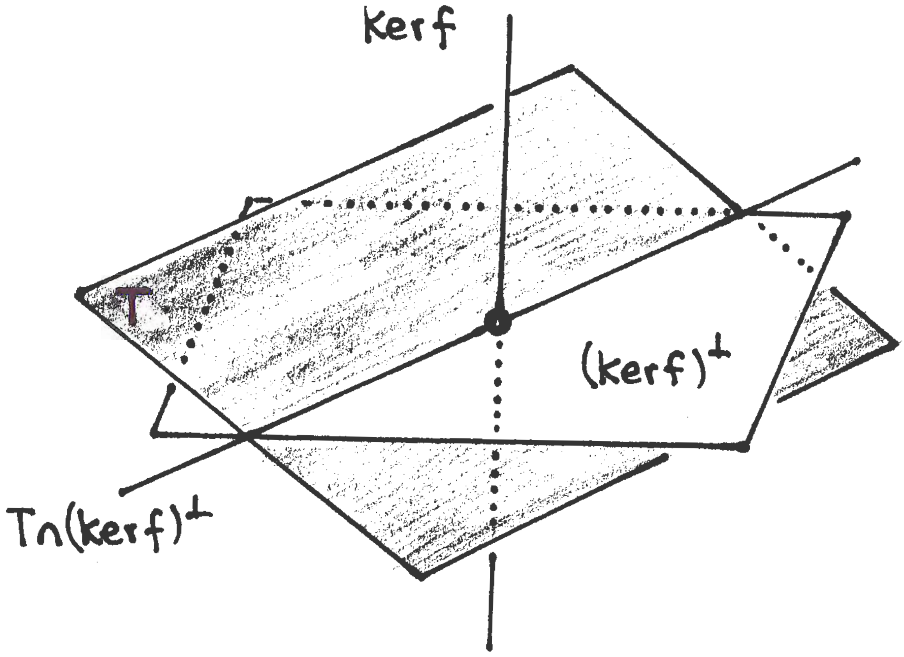
$\tilde{g} \mid S = g$.

(STACHEL 1996)

Theorem : $f \in L(V,W)$ is decomposable into a parallel projection $V \to T$ $(T \subset V)$ and a similarity $T \to W$ $\iff$ the smallest singular value λ_1 of f has multiplicity

$$\geq 2m-n.$$

(V. HAVEL, K. VALA, ~1960)



$\dim (\ker f)^\perp = \dim T = m$

$\dim (T \cap (\ker f)^\perp) \geq 2m-n$