A Dimension Formula for the Nucleus of a Veronese Variety

J. Gmainer H. Havlicek

TU Wien

Veronese mapping

$$\gamma : PG(m, F) \longrightarrow PG(M, F)$$

$$F(x_0, x_1, \ldots, x_m) \longmapsto F(\ldots, x_0^{e_0} x_1^{e_1} \cdot \ldots \cdot x_m^{e_m}, \ldots)$$

$$e_0 + e_1 + \cdots + e_m = t$$

$$M = \binom{m+t}{t} - 1$$

$\operatorname{im} \gamma$ ... Veronese variety V_m^t $(t \geq 2)$

Examples:

$m = 1$ ... Normal rational curve

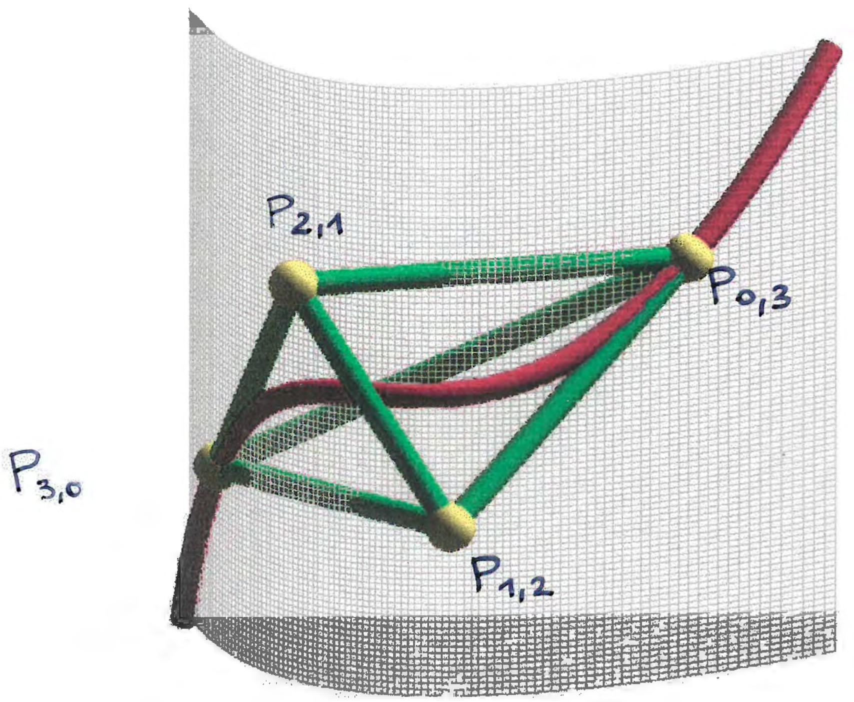
$F(x_0, x_1) \longmapsto F(x_0^2, x_0 x_1, x_1^2)$ conic $(M=2)$

$F(x_0, x_1) \longmapsto F(x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$ twisted cubic $(M=3)$

$m = 2$

$$F(x_0, x_1, x_2) \longmapsto F(x_0^2, x_0 x_1, x_0 x_2, x_1^2, x_1 x_2, x_2^2)$$

Veronese surface $(M=5)$



$$F(x_0, x_1) \mapsto F(\underset{3,0}{x_0^3}, \underset{2,1}{x_0^2 x_1}, \underset{1,2}{x_0 x_1^2}, \underset{0,3}{x_1^3})$$

Dual Veronese mapping

$$\gamma^*: PG(m,F)^* \longrightarrow PG(M,F)^*$$

$$F[a_0, a_1, \dots, a_m] \mapsto F\left[\dots, \binom{t}{e_0, e_1, \dots, e_m} a_0^{e_0} a_1^{e_1} \cdot \ldots \cdot a_m^{e_m}, \dots\right]$$

multinomial coefficient: $\dfrac{t!}{e_0!e_1!\dots e_m!}$, $e_0+e_1+\dots+e_m=t$

im γ^* ... osculating (contact) hyperplanes of V_m^t

Examples:

$m=1$

$F[a_0, a_1] \longmapsto F[a_0^2, 2\, a_0 a_1, a_1^2]$ tangents

$F[a_0, a_1] \longmapsto F[a_0^3, 3\, a_0^2 a_1, 3\, a_0 a_1^2, a_1^3]$

osculating planes

$m=2$

$F[a_0, a_1, a_2] \longmapsto F[a_0^2, 2\, a_0 a_1, 2\, a_0 a_2, a_1^2, 2\, a_1 a_2, a_2^2]$

osc. hyperplanes

Nucleus of a Veronese variety ...

intersection over all its osculating hyperplanes.

Timmermann (1977, 1978)
Herzer (1982)
Karzel (1987)
...

Theorem 1:

The nucleus of a Veronese variety V_m^t contains exactly those base points $P_{e_0,e_1,\dots,e_m}$ (of the standard basis) satisfying

$$\binom{t}{e_0,e_1,\dots,e_m} \equiv 0 \ (\mathrm{mod}\ \mathrm{Char}\, F).$$

If $\#F \geq t$, then the nucleus is spanned by those base points.

$\mathrm{Char}\, F = p > 0$

$$t = \sum_{\lambda \in \mathbb{N}} t_\lambda p^\lambda \qquad e_i = \sum_{\lambda \in \mathbb{N}} e_{i,\lambda} p^\lambda$$

$$\boxed{\binom{t}{e_0, e_1, \dots, e_m} \equiv \prod_{\lambda \in \mathbb{N}} \binom{t_\lambda}{e_{0,\lambda}, e_{1,\lambda}, \dots, e_{m,\lambda}} \pmod{p}}$$

(Lucas)

Example

$$\binom{3}{1,1,1} \equiv \underbrace{\binom{11}{01,01,01} \equiv \underbrace{\binom{1}{1,1,1}}_{=1} \cdot \underbrace{\binom{1}{0,0,0}}_{=0}}_{\text{BINARY}} \equiv 0 \pmod{2}$$

Theorem 2:

The nucleus of V_m^t has (projective) dimension

$$\binom{m+t}{t} - 1 - \prod_{\lambda \in \mathbb{N}} \binom{m+t_\lambda}{t_\lambda}$$

$(\#F \geq t, \; p = \mathrm{Char}\, F > 0)$.

Howard 1974

Volodin 1989