Recent Results in Chain Geometry

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Part 1
Basic Concepts

W. BENZ. *Vorlesungen über Geometrie der Algebren*. Springer, Berlin, 1973.

A. HERZER. Chain geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*. Elsevier, Amsterdam, 1995.

A. BLUNCK and H. HAVLICEK. Extending the concept of chain geometry. *Geom. Dedicata* **83** (2000), 119–130.

A. BLUNCK and H. HAVLICEK. The connected components of the projective line over a ring. *Adv. Geom.* **1** (2001), 107–117.

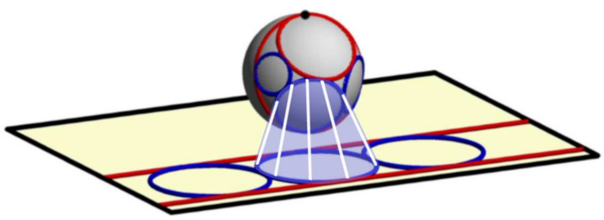
The Real Möbius Plane

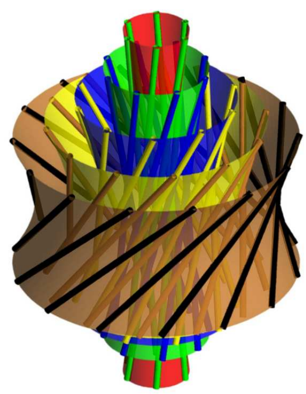
Algebraic definition

Points: $\mathbb{C} \cup \{\infty\}$ (complex projective line)
Circles: Images of $\mathbb{R} \cup \{\infty\}$ under $\mathrm{PGL}_2(\mathbb{C})$

Other models

- Elliptic quadric / conics
- Euclidean plane + one point / circles and lines
- Elliptic linear congruence of lines (regular spread) / reguli





The Projective Line over a Ring

All our rings are associative, with unit element 1 which is inherited by subrings and acts unitally on modules.

Let $\mathrm{GL}_2(R)$ be the group of invertible (2×2)-matrices with entries in a ring R.

A pair $(a, b) \in R^2$ is called *admissible* if (a, b) is the first row of a matrix in $\mathrm{GL}_2(R)$.

Projective line over R:

$$\mathbb{P}(R) := \{R(a, b) \mid (a, b) \text{ admissible } \}$$

Chain Geometries

Assume that F is a field (not necessarily commutative) contained in a ring R. There is the natural embedding

$$\mathbb{P}(F) \to \mathbb{P}(R) : F(a,b) \mapsto R(a,b).$$

The images of $\mathbb{P}(F)$ under $\mathrm{PGL}_2(R)$ are the *chains* of the *chain geometry* $\Sigma(F,R)$.

$\mathrm{PGL}_2(R)$ is a group of automorphisms of $\Sigma(F,R)$.

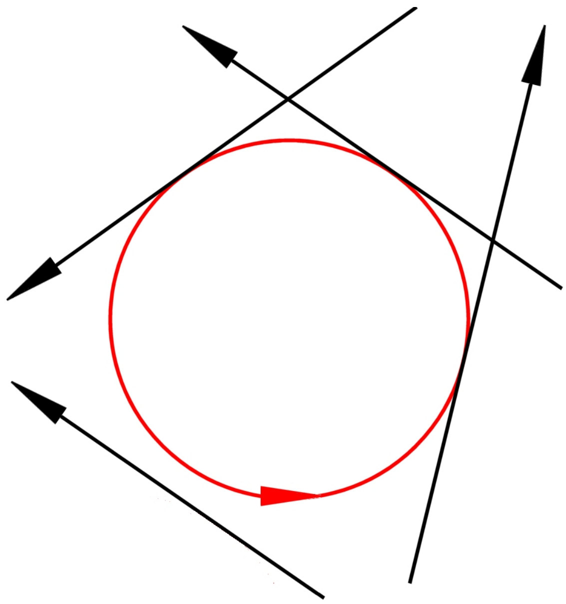
Let R^* be the group of units in R. Then $u \in R^*$ implies

$$\Sigma(F,R) = \Sigma(u^{-1}Fu, R).$$

The Real Laguerre Plane

$\mathbb{D} := \mathbb{R}[\varepsilon]$ with $\varepsilon^2 = 0$ is the ring of *dual numbers* over $\mathbb{R}$.

Up to isomorphism, $\Sigma(\mathbb{R}, \mathbb{D})$ is the geometry of *spears* and *oriented circles* in the Euclidean plane.



The Distant Graph

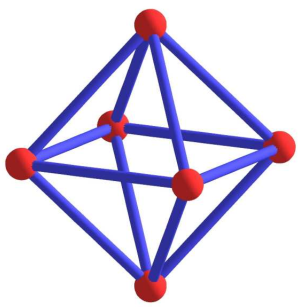
Distant points of $\mathbb{P}(R)$:

$$R(a,b) \vartriangle R(c,d) :\Longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

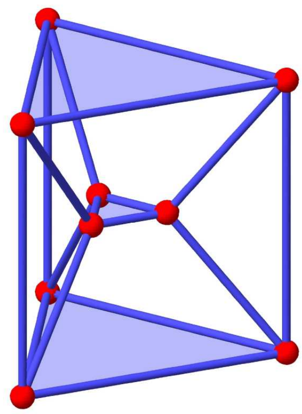
Non-distant points are also called *parallel*.

$(\mathbb{P}(R), \vartriangle)$ is the *distant graph* of $\mathbb{P}(R)$.

Examples



$R = \mathrm{GF}(2)[\varepsilon]$



$R = \mathrm{GF}(2) \times \mathrm{GF}(2)$

Two points of $\Sigma(F, R)$ are distant exactly if they are different and joined by a chain.

Chains through three points

Theorem. *There are as many chains through three mutually distant points of $\Sigma(F, R)$ as there are subfields $u^{-1}Fu$, where u is a unit in R.*

Examples

There is a unique chain through three mutually distant points if

- F is in the centre of R,
- $F^* = R^*$,
- $F = \mathrm{GF}(4)$, $R = \mathrm{M}(2, 2, \mathrm{GF}(2))$ (sporadic example).

There is more than one chain through three mutually distant points if

- $R = \mathbb{H}$ and $F = \mathbb{C}$ (4-sphere / 2-spheres),
- $F = \mathrm{GF}(q^2)$, $R = \mathrm{M}(2, 2, \mathrm{GF}(q))$ with $q > 2$.

Part 2
Connectedness

P.M. Cohn. On the structure of the GL_2 of a ring. *Inst. Hautes Etudes Sci. Publ. Math.* **30** (1966), 5–53.

A. Blunck and H. Havlicek. The connected components of the projective line over a ring. *Adv. Geom.* **1** (2001), 107–117.

A Characterization

$\mathrm{GE}_2(R)$ denotes the subgroup of $\mathrm{GL}_2(R)$ which is generated by the set of all matrices of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

with $t, u, v \in R$ and u, v invertible.

Theorem. *The projective line $\mathbb{P}(R)$ is connected if, and only if,*

$$\mathrm{GL}_2(R) = \mathrm{GE}_2(R),$$

i.e., R is a GE_2-ring.

Examples

$\mathbb{P}(R)$ is connected if R is a

- local ring,
- endomorphism ring of a vector space,
- finite-dimensional algebra,
- polynomial ring $F[X]$, X a central indeterminate.

However, $F[X_1, X_2, \ldots, X_n]$ with $n \geq 2$ central indeterminates is not a GE_2-ring.

Part 3
Residues

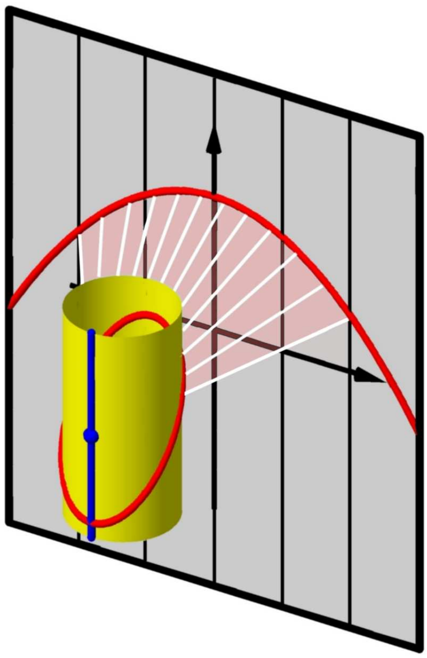
A. Blunck and H. Havlicek. Affine spaces within projective spaces. *Res. Math.* **36** (1999), 237–251.

A. Blunck and H. Havlicek. Extending the concept of chain geometry. *Geom. Dedicata* **83** (2000), 119–130.

A. Blunck and H. Havlicek. The dual of a chain geometry. *J. Geom.* (to appear).

Blaschke's Cone

A quadratic cone (without its vertex) in the real projective 3-space is a point model for the projective line over $\mathbb{R}[\varepsilon]$. Two points are parallel exactly if they are on a common generator.



Under a stereographic projection all points that are distant to the centre of projection are mapped bijectively onto the plane of dual numbers (*isotropic plane*).

Residue at a point

We fix one point of $\Sigma(F,R)$, say $R(1,0) =: \infty$ and put

$$\begin{aligned} \mathbb{P}_\infty &:= \{R(a,b) \in \mathbb{P}(R) \mid R(a,b) \vartriangle \infty\}, \\ \mathbf{B}_\infty &:= \{\mathcal{C} \setminus \{\infty\} \mid \mathcal{C} \text{ is a chain },\ \infty \in \mathcal{C}\}. \end{aligned}$$

$(\mathbb{P}_\infty, \mathbf{B}_\infty)$ is the *residue* of $\Sigma(F,R)$ at ∞. The elements of $\mathbf{B}_\infty$ are called *blocks*.

We shall identify R and $\mathbb{P}_\infty$ via the bijection

$$R \to \mathbb{P}_\infty : r \mapsto R(r,1).$$

Left and Right Affine Spaces

R is a left and a right vector space over $u^{-1}Fu$ for each $u \in R^*$.

So we get (in general a lot of) left and right affine spaces

$$\mathbb{A}(R, u^{-1}Fu)_{\text{left}},\ \mathbb{A}(R, u^{-1}Fu)_{\text{right}}$$

with common point set $\mathbb{P}_\infty = R$, each with two types of lines:

- *regular* lines (direction vector in R^*)
- *singular* lines (otherwise)

The elements of $\mathbf{B}_\infty$ are exactly the regular lines of all left (right) affine spaces from above.

Open Problem

Is it possible to characterize, in terms of $\Sigma(F, R)$, those subsets of $\mathbf{B}_\infty$ which are formed by all regular lines coming from a fixed affine space $\mathbb{A}(R, u^{-1}Fu)_{\text{left}}$ or $\mathbb{A}(R, u^{-1}Fu)_{\text{right}}$?

Part 4
Projective Representations

A. Blunck. Reguli and chains over skew fields. *Beiträge Algebra Geom.* **41** (2000), 7–21.

A. Blunck and H. Havlicek. Projective representations I. Projective lines over rings. *Abh. Math. Sem. Univ. Hamburg* **70** (2000), 287–299.

A. Blunck and H. Havlicek. Projective representations II. Generalized chain geometries. *Abh. Math. Sem. Univ. Hamburg* **70** (2000), 300–313.

Endomorphism Rings

Let U be a left vector space over a field K. We consider the projective space on $U \times U$:

$\mathcal{G}$ denotes the set of all subspaces that are isomorphic to one of their complements.

Theorem. *For* $S := \mathrm{End}_K(U)$ *the mapping*

$$\Psi : \mathbb{P}(S) \to \mathcal{G} : S(\alpha, \beta) \mapsto \{(u^\alpha, u^\beta) \mid u \in U\}$$

is bijective. Distant points and complementary subspaces are in bijective correspondence.

Example
If $\dim U = 2$ then $\mathcal{G}$ is the set of lines in the projective 3-space over K.

Arbitrary Rings

Let U be a (K, R)-bimodule and $S = \mathrm{End}_K(U)$. For each $a \in R$ the mapping

$$R \to S : a \mapsto (\rho_a : u \mapsto ua)$$

is a K-linear representation.

Theorem. *The mapping*

$$\mathbb{P}(R) \to \mathbb{P}(S) : R(a,b) \mapsto S(\rho_a, \rho_b)$$

is well defined and takes distant points to distant points. The mapping is injective exactly if U is faithful (as right R-module).

Altogether we get the *projective representation*

$$\mathbb{P}(R) \to \mathcal{G} : R(a,b) \mapsto \{ua, ub \mid u \in U\}.$$

Chain Geometries

We obtain a projective representation of $\Sigma(F, R)$ from a (K, R)-bimodule U. So U is a K-left vector space and an F-right vector space.

If R is a finite-dimensional F-algebra, $U \neq \{0\}$, and $F = K$ then the chains appear as reguli (Segre manifolds).

In general, a unified geometric description of chains seems hopeless.

- It depends on "how" the field F is embedded in the ring R.
- The link between F and K is rather weak:

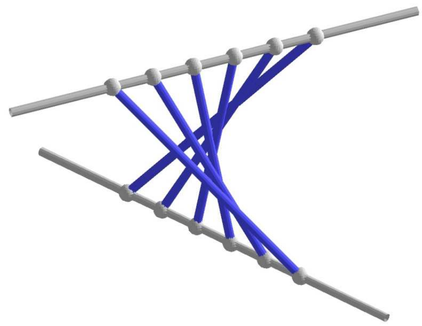
$$\operatorname{char} F = \operatorname{char} K \text{ if } U \neq \{0\}$$

Field

Let ζ_1, ζ_2 be monomorphisms of $F = K$. The mapping

$$k \mapsto \operatorname{diag}\bigl(k^{\zeta_1}, k^{\zeta_2}\bigr)$$

is a faithful representation of K. (We use matrix rings over K instead of $\operatorname{End}_K(U)$.)



2 weak transversals

Particular cases

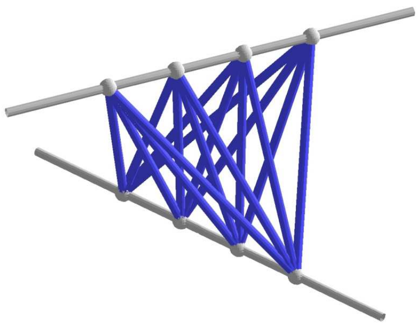
- $\zeta_1 = \zeta_2 = \mathrm{id}_K$: Regulus
- $K = \mathbb{C}$, $\zeta_1 = \mathrm{id}_{\mathbb{C}}$, $\zeta_2 =$ conjugation: Elliptic linear congruence (regular spread) of a real subgeometry.

Double numbers

Let ζ_1, ζ_2 be monomorphisms of K and let $R = K \times K$. The representation

$$(k_1, k_2) \mapsto \begin{pmatrix} k_1^{\zeta_1} & 0 \\ 0 & k_2^{\zeta_2} \end{pmatrix}$$

is faithful.



2 weak transversals

Particular case

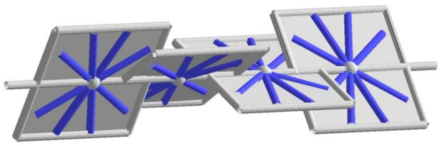
- $\zeta_1, \zeta_2 \in \mathrm{Aut}(K)$: Hyperbolic linear congruence of lines.

Twisted dual numbers

$R = K[\varepsilon]$ with $\varepsilon^2 = 0$, $\varepsilon k = k^\zeta \varepsilon$, and $\zeta \in \mathrm{Aut}(K)$. The representation

$$k_1 + k_2\varepsilon \mapsto \begin{pmatrix} k_1 & k_2 \\ 0 & k_1^\zeta \end{pmatrix}$$

is faithful.



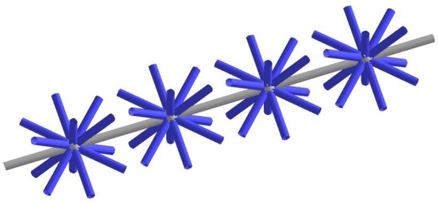
1 weak transversal

Particular cases

- $\zeta = \mathrm{id}_K$: Dual numbers, parabolic linear congruence of lines without its axis.
- $K = \mathbb{C}$, $\zeta =$ conjugation: Ring of *Study's quaternions*.

Upper triangular matrices

Let R be the ring of upper triangular (2×2)-matrices over K.



1 transversal

Special linear complex of lines without its axis.

Particular case

- $K = \mathbb{R}$: R is the ring of *ternions*.