A Parallelism Based on the Jacobson Radical of a Ring

Hans Havlicek

Institut für Geometrie
Technische Universität Wien
Vienna, Austria

A. Blunck and H. Havlicek. Radical parallelism on projective lines and non-linear models of affine spaces. *Math. Pannonica* **14** (2003), 113–127.

The Jacobson Radical

All our rings are associative, with unit element $1 \neq 0$ which is inherited by subrings and acts unitally on modules.

Jacobson radical of a ring R:

$$\operatorname{rad} R := \bigcap \text{ all maximal left (or right) ideals of } R$$

The Jacobson radical $\operatorname{rad} R$ is a two sided ideal of R and

$$\overline{R} := R/\operatorname{rad} R$$

has a zero radical.

The Meaning of the Jacobson Radical

Let R^* be the group of invertible elements of R.

In terms of R:

$$\begin{aligned} b \in \operatorname{rad} R \quad &\Leftrightarrow \quad 1 - ab \in R^* \text{ for all } a \in R \\ &\Leftrightarrow \quad 1 - ba \in R^* \text{ for all } a \in R \end{aligned}$$

In terms of matrices over R:

$$b \in \operatorname{rad} R \Leftrightarrow \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} \in \mathrm{GL}_2(R) \text{ for all } a \in R$$

Observe that we cannot use determinants in order to invert a matrix over a non-commutative ring.

Examples

Let R be a *ring of matrices* over a (skew-)field or a *direct product* of such rings:

$$\operatorname{rad} R = \{0\}$$

E.g.: $\mathbb{R}^{2\times 2}$, $\mathbb{R}\times\mathbb{R}$, $\mathbb{R}\times\mathbb{C}$, ...

Let R be a *local ring*:

$$\operatorname{rad} R = R \setminus R^*$$

E.g.: $R = \mathbb{D} = \mathbb{R} + \mathbb{R}\varepsilon$, the real *dual numbers*.

Let R be the ring of *upper triangular* 2×2*-matrices* over a field $\mathbb{F}$ (ring of *ternions*): It has an $\mathbb{F}$-basis

$$j_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ j_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \varepsilon := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Maximal left ideals in R are $\mathbb{F}j_1 + \mathbb{F}\varepsilon$ and $\mathbb{F}j_2 + \mathbb{F}\varepsilon$;

$$\operatorname{rad} R = \mathbb{F}\varepsilon.$$

The Projective Line over a Ring

A pair $(a,b) \in R^2$ is called *admissible* if (a,b) is the first row of a matrix in $\mathrm{GL}_2(R)$.

Projective line over R:

$$\begin{aligned} \mathbb{P}(R) &:= \{R(a,b) \mid (a,b) \in R^2 \text{ is admissible}\} \\ &= R(1,0)^{\mathrm{GL}_2(R)} \end{aligned}$$

Distant relation ($\triangle$) on $\mathbb{P}(R)$:

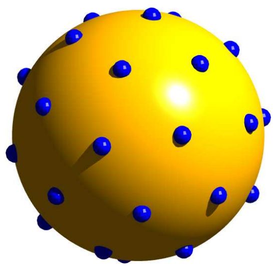
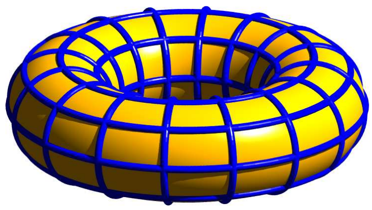
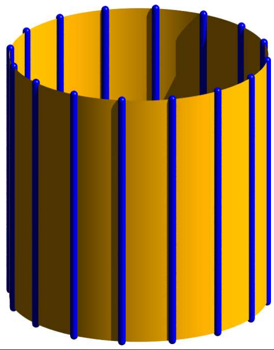
$$\triangle := (R(1,0), R(0,1))^{\mathrm{GL}_2(R)}$$

It is symmetric and anti-reflexive.
Letting $p = R(a,b)$ and $q = R(c,d)$ gives

$$p \triangle q \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R).$$

Non-distant points are also called *parallel*.

Three Classical Examples

Complex numbers $\mathbb{C}$: The relations '$\triangle$' and '$\neq$' coincide. Parallel points are identical.	
Double numbers $\mathbb{R} \times \mathbb{R}$: The parallelism is the union of two equivalence relations (meridians and parallel circles on the torus.)	
Real dual numbers $\mathbb{D}$: The parallelism is an equivalence relation (generators on the cylinder.)	

The Radical Parallelism

$p, q \in \mathbb{P}(R)$ said to be *radically parallel* ($p \parallel q$) if

$$x \mathrel{\triangle} p \;\Rightarrow\; x \mathrel{\triangle} q \text{ for all } x \in \mathbb{P}(R).$$

Properties:

- The relation $\parallel$ is reflexive and transitive.
- The relation $\parallel$ is finer than $\not\triangle$, i.e. $p \parallel q$ implies $p \not\triangle q$. (Let $x = q$ in the definition.)
- The relation $\parallel$ is invariant under the action of $\mathrm{GL}_2(R)$.

We shall see that $\parallel$ is in fact an equivalence relation.

Algebraic Description

Theorem. *The point* $R(1,0)$ *is radically parallel to* $q \in \mathbb{P}(R)$ *exactly if there is an element* b *in the Jacobson radical* $\operatorname{rad} R$ *such that*

$$q = R(1, b).$$

Recall that $\overline{R} := R/\operatorname{rad} R$.

Theorem. *The mapping*

$$\mathbb{P}(R) \to \mathbb{P}(\overline{R}) : p = R(a,b) \mapsto \overline{R}(\overline{a}, \overline{b}) =: \overline{p}$$

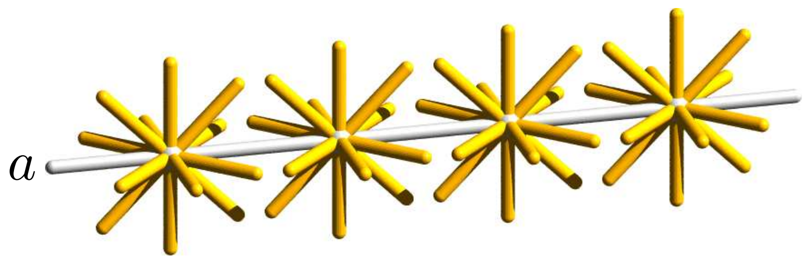
is well defined and surjective. It has the property

$$p \parallel q \Leftrightarrow \overline{p} = \overline{q} \text{ for all } p, q \in \mathbb{P}(R).$$

Therefore, $\parallel$ is an equivalence relation.

An Example

The projective line over the ring R of upper triangular matrices over a field $\mathbb{F}$ can be identified with a *special linear complex of lines* (in a projective 3-space over $\mathbb{F}$) without its axis, say a.



$$p \bigtriangleup q \quad \Leftrightarrow \quad p, q \text{ are skew lines}$$

$$p \parallel q \quad \Leftrightarrow \quad a, p, q \text{ are in a pencil}$$

Remark: $R/\mathrm{rad}\, R = \overline{R} \cong \mathbb{F} \times \mathbb{F}$.

An Application

Let A be an algebra over a field $\mathbb{F}$. Then

$$y \mapsto A(y, 1)$$

is a bijection of A onto the set of all points that are distant to $A(1, 0)$. We shall identify these sets.

Every projectivity of $\mathbb{P}(A)$ such that $A(1, 0)$ goes over to a distinct radically parallel point induces a *bijective non-linear Cremona transformation* on A.

$\Rightarrow$ Genereralizations of the *parabola model* of the real affine plane to higher dimensions.