

A Spread from Cayley's Ruled Cubic Surface

Joint work with Rolf Riesinger (Wien)

Varna, September 7th, 2007



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DIFFERENTIALGEOMETRIE UND
GEOMETRISCHE STRUKTUREN

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Spreads

Let $\mathbb{P}_3(K)$ be the 3-dimensional projective space over a field K , and let \mathcal{L} be its set of lines.

Definition. Let $\mathcal{S} \subset \mathcal{L}$ be a set of lines satisfying some of the following conditions:

1. Any two distinct lines of \mathcal{S} are skew.
2. Each point is incident with at least one line of \mathcal{S} .
3. Each plane is incident with at least one line of \mathcal{S} .

A *partial spread* is characterised by condition 1.

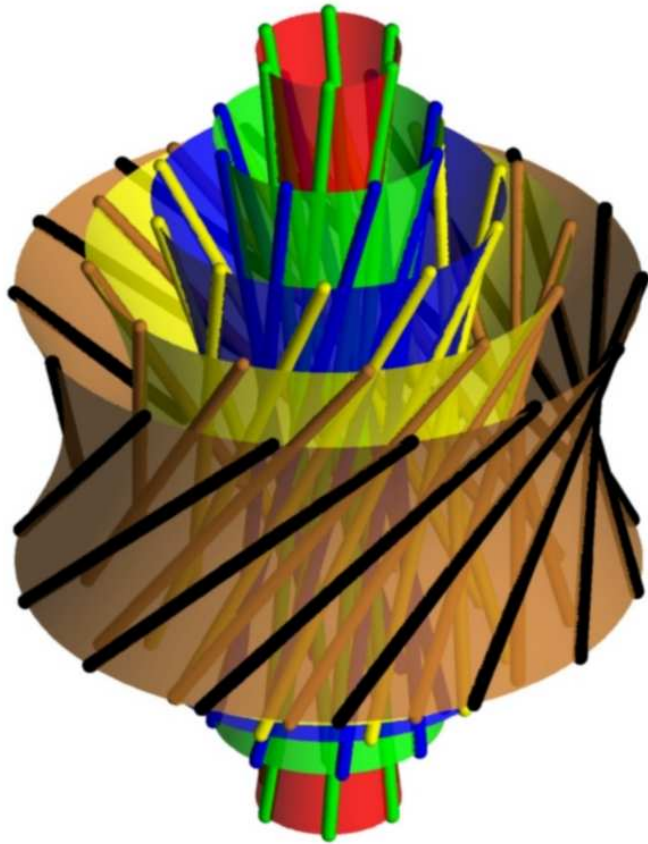
A *spread* is characterised by conditions 1 and 2.

A *dual spread* is characterised by conditions 1 and 3.

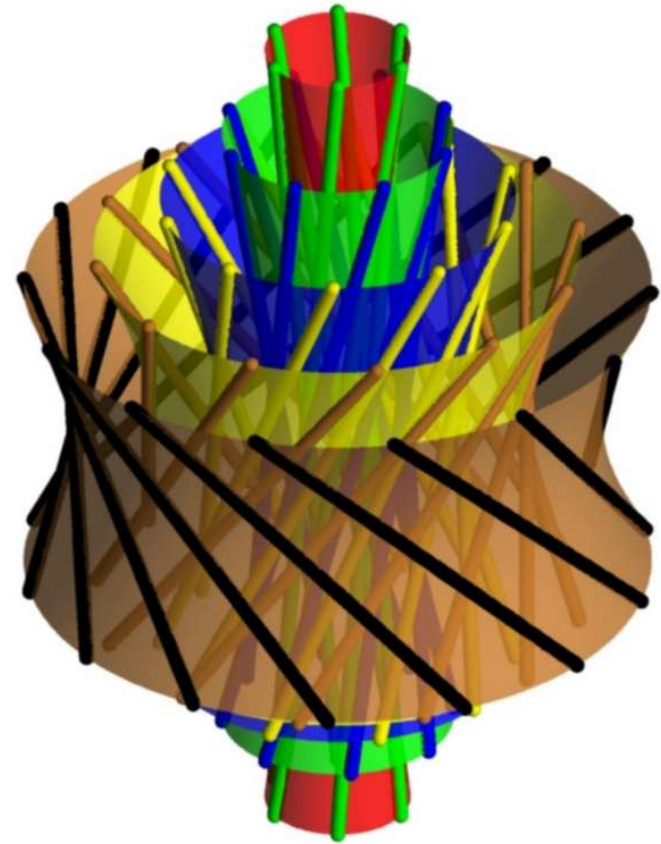
Extra Conditions

- A *regular spread* is closed under *reguli*.
- A spread is *algebraic* if its image under the Klein mapping is an *algebraic variety*.

Examples



A regular spread is an **elliptic linear congruence of lines**.



A **subregular spread** arises from a regular spread by replacing “some” reguli with their opposite reguli.

Applications

Applications of spreads to be found in the literature:

- **Foundations of geometry.**

Construction of translation planes. Uses a spread in the hyperplane at infinity of a 4-dimensional affine space ...

J. André (1956), R. H. Bruck and R. C. Bose (1963), ...

- **Parallelisms.**

Generalizations of the Clifford-parallelism.

W. K. Clifford (1873), ...

- **Descriptive geometry, computer vision.**

Non linear mappings on a plane. Parallel projection in 3-dimensional elliptic space. Non-central cameras, ...

L. Tuschel (1911), ...

Cayley's Surface

Cayley's ruled cubic surface or, for short, the *Cayley surface* is, to within projective collineations, the point set

$$F := \mathcal{V}(f(\mathbf{X})) := \{K(p_0, p_1, p_2, p_3)^T \in \mathbb{P}_3(K) \mid f(p_0, p_1, p_2, p_3) = 0\},$$

where

$$f(\mathbf{X}) := X_0X_1X_2 - X_1^3 - X_0^2X_3 \in K[\mathbf{X}] = K[X_0, X_1, X_2, X_3].$$

We shall consider $\omega := \mathcal{V}(X_0)$ as *plane at infinity*. Hence the affine part of the Cayley surface is given by the parametrisation

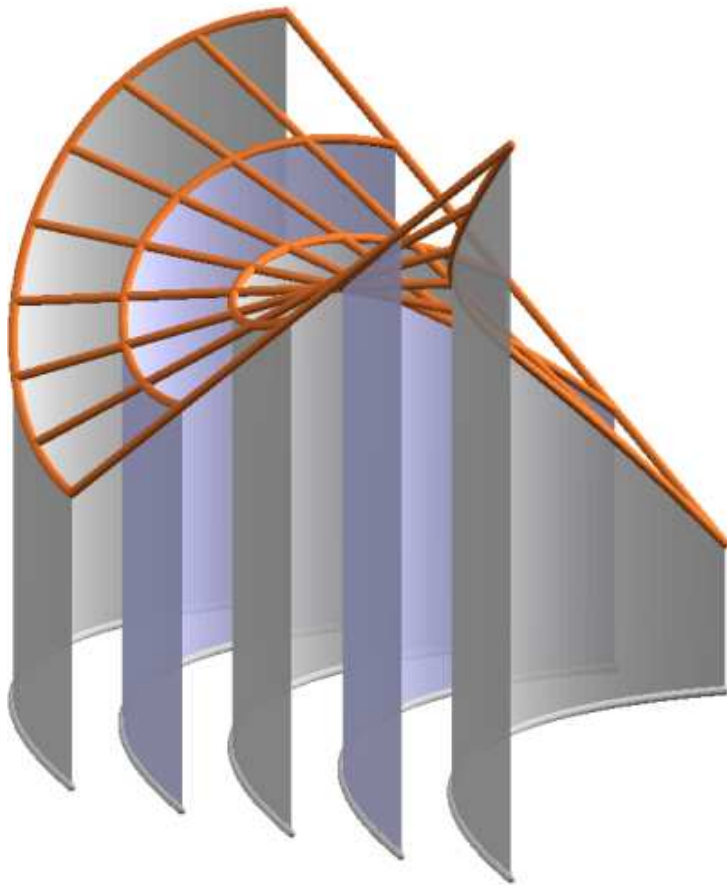
$$K^2 \rightarrow \mathbb{P}_3(K) : (u_1, u_2) \mapsto K(1, u_1, u_2, u_1u_2 - u_1^3)^T =: P(u_1, u_2).$$

The intersection of F with the plane ω is the line

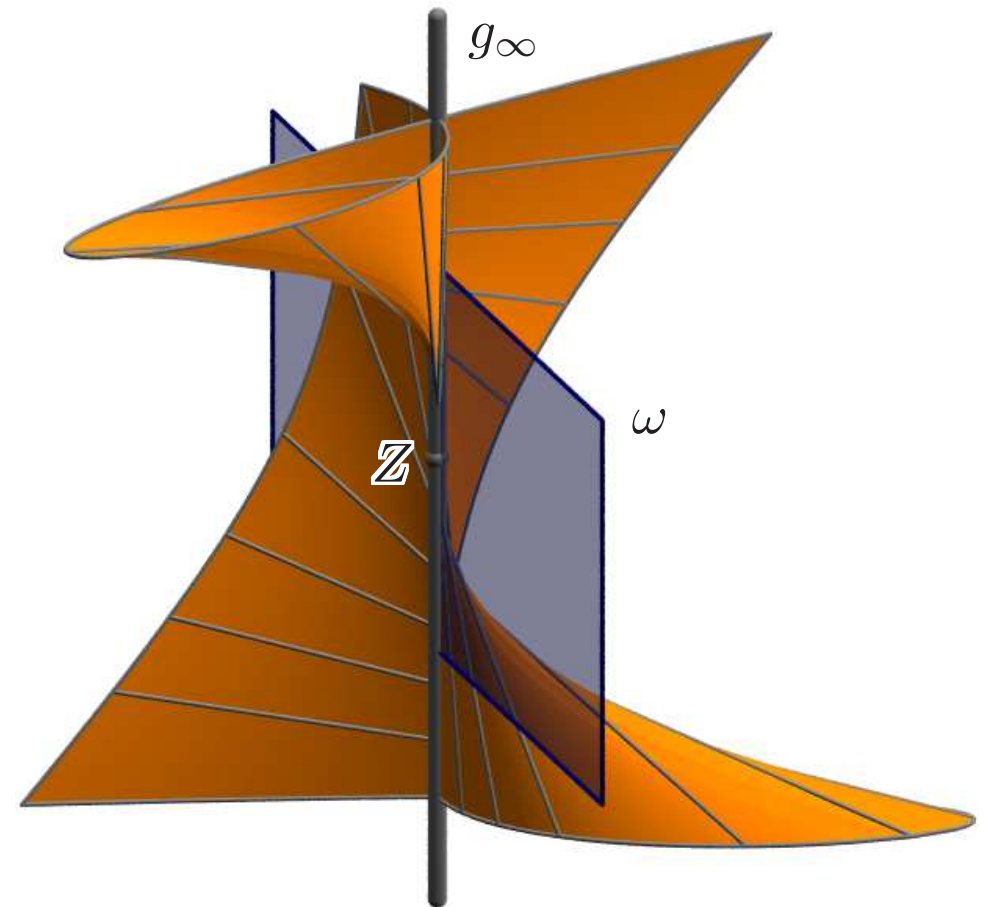
$$\mathcal{V}(X_0, X_1) =: g_\infty.$$

Pictures

All pictures illustrate the case $K = \mathbb{R}$, but in varying affine charts.



Affine point of view.
All points of $F \setminus \omega$ are **simple**.



Intersection with the plane at infinity.
All points of g_∞ are **double points**.
 $Z := K(0, 0, 0, 1)^T$ is a **pinch point**.

The Collineation Group

- The set of all matrices

$$M_{a,b,c} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & c & 0 & 0 \\ b & 3ac & c^2 & 0 \\ ab - a^3 & bc & ac^2 & c^3 \end{pmatrix}, \text{ where } a, b \in K, \text{ and } c \in K \setminus \{0\},$$

is a group, say G , under multiplication.

- Each matrix in G leaves invariant the cubic form $f(\mathbf{X}) = X_0X_1X_2 - X_1^3 - X_0^2X_3$ to within the factor $c^3 \neq 0$.
- The group G yields **all projective automorphic collineations** of F unless $\#K \leq 3$.
- Under the action of the group G , the points of F fall into three orbits:
 $F \setminus \omega$, $g_\infty \setminus \{Z\}$, and $\{Z\}$.

References



M. Chasles



A. Cayley

The name [Cayley surface](#) is not completely appropriate.

[Michel Chasles](#) published his discovery of that surface in 1861, three years before Arthur Cayley.

There is a widespread literature on the Cayley surface:

H. Brauner (1964, 1966, 1967, 1967),
J. Gmainer and H. H. (2005),
M. Husty (1984),
R. Koch (1968),
H. Neudorfer (1925),
M. Oehler (1969),
A. Wiman (1936),
H. Wresnik (1990),
W. Wunderlich (1935),
and others.

Osculating Tangents

If a line t meets F at a simple point P with multiplicity ≥ 3 then it is called an *osculating tangent* at P . Such a tangent line is either a generator or it meets F at P only. In the latter case it will be called a *proper osculating tangent* of F .

Lemma. *At each point $P(u_1, u_2) \in F \setminus g_\infty$ there is a unique proper osculating tangent, namely the line which joins $P(u_1, u_2)$ with the point $K(0, 1, 3u_1, u_2)^T$.*

Proof. The tangent plane of F at $P(0, 0)$ is $\mathcal{V}(X_3)$; this plane meets F along the line $\mathcal{V}(X_1, X_3)$ and the parabola given by

$$\mathcal{V}(X_1(X_0X_2 - X_1^2), X_3). \quad (1)$$

The tangent t of this parabola at $P(0, 0)$ is easily seen to be the only proper osculating tangent at $P(0, 0)$. The point at infinity of t is $K(0, 1, 0, 0)^T$. By the action of the matrix $M_{u_1, u_2, 1} \in G$ the assertion follows for any point $P(u_1, u_2) \in F \setminus g_\infty$. \square

Main Theorem (cont.)

Main Theorem. *The set*

$$\mathcal{O} := \{t \in \mathcal{L} \mid t \text{ is a proper osculating tangent of } F\} \cup \{g_\infty\}$$

has the following properties:

1. *\mathcal{O} is a partial spread of $\mathbb{P}_3(K)$ if, and only if, $\text{Char } K \neq 3$ and K does not contain a third root of unity other than 1.*

Proof

Ad 1. All proper osculating tangents are skew to g_∞ . The osculating tangents at $P(0,0) \neq P(u_1, u_2)$ are skew if, and only if,

$$\Delta(u_1, u_2) := \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & u_1 & 1 \\ 0 & 0 & u_2 & 3u_1 \\ 0 & 0 & u_1u_2 - u_1^3 & u_2 \end{pmatrix} = u_2^2 - 3u_1^2u_2 + 3u_1^4 \neq 0.$$

For $u_1 = 0$ we have $u_2 \neq 0$ so that $\Delta(u_1, u_2) \neq 0$.

For $u_1 \neq 0$ we substitute $u_2 = (2 + y)u_1^2$ with $y \in K$ and obtain the equivalent condition $u_1^4(y^2 + y + 1) \neq 0$. But the polynomial

$$X^2 + X + 1 \in K[X]$$

has a zero in K precisely when:

- Char $K = 3$, since in this case $X^2 + X + 1 = (X - 1)^2$;
- Char $K \neq 3$ and there exists a third root of unity $w \neq 1$ in K .

Main Theorem (cont.)

Main Theorem. *The set*

$$\mathcal{O} := \{t \in \mathcal{L} \mid t \text{ is a proper osculating tangent of } F\} \cup \{g_\infty\}$$

has the following properties:

1. *\mathcal{O} is a partial spread of $\mathbb{P}_3(K)$ if, and only if, $\text{Char } K \neq 3$ and K does not contain a third root of unity other than 1.*
2. *If \mathcal{O} is a partial spread then it is maximal, i.e., it is not a proper subset of any partial spread of $\mathbb{P}_3(K)$.*

Proof (cont.)

Ad 2. It suffices to show that each point at infinity is on a line of \mathcal{O} .

This is obviously true for any point on $g_\infty \in \mathcal{O}$.

As (u_1, u_2) varies in K^2 , the osculating tangent at $P(u_1, u_2)$ contains the point

$$K(0, 1, 3u_1, u_2)^T \in \omega.$$

By part 1, we have $\text{Char } K \neq 3$. This means that each point of $\omega \setminus g_\infty$ is incident with a line of \mathcal{O} .

Main Theorem (cont.)

Main Theorem. *The set*

$$\mathcal{O} := \{t \in \mathcal{L} \mid t \text{ is a proper osculating tangent of } F\} \cup \{g_\infty\}$$

has the following properties:

1. *\mathcal{O} is a partial spread of $\mathbb{P}_3(K)$ if, and only if, $\text{Char } K \neq 3$ and K does not contain a third root of unity other than 1.*
2. *If \mathcal{O} is a partial spread then it is maximal, i.e., it is not a proper subset of any partial spread of $\mathbb{P}_3(K)$.*
3. *\mathcal{O} is a covering of $\mathbb{P}_3(K)$ if, and only if, $\text{Char } K \neq 3$ and each element of K has a third root in K .*

Proof (cont.)

Ad 3.

- Let $\text{Char } K = 3$.

By part 2, \mathcal{O} is not even a covering of the plane at infinity.

- Let $\text{Char } K \neq 3$.

A point $K(1, p_1, p_2, p_3)$ is on a line of \mathcal{O} if, and only if, there is a pair $(u_1, u_2) \in K^2$ and an $s \in K$ such that

$$(1, p_1, p_2, p_3)^T = (1, u_1, u_2, u_1 u_2 - u_1^3)^T + s(0, 1, 3u_1, u_2)^T.$$

So we obtain the following system of equations in the unknowns $u_1, u_2, s \in K$:

$$u_1 = p_1 - s, \quad u_2 = p_2 - 3s(p_1 - s), \quad s^3 = p_3 - (p_1 p_2 - p_1^3).$$

This system has a solution, because $p_3 - (p_1 p_2 - p_1^3)$ has a third root in K . \square

Remarks

The line set \mathcal{O} has the following further properties.

- Let $\text{Char } K \neq 3$.

\mathcal{O} admits a projective correlation which leaves \mathcal{O} invariant, as a set. Hence \mathcal{O} is a **spread** if, and only if, it is a **dual spread**.

- Let $\text{Char } K = 3$.

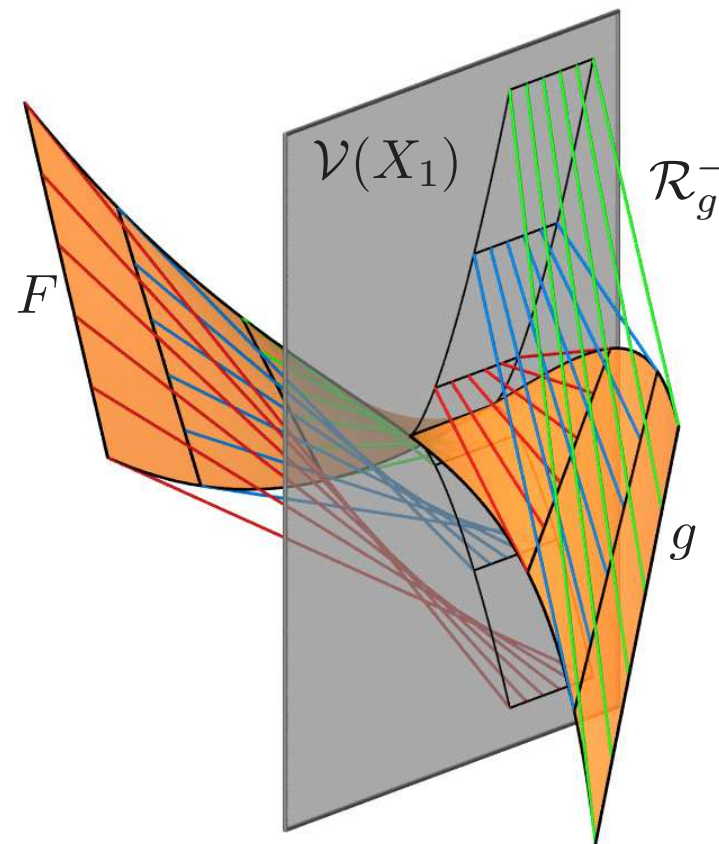
Here \mathcal{O} is part of a **parabolic linear congruence**. The axis of this congruence is a **line of nuclei** of the Cayley surface.

(Cf. M. de Finis, M.-J. de Resmini, 1983.)

Covering \mathcal{O} with Reguli

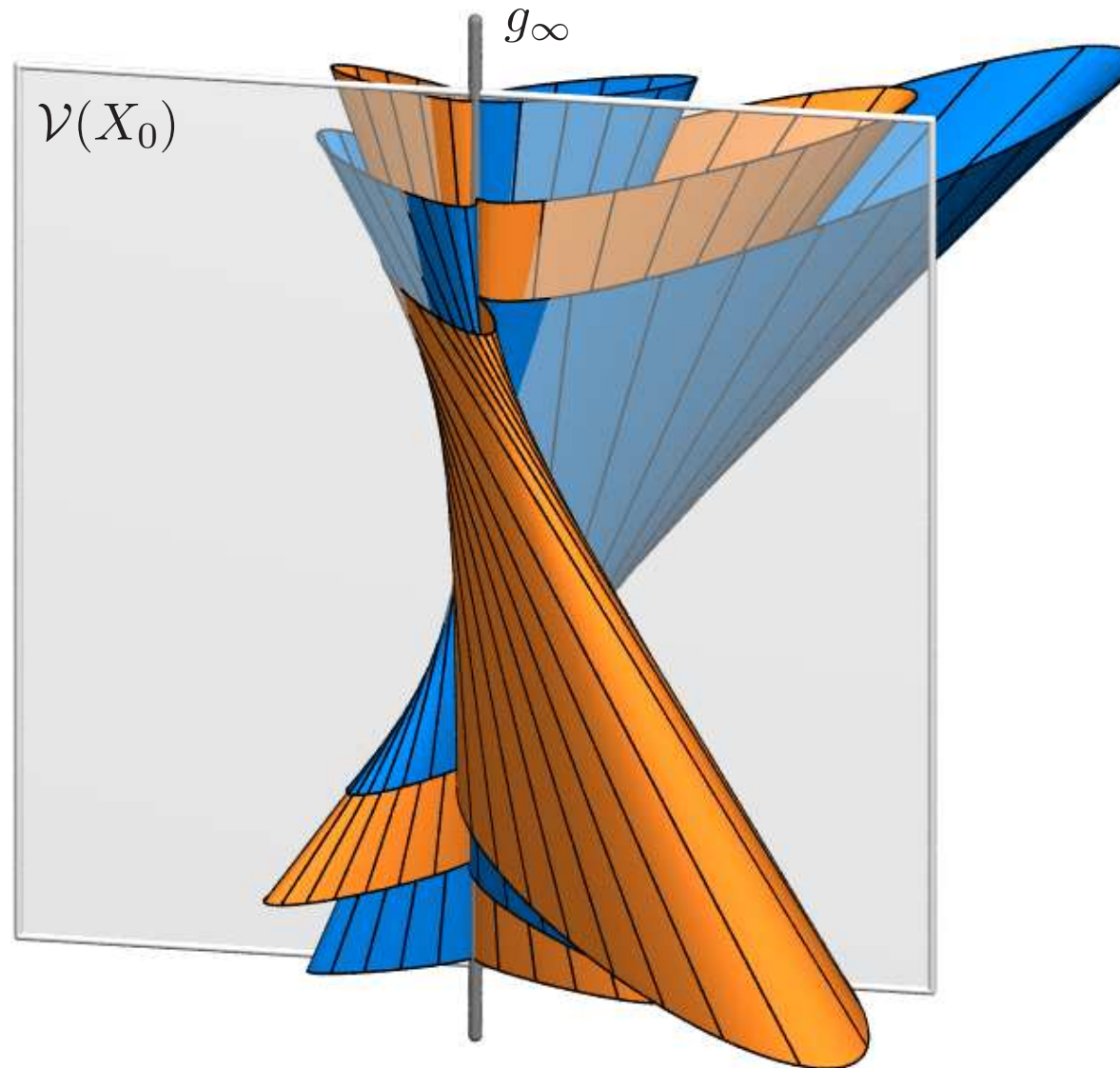
The following assertions can be verified by straightforward calculations:

- All osculating tangents at the points of a generator $g \neq g_\infty$ together with g_∞ form a **regulus** \mathcal{R}_g^- , say. In affine terms this regulus is one family of generators on a hyperbolic paraboloid \mathcal{H}_g .
- The hyperbolic paraboloid \mathcal{H}_g is the **Lie quadric** of F along the generator g .
- Given generators $g, g' \neq g_\infty$ the reguli \mathcal{R}_g^- and $\mathcal{R}_{g'}^-$ have only the line g_∞ in common.
- Given generators $g, g' \neq g_\infty$ the Lie quadrics \mathcal{H}_g and $\mathcal{H}_{g'}$ have the same tangent plane at each point of g_∞ .



Covering \mathcal{O} with Reguli

Here is another picture, where $\mathcal{V}(X_3)$ appears at infinity:



Final Remarks

- The *Betten-Walker spread* \mathcal{O} appears in the literature under various names.
- The Betten-Walker spread in $\mathbb{P}_3(\mathbb{R})$ yields a 4-dimensional translation plane. (D. Betten).
- Let K be infinite. The union of \mathcal{O} with the pencil $\mathcal{L}(Z, \omega)$ is the smallest algebraic set of lines containing \mathcal{O} .

So, for example, the Betten-Walker spread in $\mathbb{P}_3(\mathbb{R})$ is **not an algebraic spread**, but it is very “close” to being algebraic.

- Only few algebraic spreads of $\mathbb{P}_3(\mathbb{R})$ seem to be known. Non-regular examples are due to R. Riesinger.
- For further details see: H. H. and R. Riesinger, The Betten-Walker spread and Cayley’s ruled cubic surface, *Beitr. Algebra Geometrie* **47** (2006), 527–541.