A note on Segre varieties in characteristic two

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Our Segre varieties

Let $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_m$ be $m \geq 1$ two-dimensional vector spaces over a commutative field F.

$\mathbb{P}(\boldsymbol{V}_k) = \mathrm{PG}(1, F)$ are projective lines over F for $k \in \{1, 2, \ldots, m\}$.

The non-zero decomposable tensors of $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ determine the *Segre variety*

$$\mathcal{S}_{\underbrace{1,1,\ldots,1}_{m}}(F) = \mathcal{S}_{(m)}(F) = \left\{ F\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \cdots \otimes \boldsymbol{a}_m \mid \boldsymbol{a}_k \in \boldsymbol{V}_k \setminus \{0\} \right\}$$

with ambient projective space $\mathbb{P}\left(\bigotimes_{k=1}^{m} \boldsymbol{V}_k\right) = \mathrm{PG}(2^m - 1, F)$.

Bases

Given a basis $(\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)})$ for each vector space $\boldsymbol{V}_k$, $k \in \{1, 2, \ldots, m\}$, the tensors

$$\boldsymbol{E}_{i_1, i_2, \ldots, i_m} := \boldsymbol{e}_{i_1}^{(1)} \otimes \boldsymbol{e}_{i_2}^{(2)} \otimes \cdots \otimes \boldsymbol{e}_{i_m}^{(m)} \quad \text{with} \quad (i_1, i_2, \ldots, i_m) \in I_m := \{0, 1\}^m \tag{1}$$

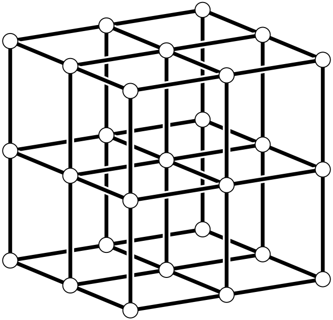
constitute a basis of $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$.

For any multi-index $\boldsymbol{i} = (i_1, i_2, \ldots, i_m) \in I_m$ the *opposite* multi-index $\boldsymbol{i}' \in I_m$ is characterised by

$$i_k \neq i'_k \quad \text{for all} \quad k \in \{1, 2, \ldots, m\}.$$

Examples

- $\mathcal{S}_1(F) = \mathrm{PG}(1, F)$.
- $\mathcal{S}_{1,1}(F)$ is a hyperbolic quadric of $\mathrm{PG}(3, F)$.
- $\mathcal{S}_{1,1,1}(2)$ has 27 points and contains precisely 27 lines (three through each point). The ambient $\mathrm{PG}(7, 2)$ has 255 points.



Collineations

The subgroup of $\mathrm{GL}\left(\bigotimes_{k=1}^{m} \boldsymbol{V}_k\right)$ preserving decomposable tensors is generated by the following transformations:

$$f_1 \otimes f_2 \otimes \cdots \otimes f_m \quad \text{with} \quad f_k \in \mathrm{GL}(\boldsymbol{V}_k) \text{ for } k \in \{1, 2, \ldots, m\}. \qquad (2)$$

$$f_\sigma \quad \text{with} \quad \boldsymbol{E}_{(i_1, i_2, \ldots, i_m)} \mapsto \boldsymbol{E}_{(i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \ldots, i_{\sigma^{-1}(m)})} \quad \text{for all} \quad \boldsymbol{i} \in I_m, \qquad (3)$$

where $\sigma \in S_m$ is arbitrary.

This subgroup induces the stabiliser $G_{\mathcal{S}_{(m)}(F)}$ of the Segre $\mathcal{S}_{(m)}(F)$ within the projective group $\mathrm{PGL}\left(\bigotimes_{k=1}^{m} \boldsymbol{V}_k\right)$.

Bilinear forms

Each of the vector spaces $\boldsymbol{V}_k$ admits a symplectic bilinear form

$$[\cdot,\cdot] : \boldsymbol{V}_k \times \boldsymbol{V}_k \to F.$$

Consequently, $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ is equipped with a bilinear form which is given by

$$\left[\boldsymbol{a}_1 \otimes \boldsymbol{a}_2 \otimes \cdots \otimes \boldsymbol{a}_m, \boldsymbol{b}_1 \otimes \boldsymbol{b}_2 \otimes \cdots \otimes \boldsymbol{b}_m\right] := \prod_{k=1}^{m} [\boldsymbol{a}_k, \boldsymbol{b}_k] \quad \text{for } \boldsymbol{a}_k, \boldsymbol{b}_k \in \boldsymbol{V}_k, \qquad (4)$$

and extending bilinearly.

All these bilinear forms are unique up to a non-zero factor in F.

Bilinear forms (cont.)

Given $\boldsymbol{i}, \boldsymbol{j} \in I_m$ we have

$$
\begin{aligned}
[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] &= \prod_{k=1}^{m} [\boldsymbol{e}^{(k)}_{i_k}, \boldsymbol{e}^{(k)}_{i'_k}] = (-1)^m [\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{E}_{\boldsymbol{i}}] \neq 0, \qquad (5) \\
[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{j}}] &= 0 \quad \text{for all} \quad \boldsymbol{j} \neq \boldsymbol{i}'. \qquad (6)
\end{aligned}
$$

Hence the form $[\cdot, \cdot]$ on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ is non-degenerate.
Furthermore, it is

- symmetric when m is even and $\operatorname{Char} F \neq 2$;
- alternating otherwise (*i. e.*, when m is odd or $\operatorname{Char} F = 2$).

The fundamental polarity

In projective terms the form $[\cdot,\cdot]$ on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ (or any proportional one) determines the fundamental polarity of the Segre $\mathcal{S}_{(m)}(F)$, *i. e.*, a polarity of $\mathbb{P}(\bigotimes_{k=1}^{m} \boldsymbol{V}_k)$ which sends $\mathcal{S}_{(m)}(F)$ to its dual.

This polarity is

- associated with a regular quadric when m is even and $\mathrm{Char}\, F \neq 2$;
- null otherwise (*i. e.*, when m is odd or $\mathrm{Char}\, F = 2$).

Characteristic two

Let $\mathrm{Char}\, F = 2$.

Here $[\cdot, \cdot]$ is a symplectic bilinear form on $\bigotimes_{k=1}^{m} \boldsymbol{V}_k$ for all $m \geq 1$, whence the fundamental polarity of the Segre $\mathcal{S}_{(m)}(F)$ is always null.

Furthermore, (5) simplifies to

$$[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] = \prod_{k=1}^{m} [\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] = [\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{E}_{\boldsymbol{i}}] \neq 0. \tag{7}$$

A quadratic form

Proposition

Let $m \geq 2$ and $\mathrm{Char}\, F = 2$. Then there is a unique quadratic form

$$Q : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F$$

satisfying the following two properties:

1. *Q vanishes for all decomposable tensors.*
2. *The symplectic bilinear form*

$$[\cdot,\cdot] : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \times \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F$$

is the polar form of Q.

Proof (sketched)

We denote by $I_{m,0}$ the set of all multi-indices $(i_1, i_2, \ldots, i_m) \in I_m$ with $i_1 = 0$.

In terms of our basis (1) a quadratic form is given by

$$Q : \bigotimes_{k=1}^{m} \boldsymbol{V}_k \to F : \boldsymbol{X} \mapsto \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{X}][\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{X}]}{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}]}. \tag{8}$$

Proof (cont.)

Given an arbitrary decomposable tensor we have

$$\begin{aligned} Q(\boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m) &= \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m][\boldsymbol{E}_{\boldsymbol{i}'}, \boldsymbol{a}_1 \otimes \cdots \otimes \boldsymbol{a}_m]}{[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}]} \\ &= \sum_{\boldsymbol{i} \in I_{m,0}} \frac{[\mathbf{e}_0^{(1)}, \boldsymbol{a}_1][\mathbf{e}_1^{(1)}, \boldsymbol{a}_1] \cdots [\mathbf{e}_0^{(m)}, \boldsymbol{a}_m][\mathbf{e}_1^{(m)}, \boldsymbol{a}_m]}{[\mathbf{e}_0^{(1)}, \mathbf{e}_1^{(1)}] \cdots [\mathbf{e}_0^{(m)}, \mathbf{e}_1^{(m)}]} \\ &= 2^{m-1} \frac{[\mathbf{e}_0^{(1)}, \boldsymbol{a}_1][\mathbf{e}_1^{(1)}, \boldsymbol{a}_1] \cdots [\mathbf{e}_0^{(m)}, \boldsymbol{a}_m][\mathbf{e}_1^{(m)}, \boldsymbol{a}_m]}{[\mathbf{e}_0^{(1)}, \mathbf{e}_1^{(1)}] \cdots [\mathbf{e}_0^{(m)}, \mathbf{e}_1^{(m)}]} \\ &= 0, \end{aligned}$$

where we used $\# I_{m,0} = 2^{m-1}$, $m \geq 2$, and $\operatorname{Char} F = 2$.

Explicit equation

From (8), the quadratic form Q can be written in terms of tensor coordinates $x_{\boldsymbol{j}} \in F$ as

$$Q\Big(\sum_{\boldsymbol{j} \in I_m} x_{\boldsymbol{j}} \boldsymbol{E}_{\boldsymbol{j}}\Big) = \sum_{\boldsymbol{i} \in I_{m,0}} [\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] x_{\boldsymbol{i}} x_{\boldsymbol{i}'} = \prod_{k=1}^{m} [\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] \cdot \sum_{\boldsymbol{i} \in I_{m,0}} x_{\boldsymbol{i}} x_{\boldsymbol{i}'}. \tag{9}$$

Remarks

The previous results may be slightly simplified by taking symplectic bases, *i. e.*,

$$[\boldsymbol{e}_0^{(k)}, \boldsymbol{e}_1^{(k)}] = 1 \text{ for all } k \in \{1, 2, \ldots, m\},$$

whence also

$$[\boldsymbol{E}_{\boldsymbol{i}}, \boldsymbol{E}_{\boldsymbol{i}'}] = 1 \text{ for all } \boldsymbol{i} \in I_m.$$

Proposition 1 fails to hold for $m = 1$: A quadratic form Q vanishing for all decomposable tensors of $\boldsymbol{V}_1$ is necessarily zero, since any element of $\boldsymbol{V}_1$ is decomposable. Hence the polar form of such a Q cannot be non-degenerate.

Main result

Theorem

Let $m \geq 2$ and $\mathrm{Char}\, F = 2$. There exists in the ambient space of the Segre $\mathcal{S}_{(m)}(F)$ a regular quadric $\mathcal{Q}(F)$ with the following properties:

1. *The projective index of $\mathcal{Q}(F)$ is $2^{m-1} - 1$.*
2. *$\mathcal{Q}(F)$ is invariant under the group of projective collineations stabilising the Segre $\mathcal{S}_{(m)}(F)$.*

Conclusion

We call $\mathcal{Q}(F)$ the *invariant quadric* of the Segre $\mathcal{S}_{(m)}(F)$.

The case $m = 2$ deserves special mention, as the Segre $\mathcal{S}_{1,1}(F)$ coincides with its invariant quadric $\mathcal{Q}(F)$ given by

$$Q\Big(\sum_{\boldsymbol{j}\in I_2} x_{\boldsymbol{j}} \boldsymbol{E}_{\boldsymbol{j}}\Big) = x_{00}x_{11} + x_{01}x_{10} = 0.$$

This result parallels the situation for $\operatorname{Char} F \neq 2$.

Problem: Is there a "better" definition of the quadratic form Q?

References

This presentation:

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On invariant notions of Segre varieties in binary projective spaces.
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