From geometry to invertibility preservers

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Geometry of Matrices

Let $M_{m,n}$, $m, n \geq 2$, be the vector space of all $m \times n$ matrices over a field $\mathbb{F}$.

Two matrices (linear operators) A and B are *adjacent* if $A - B$ is of rank one.

We may consider $M_{m,n}$ as an undirected graph the edges of which are precisely the (unordered) pairs of adjacent matrices.

Two matrices A and B are at the graph-theoretical distance $k \geq 0$ if, and only if

$$\operatorname{rank}(A - B) = k.$$

On the other hand we may consider $M_{m,n}$ as an affine space. Its lines fall into $\min\{m, n\}$ classes, according to the rank of a "direction vector".

Hua's Theorem

Fundamental Theorem (1951). Every bijective map $\varphi : M_{m,n} \to M_{m,n} : A \mapsto \varphi(A)$ preserving adjacency in both directions is of the form

$$A \mapsto TA_\sigma S + R,$$

where T is an invertible $m \times m$ matrix, S is an invertible $n \times n$ matrix, R is an $m \times n$ matrix, and σ is an automorphism of the underlying field.

If $m = n$, then we have the additional possibility that

$$A \mapsto TA_\sigma^t S + R$$

where T, S, R and σ are as above, and A^t denotes the transpose of A.

The assumptions in Hua's fundamental theorem can be weakened.
W.-l. Huang and Z.-X. Wan: Beiträge Algebra Geom. 45 (2004), no. 2, 435–446.

Grassmann Spaces

Let m, n be integers ≥ 2. We consider the Grassmannian $G_{m+n,m}$ whose elements are vector subspaces of $\mathbb{F}^{m+n}$ of dimension m. Alternatively, the point of view of projective geometry may be adopted.

Two m-dimensional subspaces U and V are *adjacent* if $\dim(U+V) = m+1$.

As before, we obtain a graph known as the *Grassmann graph* of $G_{m+n,m}$.

Two subspaces U and V are at graph-theoretical distance k if, and only if,

$$\dim(U+V) = m+k,$$

whence $k \leq \min\{m, n\}$.

On the other hand, we may consider $G_{m+n,m}$ as the "point set" of a *Grassmann space*. Its "lines" are the pencils of k-subspaces.

Chow's Theorem

Chow's Theorem (1949). Every bijective map $\varphi : G_{m+n,n} \to G_{m+n,n} : U \mapsto \varphi(U)$ preserving adjacency in both directions is induced by a semilinear mapping

$$f : \mathbb{F}^{m+n} \to \mathbb{F}^{m+n} : x \mapsto Lx_\sigma \text{ such that } \varphi(U) = f(U),$$

where L is an invertible $(m+n) \times (m+n)$ matrix, and σ is an automorphism of the underlying field.

If $m = n$ we have the additional possibility that φ is induced by a sesquilinear form

$$g : \mathbb{F}^{m+n} \times \mathbb{F}^{m+n} \to \mathbb{F} : (x, y) \mapsto x_\sigma^t Ly \text{ such that } U \perp_g \varphi(U),$$

where L and σ are as above.

The assumptions in Chow's theorem can be weakened.
W.-l. Huang: Abh. Math. Sem. Univ. Hamburg 68 (1998), 65–77.

Coordinates

To each m-dimensional subspace U of $\mathbb{F}^{m+n}$ we can associate an $m \times (m+n)$ matrix whose rows form a basis of U. This matrix can be written in block form as

$$[X\ Y]$$

where X, Y are of size $m \times n$ and $m \times m$, respectively.

Two matrices $[X\ Y]$ and $[X'\ Y']$, each with rank m, are associated to the same U if, and only if,

$$[X\ Y] = P[X'\ Y']$$

for some invertible $m \times m$ matrix P.

This gives *"homogeneous coordinates"* for the Grassmann space. For $m = n$ we obtain the projective line over the ring of $m \times m$ matrices.

Connection

Let U be a point of the Grassmann space and $[X\ Y]$ an associated matrix:

- U is *at infinity* if Y is not invertible.
- U is a *finite* point otherwise. Hence it can be written uniquely in the form $[A\ I]$, where A is an $m \times n$ matrix and I is the identity matrix.

The mapping $U \mapsto A$ is a bijection from the set of finite points of the Grassmann space $G_{m+n,n}$ onto the space $M_{m,n}$; adjacency is preserved in both directions.

Alternative point of view: Stereographic projection of a Grassmann variety (folklore). Cf. also: R. Metz: Geom. Dedicata 10 (1981), no. 1-4, 337–367.

Full Rank Differences

Let $\mathbb{F}$ be a field with at least three elements and m, n integers with $m \geq n \geq 2$.

Given $A, B \in M_{m,n}$ we write $A \mathbin{\triangle} B$ if $A - B$ is of full rank (i.e., the rank equals n).

For two finite points U, V of the Grassmann space $\mathbb{F}^{m+n}$ the sum

$$U + V \quad \text{is direct}$$

(i. e. they meet at 0 only) if, and only if, their associated matrices A, B satisfy

$$A \mathbin{\triangle} B.$$

Full Rank Preservers

Theorem 1. Assume that $\varphi : M_{m,n} \to M_{m,n}$ is a bijective map such that for every pair $A, B \in M_{m,n}$ we have

$$A \vartriangle B \Leftrightarrow \varphi(A) \vartriangle \varphi(B).$$

Then adjacency is preserved under φ in both directions.

Consequently, Hua's theorem can be applied and all such mappings can be described explicitly as before.

Cf. A. Blunck, H. H.: Discrete Math. 301 (2005), no. 1, 46–56.

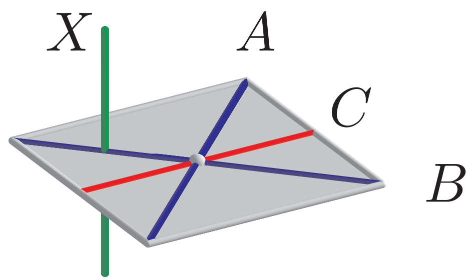
Sketch of the Proof

Proposition. Let $A, B \in M_{m,n}$ be matrices with $A \neq B$. Then the following are equivalent:

1. A and B are adjacent.
2. There exists $C \in M_{m,n}$, $C \neq A, B$, such that for every $X \in M_{m,n}$

$$\text{the relation } X \vartriangle C \text{ yields } X \vartriangle A \text{ or } X \vartriangle B.$$

Geometric idea behind the proof $(m = n = 2)$:



Hilbert Spaces

Let H be an infinite-dimensional complex Hilbert space and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H.

Given $A, B \in \mathcal{B}(H)$ we write $A \vartriangle B$ if $A - B$ is invertible.

Then it is possible to characterise all *invertibility preservers*, i. e., all bijective mappings $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ with the following property:

For every pair $A, B \in \mathcal{B}(H)$ we have

$$A - B \text{ is invertible } \Leftrightarrow \varphi(A) - \varphi(B) \text{ is invertible.}$$

Invertibility Preservers

Theorem 2. Let H be an infinite-dimensional complex Hilbert space and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. Assume that $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ is an invertibility preserver.

Then there exist $R \in \mathcal{B}(H)$ and invertible $T, S \in \mathcal{B}(H)$ such that either

$$\varphi(A) = TAS + R$$

for every $A \in \mathcal{B}(H)$, or

$$\varphi(A) = TA^tS + R$$

for every $A \in \mathcal{B}(H)$, or

$$\varphi(A) = TA^*S + R$$

for every $A \in \mathcal{B}(H)$, or

$$\varphi(A) = T(A^t)^*S + R$$

for every $A \in \mathcal{B}(H)$.

Here, A^t and A^* denote the transpose with respect to an arbitrary but fixed orthonormal basis, and the usual adjoint of A in the Hilbert space sense, respectively.