Möbius differential geometry

(Version: July 17, 2003; letter format)

Basics

Möbius geometry is the geometry of the group of Möbius transformations, that is, hypersphere preserving (point) transformations, acting on the n-sphere S^n as a base manifold. The elements of Möbius geometry are points (elements of the first kind) and hyperspheres (elements of the second kind).

Models

Models serve a uniform description of the elements of (Möbius) geometry (points, hyperspheres) and derived objects (for example, k-spheres) as well as a description of the Möbius transformations as linear, fractional linear, or spin transformations.

The **classical (projective) model**: the conformal n-sphere as an absolute quadric $S^n \cong \{\mathbb{R}v \subset \mathbb{R}^{n+2}_1 \,|\, |v|^2 = 0\} \subset \mathbb{R}P^{n+1}$, the space of hyperspheres as the "outer space" $S^{n+1}_1/_{\pm 1} \subset \mathbb{R}P^{n+1}$; the Lorentz sphere $S^{n+1}_1 = \{v \in \mathbb{R}^{n+2}_1 \,|\, |v|^2 = 1\}$ can be interpreted as the space of *oriented* hyperspheres. Möbius transformations become Lorentz transformations, resp. projective transformations that preserve $S^n \subset \mathbb{R}P^{n+1}$.

The **quaternionic approach**: the conformal 4-sphere as the quaternionic projective line, $S^4 \cong \mathbb{H}P^1$, and the space of quaternionic Hermitian forms $\mathfrak{H}(\mathbb{H}^2) \cong \mathbb{R}^6_1$ with $|h|^2 = -\det h$ (w.r.t. some basis) so that 3-spheres are quaternionic Hermitian forms. Möbius involutions $S \in \mathfrak{S}(\mathbb{H}^2)$, $S^2 = -id$, are 2-spheres. Orientation preserving Möbius transformations are fractional linear transformations, or special linear transformations (on homogeneous coordinates $v \in \mathbb{H}^2$).

A **Clifford algebra model**: the coordinate Minkowski space $\mathbb{R}^{n+2}_1$ of the projective model is embedded into its Clifford algebra $\mathcal{A}\mathbb{R}^{n+2}_1$. Möbius transformations are (s)pin transformations.

The **Vahlen matrix approach**: the Clifford algebra $\mathcal{A}\mathbb{R}^{n+2}_1$ is described in terms of 2×2-matrices with entries from the Clifford algebra $\mathcal{A}\mathbb{R}^n$ of Euclidean n-space. Möbius transformations are fractional linear transformations, given by Vahlen matrices.

Points

We consider $\mathbb{R}^{n+2}_1 = \mathbb{R}\times\mathbb{R}^{n+1} = \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$ with the Minkowski product $\langle (y_0, y), (y_0, y)\rangle = -y_0^2 + |y|^2$. The following are descriptions of points in different models.

As points of the absolute quadric in the projective model:

$$\left.\begin{array}{rcl} \mathbb{R}^{n+1} \supset S^n \ni y & \leftrightarrow & \mathbb{R}\,(1, y) \\ \mathbb{R}^n \ni x & \mapsto & \mathbb{R}\left(\frac{1+|x|^2}{2}, x, \frac{1-|x|^2}{2}\right) \end{array}\right\} \in S^n \subset \mathbb{R}P^{n+1}.$$

As quaternionic Hermitian forms, in the quaternionic approach ($\mathbb{R}^4 \cong \mathbb{H}$ can be identified with the affine slice $v_2 = 1$):

$$S^4 \cong \mathbb{H}P^1 \ni \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\mathbb{H} \quad \leftrightarrow \quad \mathbb{R}\begin{pmatrix} |v_2|^2 & -v_1\bar v_2 \\ -v_2 \bar v_1 & |v_1|^2 \end{pmatrix} \subset \mathfrak{H}(\mathbb{H}^2).$$

As 2×2-Clifford algebra matrices in the Vahlen matrix approach:

$$\mathbb{R}^n \ni x \quad \mapsto \quad \mathbb{R}\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} \subset \mathcal{A}\mathbb{R}^{n+2}_1 \cong M(2\times 2, \mathcal{A}\mathbb{R}^n).$$

Hyperspheres

A hypersphere with center $m \in S^n \subset \mathbb{R}^{n+1}$ and radius $\varrho \in (0, \pi)$:

$$S = \tfrac{1}{\sin\varrho}\,(\cos\varrho, m) \in S^{n+1}_1;$$

a change to $-m$ and $\pi - \varrho$ reverts the orientation.

A hypersphere with center $m \in \mathbb{R}^n$ and radius $r \neq 0$:

$$S = \tfrac{1}{r}\left(\tfrac{1+(|m|^2-r^2)}{2}, m, \tfrac{1-(|m|^2-r^2)}{2}\right) \in S^{n+1}_1,$$

and a hyperplane with normal $n \in S^{n-1} \subset \mathbb{R}^n$ and (directed) distance $d \in \mathbb{R}$ from the origin:

$$T = (d, n, -d) \in S^{n+1}_1;$$

as Vahlen matrices:

$$S = \tfrac{1}{r}\begin{pmatrix} m & -m^2-r^2 \\ 1 & -m \end{pmatrix}, \quad T = \begin{pmatrix} n & 2d \\ 0 & -n \end{pmatrix} \in \mathcal{A}\mathbb{R}^{n+2}_1;$$

and as quaternionic Hermitian forms:

$$S = \tfrac{1}{r}\begin{pmatrix} 1 & -m \\ -\bar m & |m|^2-r^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -n \\ -\bar n & 2d \end{pmatrix} \in \mathfrak{H}(\mathbb{H}^2).$$

2-spheres (or planes) in $\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$ as Möbius involutions:

$$S = \tfrac{1}{r}\begin{pmatrix} m & |m|^2-r^2 \\ 1 & \bar m \end{pmatrix}, \quad T = \begin{pmatrix} n & 2d \\ 0 & \bar n \end{pmatrix} \in \mathfrak{S}(\mathbb{H}^2).$$

Note that S (and T) are symmetric w.r.t. $\mathbb{R}^3 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: more generally, a 2-sphere $S \in \mathfrak{S}(\mathbb{H}^2)$ lies inside a 3-sphere $S^3 \in \mathfrak{H}(\mathbb{H}^2)$ iff S is symmetric w.r.t. S^3, $S^3(.,S.) = S^3(S.,.)$.

Incidence and intersection angle

A point $p \in S^n \subset \mathbb{R}P^{n+1}$ lies on a hypersphere $S \in S^{n+1}_1$ iff p is in the polar hyperplane of S w.r.t. S^n; in homogeneous coordinates, this is orthogonality:

$$p = \mathbb{R}v \in S \quad \Leftrightarrow \quad \langle v, S\rangle = 0.$$

In the Vahlen matrix description or the description of 2-spheres in $\mathbb{H}P^1$ as involutions, incidence can be expressed as

$$p \in S \quad \Leftrightarrow \quad p = S\cdot p$$

that is, $p \in \mathbb{R}^n \cup \{\infty\}$ (or $p \in \mathbb{H} \cup \{\infty\}$) is a fixed point of the inversion at S; in case $p = v\mathbb{H} \in \mathbb{H}P^1$ this can also be written

$$p = v\mathbb{H} \in S \quad \Leftrightarrow \quad \exists \lambda \in \mathbb{H} : Sv = v\lambda,$$

that is, $v \in \mathbb{H}^2$ is an eigenvector of $S \in \mathfrak{S}(\mathbb{H}^2)$. Incidence of a point $p = v\mathbb{H} \in \mathbb{H}P^1$ and a 3-sphere $S \in \mathfrak{H}(\mathbb{H}^2)$ is isotropy,

$$p = v\mathbb{H} \in S \quad \Leftrightarrow \quad S(v, v) = 0.$$

The intersection angle α of two hyperspheres $S_1, S_2 \in S^{n+1}_1$ is given by

$$\cos\alpha \;=\; \langle S_1, S_2\rangle \;=\; -\tfrac{1}{2}\{S_1, S_2\},$$

where $\{.,.\}$ is the anti-commutator in $\mathcal{A}\mathbb{R}^{n+2}_1$; in particular, orthogonal intersection becomes orthogonality.

Inversions

The inversion at a hypersphere $S \subset S^n$ is the polar reflection at $S \in \mathbb{R}P^{n+1}$; in homogeneous coordinates, $p = \mathbb{R}v$ and $S \in S^{n+1}_1$:

$$\mathbb{R}^{n+1}_1 \ni v \mapsto v - 2\langle v, S\rangle S = SvS \in \mathbb{R}^{n+1}_1 \subset \mathcal{A}\mathbb{R}^{n+1}_1.$$

In terms of Vahlen matrices,

$$\mathbb{R}^n \cup \{\infty\} \ni p \mapsto S\cdot p = \left\{\begin{array}{l} m - r^2(p-m)^{-1} \\ npn + 2dn \end{array}\right\} \in \mathbb{R}^n \cup \{\infty\}.$$

$Sl(2,\mathbb{H})$ does not provide (orientation reversing) inversions.

The Möbius group

The Möbius group $M\ddot{o}b(S^n)$ is the conformal group $Conf(S^n)$ of S^n; in the classical (projective) picture, this is the group of projective transformations that map $S^n \subset \mathbb{R}P^{n+1}$ to itself.

$O_1(n+2)$ is a (trivial) double cover of $M\ddot{o}b(S^n)$ with kernel $\{\pm id\}$; its identity component $SO^+_1(n+2)$ is isomorphic to the group $M\ddot{o}b^+(S^n)$ of orientation preserving Möbius transformations.

$Pin_1(n+2)$ is a double cover of $O_1(n+2)$ via the twisted adjoint action

$$Pin_1(n+2) \times \mathbb{R}^{n+2}_1 \ni (\mathfrak{s}, v) \mapsto \mathfrak{s}v\hat{\mathfrak{s}}^{-1} \in \mathbb{R}^{n+2}_1,$$

where $\hat{\ }$ is the order involution on $\mathcal{A}\mathbb{R}^{n+2}_1$,

$$\hat{\mathfrak{s}} = (-1)^k\mathfrak{s} \quad \text{for} \quad \mathfrak{s} = s_1\cdots s_k, \quad s_j \in \mathbb{R}^{n+1}_1;$$

$Spin^+_1(n+2)$ is the universal cover of $SO^+_1(n+2) \cong M\ddot{o}b^+(S^n)$; in terms of Vahlen matrices, Möbius transformations are fractional linear:

$$\mathbb{R}^n \cup \{\infty\} \ni p \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot p = (ap+b)(cp+d)^{-1} \in \mathbb{R}^n \cup \{\infty\}.$$

$Sl(2,\mathbb{H})$ is the double universal cover of $M\ddot{o}b^+(S^4)$; its action on $\mathbb{H}P^1 \cong \mathbb{H} \cup \{\infty\}$ is by fractional linear transformations,

$$Sl(2,\mathbb{H}) \times \mathbb{H}P^1 \ni (\mu, v\mathbb{H}) \mapsto (\mu v)\mathbb{H} \in \mathbb{H}P^1,$$

and on $\mathfrak{H}(\mathbb{H}^2)$ it is given by

$$Sl(2,\mathbb{H}) \times \mathfrak{H}(\mathbb{H}^2) \ni (\mu, S) \mapsto S(\mu^{-1}., \mu^{-1}.) \in \mathfrak{H}(\mathbb{H}^2).$$

Any (orientation preserving) Möbius transformation is the composition of (an even number of) inversions at hyperspheres.

Spheres of arbitrary dimension

A sphere $S \subset S^n$ of dimension $k < n$ can be identified with

– the projective $(k+1)$-plane that intersects S^n in the k-sphere: this plane is spanned by $k+2$ points $p_i = \mathbb{R}v_i \in S^n$ in "general position,"
$$S = v_1 \wedge \ldots \wedge v_{k+2} \in \mathcal{A}\mathbb{R}^{n+2}.$$

– the space of all hyperspheres that contain S, or the projective $(n-k-1)$-plane that contains these hyperspheres, respectively: this plane does not intersect S^n and can be spanned by $n-k$ orthogonal hyperspheres S_j, that is, S is the orthogonal intersection of the S_j,
$$S = S_1 \wedge \ldots \wedge S_{n-k} = S_1 \cdots S_{n-k} \in Pin(\mathbb{R}^{n+1}_1) \subset \mathcal{A}\mathbb{R}^{n+1}_1;$$
S can be interpreted as a Möbius involution with
$$S \in \Lambda^{n-k}\mathbb{R}^{n+2}_1 \quad \text{and} \quad S^2 = (-1)^{\binom{n-k}{2}},$$
which conforms with the identification of $\mathfrak{S}(\mathbb{H}^2)$ with the space of 2-spheres in $S^4 \cong \mathbb{H}P^1$.

The passage from one description to the other is

– by polarity w.r.t. $S^n \subset \mathbb{R}P^{n+1}$ in the projective picture,

– by taking orthogonal complements in $\mathbb{R}^{n+2}_1$, or

– by taking the Clifford dual (or, the Hodge dual) in $\mathcal{A}\mathbb{R}^{n+2}_1$.

Sphere pencils and complexes

A sphere pencil consists of all spheres on a line in $\mathbb{R}P^{n+1}$; it is

- elliptic if the line does not intersect S^n ($\Leftrightarrow |S_1 \wedge S_2|^2 > 0$ for any two hyperspheres $S_1 \neq S_2$ in the pencil), that is, all spheres intersect in a codimension 2 sphere;
- parabolic if the line touches S^n ($\Leftrightarrow |S_1 \wedge S_2|^2 = 0$ for S_1, S_2 in the pencil), that is, all spheres touch in a point (the point of contact with S^n) and form a "contact element";
- hyperbolic if the line intersects S^n ($\Leftrightarrow |S_1 \wedge S_2|^2 < 0$ for any two hyperspheres $S_1 \neq S_2$ in the pencil), that is, all spheres have one intersection point of the line with S^n as their center when interpreting the other as ∞, $S^n \setminus \{\infty\} \cong \mathbb{R}^n$, and the pencil can be identified with this "point pair."

A (linear) sphere complex consists of all spheres S in the polar hyperplane of a point $\mathbb{R}\mathcal{K} \in \mathbb{R}P^{n+1}$, $S \perp \mathcal{K}$; it is called

- elliptic if $\mathcal{K}$ lies outside S^n, $|\mathcal{K}|^2 > 0$;
- parabolic if $\mathcal{K}$ lies on S^n, $|\mathcal{K}|^2 = 0$; and
- hyperbolic if $\mathcal{K}$ lies inside S^n, $|\mathcal{K}|^2 < 0$.

These sphere complexes describe the hyperplanes of the hyperbolic, Euclidean, and spherical subgeometries of Möbius geometry, respectively.

Quadrics of constant curvature

Given $\mathcal{K} \in \mathbb{R}^{n+2}_1 \setminus \{0\}$, the quadric

$$Q_\kappa = \{p \in \mathbb{R}^{n+2}_1 \,|\, |p|^2 = 0 \text{ and } \langle p, \mathcal{K}\rangle = -1\}$$

has constant sectional curvature $\kappa = -|\mathcal{K}|^2$. The standard ball models $B^n_\kappa = (\{x \in \mathbb{R}^n \,|\, 1 + \kappa|x|^2 > 0\}, \frac{4|dx|^2}{(1+\kappa|x|^2)^2})$ of constant curvature κ spaces are isometrically embedded by

$$B^n_\kappa \ni x \quad\mapsto\quad \left(\tfrac{1+|x|^2}{1+\kappa|x|^2}, \tfrac{2x}{1+\kappa|x|^2}, \tfrac{1-|x|^2}{1+\kappa|x|^2}\right) \in Q_\kappa,$$

where $\mathcal{K} = (\frac{\kappa+1}{2}, 0, \frac{\kappa-1}{2})$; the spheres $S^n(r)$ embed via

$$\mathbb{R}^{n+1} \supset S^n(r) \ni y \quad\mapsto\quad (r, y) \in Q_{1/r^2}, \quad \mathcal{K} = (\tfrac{1}{r}, 0, 0).$$

The (mean) curvature H of a hypersphere $S \in S^{n+2}_1$ is given by

$$H = -\langle S, \mathcal{K}\rangle,$$

in particular, S is a hyperplane in Q_κ iff S is a sphere of the sphere complex $\mathcal{K}$, $S \perp \mathcal{K}$.

A k-sphere $S = S_1 \wedge \ldots \wedge S_{n-k}$ is a k-plane in Q_κ iff all

$$S_j \perp \mathcal{K} \quad\Leftrightarrow\quad \mathcal{K} \in \operatorname{span}\{v_i \,|\, i = 1, \ldots, k+2\}$$

for $k+2$ points $p_i = \mathbb{R}v_i \in S$ in general position.

A 2-sphere $S \in \mathfrak{S}(\mathbb{H}^2)$ is a 2-plane in Q_κ given by $\mathcal{K} \in \mathfrak{H}(\mathbb{H}^2)$ iff S is skew w.r.t. $\mathcal{K}$; more generally, its mean curvature is given by

$$|H|^2 = |\mathcal{K}_S|^2, \quad \text{where} \quad \mathcal{K}_S = \tfrac{1}{2}(\mathcal{K}(.,S.) + \mathcal{K}(S.,.)).$$

A Möbius transformation that fixes the sphere complex $\mathcal{K}$ (the hyperplanes of Q_κ) is an isometry of Q_κ if $\kappa \neq 0$ or a similarity of Q_0, respectively;

$$Isom(Q_\kappa) = \{\mu \in O_1(n+2) \,|\, \mu(\mathcal{K}) = \mathcal{K}\}$$

is the group of isometries of Q_κ — in case $\kappa < 0$, it is the group of isometries that extend smoothly through the infinity sphere $\mathbb{R}\mathcal{K}$.

Stereographic projection

Let $\mathcal{K}_0 = (1, 0, -1) \in Q_1$ be the "south pole" in the round n-sphere $S^n \cong Q_1$ given by $\mathcal{K}_1 = (1, 0, 0)$;

$$Q_0 \ni \left(\tfrac{1+|x|^2}{2}, x, \tfrac{1-|x|^2}{2}\right) \quad\mapsto\quad \left(1, \tfrac{2x}{1+|x|^2}, \tfrac{1-|x|^2}{1+|x|^2}\right) \in Q_1$$

$$Q_1 \setminus \{\mathcal{K}_0\} \ni (1, y_1, y_2) \quad\mapsto\quad \left(\tfrac{1}{1+y_2}, \tfrac{y_1}{1+y_2}, \tfrac{y_2}{1+y_2}\right) \in Q_0$$

then yields the classical stereographic projection. More generally,

$$S^n \ni \mathbb{R}v \mapsto -\tfrac{v}{\langle v, \mathcal{K}\rangle} \in Q_\kappa$$

can be considered as a stereographic projection from (part of) the conformal n-sphere onto a quadric of constant curvature.

With $\nu_\infty, \nu_0 \in (\mathbb{H}^2)^*$ a notion of stereographic projection is given by

$$\mathbb{H}P^1 \setminus \{\infty\} \ni p = v\mathbb{H} \mapsto (\nu_0 v)(\nu_\infty v)^{-1} = \mathfrak{p} \in \mathbb{H},$$

where $\infty = v_\infty \mathbb{H}$ is the unique point with $\nu_\infty v_\infty = 0$.

The cross ratio

Four points $p_i \in S^n$ always lie on a 2-sphere S that can be considered as a Riemann sphere, so that their complex cross ratio $[p_1; p_2; p_3; p_4]$ can be defined up to complex conjugation (orientation of S). In the following $[p_1; p_2; p_3; p_4] \in \mathbb{C}$ is obtained by taking $[p_1; p_2; p_3; p_4] = \operatorname{Re} q + i\,|\operatorname{Im} q|$ where appropriate.

Expressing the cross ratio in terms of the distances

$$|x_i - x_j|^2 = -2\langle v_i, v_j\rangle, \quad \text{where} \quad v_k = \left(\tfrac{1+|x_k|^2}{2}, x_k, \tfrac{1-|x_k|^2}{2}\right)$$

of the four points in $\mathbb{R}^n$, one arrives at

$$q = \frac{\langle v_1,v_2\rangle\langle v_3,v_4\rangle - \langle v_1,v_3\rangle\langle v_2,v_4\rangle + \langle v_1,v_4\rangle\langle v_2,v_3\rangle + \sqrt{\det(\langle v_i,v_j\rangle)}}{2\langle v_1,v_4\rangle\langle v_2,v_3\rangle}.$$

Using the Clifford algebra setup, the cross ratio is obtained from

$$q = \frac{v_1v_2v_3v_4 - v_4v_3v_2v_1}{(v_1v_4+v_4v_1)(v_2v_3+v_3v_2)} \in \Lambda^0\mathbb{R}^{n+2}_1 \oplus \Lambda^4\mathbb{R}^{n+2}_1,$$

and the direction of the $\Lambda^4\mathbb{R}^{n+2}_1$-part defines the 2-sphere S of the four points; for $x_i \in \mathbb{R}^n$,

$$q = (x_1 - x_2)(x_2 - x_3)^{-1}(x_3 - x_4)(x_4 - x_1)^{-1} \in \Lambda^0\mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n$$

provides the cross ratio, and the same formula holds true for four points $x_i \in \mathbb{H}$ in the quaternionic setup; if $p_i = v_i\mathbb{H} \in \mathbb{H}P^1$ then

$$q = (\nu_1 v_2)(\nu_3 v_2)^{-1}(\nu_3 v_4)(\nu_1 v_4)^{-1} \in \mathbb{H}$$

gives their cross ratio, where $\nu_1, \nu_3 \in (\mathbb{H}^2)^* \setminus \{0\}$ are quaternionic linear forms with $\nu_i v_i = 0$.

The cross ratio $[p_1; p_2; p_3; p_4] \in \mathbb{R}$ is real iff the four points are concircular (form a "conformal rectangle," which is embedded iff $[p_1; p_2; p_3; p_4] < 0$) and the cross ratio $[p_1; p_2; p_3; p_4] = -1$ iff they form an (embedded) "conformal square."

The cross ratio $cr := [p_1; p_2; p_3; p_4]$ satisfies the following identities under permutations of the four points (the complex conjugate $\overline{cr}$ appears when the imaginary part is chosen to be always positive):

cr :	1234	2143	3412	4321
$1 - \overline{cr}$:	1324	2413	3142	4231
$\frac{1}{1-cr}$:	1423	2314	3241	4132
$\frac{1}{\overline{cr}}$:	1432	2341	3214	4123
$1 - \frac{1}{cr}$:	1342	2431	3124	4213
$\frac{\overline{cr}}{\overline{cr}-1}$:	1243	2134	3421	4312

Sphere congruences and envelopes

A sphere congruence is a smooth map $S : M^m \to S^{n+1}_1/\pm$, and a smooth map $f : M^m \to S^n$ is said to envelope S if

$$f(p) \in S(p) \quad\text{and}\quad d_pf(T_pM^m) \subset T_{f(p)}S(p) \quad\text{for all } p \in M^m.$$

For hypersurfaces, $m = n - 1$, this reads

$$0 = \langle f, S\rangle = \tfrac{1}{2}S(SfS - f) \quad\text{and}\quad 0 \equiv \langle df, S\rangle = \tfrac{1}{2}S(SdfS - df),$$

when considering $f, S : M^{n-1} \to \mathbb{R}^{n+2}_1 \subset \mathcal{A}\mathbb{R}^{n+2}_1$; an immersed congruence $S : M^{n-1} \to S^{n+1}_1$ has two envelopes iff $\langle dS, dS\rangle$ is positive definite. For $f : M^3 \to \mathbb{H}^2$ and $S : M^3 \to \mathfrak{H}(\mathbb{H}^2)$ the enveloping condition reads

$$0 = S(f, f) \quad\text{and}\quad 0 \equiv S(df, f) + S(f, df).$$

A 2-sphere congruence $S : M^2 \to \mathfrak{S}(\mathbb{H}^2)$ is enveloped by f iff

$$S \cdot f \parallel f \quad\text{and}\quad dS \cdot f \parallel f$$

or, equivalently, if f envelopes every hypersphere congruence (section) in the congruence of elliptic sphere pencils given by S.

Similarly, an m-sphere congruence $S : M^m \to \Lambda^{n-m}\mathbb{R}^{n+2}_1$ is enveloped by $f : M^m \to \mathbb{R}^{n+2}_1$ iff f envelopes any section of S (hypersphere congruence in S). With the contact elements

$$\mathfrak{t}(p) = f(p) \cdot d_pf(e_1) \cdots d_pf(e_m), \quad (e_1, \ldots, e_m) \text{ orthonormal},$$

of an immersion $\mathbb{R}f : M^m \to S^n$, the enveloping condition reads

$$\mathfrak{t} \parallel \mathbb{v}(Sf), \quad\text{where}\quad \mathcal{A}\mathbb{R}^{n+2}_1 \ni \mathfrak{x} \mapsto \mathbb{v}\mathfrak{x} \in \mathcal{A}\mathbb{R}^{n+2}_1$$

is the Clifford dual. Two immersion f and $\hat{f}$ envelope an m-sphere congruence iff

$$\hat{f} \cdot \mathfrak{t} \parallel \hat{\mathfrak{t}} \cdot f.$$

The central sphere congruence $Z : M^m \to \Lambda^{n-m}\mathbb{R}^{n+2}_1$ of an immersion $\mathbb{R}f : M^m \to S^n$ is given by

$$\mathbb{v}Z = \tfrac{1}{2m}(\mathfrak{t} \cdot \Delta f - (-1)^m \Delta f \cdot \mathfrak{t}).$$

Conformal change of metric

Let $S^m \subset M^n$ be a submanifold, (M^n, g) Riemannian, $\tilde{g} = e^{2u}g$ a conformal change of the ambient metric; then the geometric quantities of S^m change as follows:

$$\begin{aligned}
\tilde{\nabla}_v w &= \nabla_v w + (vu)w + (wu)v - g(v,w)\cdot\nabla u\\
\tilde{I\!I}(v,w) &= I\!I(v,w) - g(v,w)\cdot(\operatorname{grad}_M u)^\perp\\
\tilde{A}_n v &= A_n v - (nu)v\\
\tilde{\nabla}^\perp_v n &= \nabla^\perp_v n + (vu)n;
\end{aligned}$$

and the real valued curvature quantities:

$$\begin{aligned}
\tilde{s} &= s - b_u \quad (s = \tfrac{1}{n-2}(ric - \tfrac{scal}{2(n-1)}g) \quad \text{Schouten tensor})\\
\tilde{w} &= e^{2u}w \qquad (w = r - s \wedge g \quad \text{Weyl tensor})\\
\tilde{r} &= e^{2u}(r - b_u \wedge g)\\
\tilde{K}_\pi &= e^{-2u}(K_\pi - \operatorname{tr}_g b_u|_\pi) \qquad (\text{sect. curv. on } \pi \subset TS^m)\\
\tilde{K} &= e^{-2u}(K - \Delta u) \qquad (\text{Gauss curv. for } m = 2),
\end{aligned}$$

where $b_u(v,w) = (\nabla^2 u)(v,w) - (vu)(wu) + \frac{1}{2}g(\nabla u, \nabla u)g(v,w)$ and $(b_1 \wedge b_2)(v,w,x,y) = \begin{vmatrix} b_1(v,x) & b_1(v,y) \\ b_2(w,x) & b_2(w,y) \end{vmatrix} + \begin{vmatrix} b_2(v,x) & b_2(v,y) \\ b_1(w,x) & b_1(w,y) \end{vmatrix}$ is the Kulkarni-Nomizu product of two bilinear forms.

Important invariants are umbilics, the normal curvature $R^\perp$, and the trace free second fundamental form $I\!I_0 = I\!I - H \cdot g$ with the mean curvature $H = \frac{1}{m}\operatorname{tr}_g I\!I$ of S^m. A conformal metric is obtained by $g_{conf} = h^2 g$, $h^2 = \frac{1}{m}g(I\!I_0, I\!I_0)$; this is the induced metric of the conformal Gauss map in case $m = 2$ and $n = 3$.