Discrete Differential Geometry: Integrable discretization

Udo Hertrich-Jeromin, 8 Feb 2019 (Rev 5 May 2020)

Send comments and corrections to Udo Hertrich-Jeromin, TU Wien E104, Wiedner Hauptstr 8-10, 1040 Wien, Austria <udo.hertrich-jeromin@tuwien.ac.at>.

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Manifesto

Integrable discretizations yield an efficient and rather straightforward way to *discretize a theory*, not just its objects. At the core of this approach is an (integrable) transformation theory that replicates itself in a discrete setting. On the one hand, this yields an effective discretization scheme and, on the other hand, it provides deeper insights about the integrable nature of a smooth theory:

> "Since the existence of Bäcklund-like transformations with permutability properties is associated with integrability of the underlying differential equations, one is led to regard the multidimensional consistency of their discretizations as the core of integrability itself."
> [Bobenko/Suris (2008) Chap 2]

Early instances of this type of discretization date back to the mid 1900's (e.g., Sauer & Wunderlich), though these are based on great geometric intuition rather than a systematic approach based on integrable systems.

The fundamental papers by Bobenko & Pinkall (mid 1990's) laid the base for the systematic approach presented here. The principles presented here crystalized during the early 2000's and are probably most clearly formulated in [Bobenko/Suris (2008)].

Integrable discretization is a very active field of research since the late 1990's; besides the discretizations of various theories, also the relations between those different theories through *integrable reductions* or *symmetry breaking* have been intensively studied.

Organization. A large part of these introductory lectures will be devoted to the background of one integrable theory: the (smooth) differential geometry of pseudospherical surfaces and their Bäcklund transformations, as an example (Sects 1 & 3). Before turning to an integrable discretization scheme (Sect 4) we address some "intuitive discretizations" in differential geometry, and the issues/problems they raise (Sect 2).

Literature

Though discrete differential geometry is not a new area of research there has been much interest and progress over the past two decades, since a relation with integrable systems has been established through the publications by Bobenko/Pinkall.

- A Bobenko, Y Suris: *Discrete differential geometry, Integrable structure*; AMS Grad Stud Math 98, Providence (2008)

  Currently the authoritive source for this branch of mathematics, where also most of the material presented here may be found.

- T Hoffmann: *Discrete differential geometry of curves and surfaces*; COE Lect Notes 18, Kyushu Univ (2009)

  This is a very approachable and perhaps less overwhelming source to start from when first learning about integrable discretization.

- C Rogers, W Schief: *Bäcklund and Darboux transformations*; Cambridge Univ Press, Cambridge (2002)

  This book provides a wealth of background information, covering the material from differential geometry as well as integrable systems viewpoints — it is a good source for the material of Sect 3.

- U Hertrich-Jeromin: *Introduction to Möbius differential geometry*; Cambridge Univ Press, Cambridge (2003)

  This source focuses on transformations and discretization in a higher (Möbius) geometry framework, hence deals in depth with principal and isothermic nets and their subclasses.

There is a notable number of publications that model certain aspects of smooth differential geometry (in Euclidean space) in a discrete setting, often in a somewhat "intuitive" way. The following are related to integrable discretization:

- R Sauer: *Differenzengeometrie*; Springer, Berlin (1970)

  A classic, that also takes account of physical models of discrete curves and surfaces (sadly only in German). A work of formidable vision.

- A Bobenko, H Pottmann, J Wallner: *A curvature theory for discrete surfaces based on mesh parallelity*; Math Ann 348, 1-24 (2010)

  Integrable and intuitive discretizations are closely related.

- T Hoffmann, A Sageman-Furnas, M Wardetzky: *A discrete parametrized surface theory in* $\mathbb{R}^3$; Math Res Notices 14, 4217-4258 (2017)

  Informed by integrable discretization, certain effects that could not be handled before are treated.

Since the mid 1990ies a wealth of research papers has been published in the area of integrable discretization. In what follows we give pointers to those papers that we consider useful as seminar topics for the present course.

Pseudospherical surfaces.

- A Bobenko, U Pinkall: *Discrete surfaces with constant negative Gaussian curvature and the Hirota equation*; J Differ Geom 43, 527-611 (1996)

  The original paper: a large amount of material is covered, most relevant in the context of this course are Sects 1-8.

- L Bianchi: *Lezioni di geometria differenziale (3rd ed)*; Enrico Spoerri, Pisa (1923)

  The classical textbook: this is *the* source on surfaces and their transformations from a viewpoint of the classical differential geometry of curves and surfaces, including the Bäcklund transformations of pseudospherical and constant mean curvature surfaces as well as the Darboux transformation for isothermic surfaces. The second volume contains a comprehensive discussion of the Ribaucour transformation.

Isothermic and minimal surfaces.

- A Bobenko, U Pinkall: *Discrete isothermic surfaces*; J reine angew Math 475, 187-208 (1996)

  The original paper: a nice approach to discrete isothermic surfaces is discussed, and applied to a discretization of minimal surfaces and their Weierstrass representation.

- U Hertrich-Jeromin: *The surfaces capable of division into infinitesimal squares by their curves of curvature*; Math Intell 22, 54-61 (2000) Erratum in Math Intell 24, 4 (2002)

  This complements the introduction of discrete isothermic surfaces by Bobenko/Pinkall (1996) by drawing a connection to Nonstandard Analysis.

- U Hertrich-Jeromin, E Musso, L Nicolodi: *Möbius geometry of surfaces of constant mean curvature 1 in hyperbolic space*; Ann Global Anal Geom 19, 185-205 (2001)

  The Darboux transformation of (smooth) isothermic surfaces is discussed; hence the developed theory is applied to describe horospherical surfaces in hyperbolic space. This relates to a description of those surfaces first given by Bianchi and Calo.

- F Burstall, U Hertrich-Jeromin, C Müller, W Rossman: *Semi-discrete isothermic surfaces*; Geom Dedicata 183, 43-58 (2016)

  Semi-discrete isothermic surfaces are introduced purely in terms of (Darboux) transformations of curves: this paper demonstrates the viewpoint of this course rather clearly.

Surfaces of constant mean curvature.

- U Hertrich-Jeromin, T Hoffmann, U Pinkall: *A discrete version of the Darboux transform for isothermic surfaces*; Oxf Lect Ser Math Appl 16, 59-81 (1999)

  The Darboux transformation of isothermic surfaces is discussed in terms of quaternions. The paper contains a section on discrete surfaces of constant mean curvature, defined in terms of transformations.

- U Hertrich-Jeromin, F Pedit: *Remarks on the Darboux transform of isothermic surfaces*; Doc Math J DMV 2, 313-333 (1997)

  The Darboux transformation of (smooth) isothermic surfaces is discussed in terms of quaternions and a Riccati-type PDE; the results are then applied to surfaces of constant mean curvature and their Bäcklund transformations.

- H Pottmann, Y Liu, H Wallner, A Bobenko, W Wang: *Geometry of multi-layer freeform structures for architecture*; ACM Trans Graphics 26:65 (2007)

  The application of discrete surfaces in architecture is discussed; this paper also discusses Gauss and mean curvatures of discrete surfaces using the Steiner formula approach.

Surfaces with curvature line coordinates.

- P Calapso: *Sulla teoria generale delle trasformazioni di Ribaucour, e sue applicazioni alla generalizzazione delle trasformazioni di Darboux*; Ann Mat Pura Appl 29, 17-69 (1921)

  A classical paper by an eminent Italian mathematician that discusses details of the Ribaucour transformation of surfaces, and the Darboux transformation as a reduction.

- F Burstall, U Hertrich-Jeromin, M Lara Miro: *Ribaucour coordinates*; Beitr Alg Geom (2018)

  Discrete and semi-discrete curvature line nets are described in terms of Ribaucour transformations; several results of interest for applications are presented, including a method to solve a discrete boundary value problem and a way of "smoothing" for semi-discrete curvature line nets. The first half of the paper should be rather accessible.

1 Curves & Surfaces

We start with a (very condensed) review of some elementary differential geometry of curves and surfaces: in order to discretize it is *paramount* to have a *good understanding* of the objects to be discretized!

1.1 Parametrization

We will discretize *parametrized* curves and surfaces:

Def. *A map* $X : \mathbb{R}^m \supset U \to \mathcal{E}^3$ $(m = 1, 2)$ *is* regular *if*

$$\forall x \in U : \mathrm{rk}\, d_x X = m;$$

- *a* (parametrized) curve *is a regular map* $X : \mathbb{R} \supset I \to \mathcal{E}^3$*;*
- *a* (parametrized) surface *is a regular map* $X : \mathbb{R}^2 \supset U \to \mathcal{E}^3$*.*

Examples. A curve and its corresponding surface of revolution:

- the *tractrix* $X : \mathbb{R} \to \mathcal{E}^3$,

  $$t \mapsto X(t) := O + e_1 \tfrac{1}{\cosh t} + e_3(t - \tanh t),$$

  is not a curve since $X'(t) = (-e_1 + e_3 \sinh t)\frac{\sinh t}{\cosh^2 t} = 0$ for $t = 0$; however, $X|_{(0,\infty)}$ or $X|_{(-\infty,0)}$ are curves.
- after restriction to $(0, \infty) \times \mathbb{R}$ the *pseudosphere* $X : \mathbb{R}^2 \to \mathcal{E}^3$,

  $$(x, t) \mapsto X(x, t) := O + \tfrac{e_1 \cos x + e_2 \sin x}{\cosh t} + e_3(t - \tanh t),$$

  is a surface, but it is not on $\mathbb{R}^2$.

Remark. A *reparametrization* of a curve/surface $X : U \to \mathcal{E}^3$ is a new curve/surface

$$\tilde{X} = X \circ \xi \text{ where } \xi : \tilde{U} \to U$$

is a diffeomorphism, i.e., ξ is bijective and ξ and ξ^{-1} are smooth. This changes the parametric description of a curve/surface but not its shape in space.

Example. With $(u, v) = (\frac{x+t}{2}, \frac{x-t}{2})$, hence $(x, t) = (u + v, u - v)$, a reparametrization

$$\tilde{X}(u, v) = X(u + v, u - v)$$

of the pseudosphere, defined on $\tilde{U} = \{(u,v) \,|\, u > v\}$, is obtained.

Arc length. The *arc length* $s(t) = \int_o^t |X'(t)|\,dt$ of a curve X measures the length of the curve $X|_{[o,t]}$; we denote the *arc length differential* by

$$ds|_t = |X'(t)|\,dt.$$

Def. *The* induced metric *or* first fundamental form *of a surface X is the pull back of the Euclidean inner product on $\mathcal{E}^3$,*

$$ds^2 = \langle dX, dX\rangle = E\,du^2 + 2F\,dudv + G\,dv^2$$

with $E = |X_u|^2$, $F = \langle X_u, X_v\rangle$, $G = |X_v|^2$.

Remark. $ds^2|_{(u,v)}$ is an inner product on $\mathbb{R}^2$ for every $(u,v) \in U$.

Example. The pseudosphere $X(x,t)$ has induced metric

$$ds^2 = E\,dx^2 + G\,dt^2 \text{ with } E(x,t) = \tfrac{1}{\cosh^2 t} \text{ and } G(x,t) = \tfrac{\sinh^2 t}{\cosh^2 t};$$

note that $E + G \equiv 1$. With $(x,t)(u,v) = (u+v, u-v)$ its reparametrization $\tilde{X} = X \circ (x,t)$ has induced metric

$$d\tilde{s}^2 = du^2 + \tilde{F}\,dudv + dv^2 \text{ with } \tilde{F}(u,v) = \tfrac{1-\sinh^2 t}{\cosh^2 t}.$$

Thus the *parameter curves* $u \mapsto \tilde{X}(u,v)$ and $v \mapsto \tilde{X}(u,v)$ are parametrized by arc length and their intersection angle is given by $\tilde{F} = \cos\vartheta$, where

$$\vartheta = 2\arctan\sqrt{\tfrac{G}{E}} \quad\Leftrightarrow\quad \begin{cases} \cos\frac{\vartheta}{2} = \frac{1}{\cosh t} \\ \sin\frac{\vartheta}{2} = \tanh t \end{cases}$$

In particular, the parameter lines form a *Chebyshev net*.

1.2 Curvature

Curvature of a curve can be measured by "how fast" its tangent direction changes:

Lemma & Def. *$T' \perp T$ for the* unit tangent field $T := \frac{d}{ds}X = \frac{X'}{|X'|}$ *of a curve $t \mapsto X(t)$; and the* curvature *of X is*

$$\kappa := |\tfrac{d}{ds}T| = \tfrac{|T'|}{|X'|}.$$

Example. The tractrix $t \mapsto O + e_1 \frac{1}{\cosh t} + e_3(t - \tanh t)$ has unit tangent field

$$T(t) = \tfrac{X'}{|X'|}(t) = (-e_1 + e_3 \sinh t)\tfrac{1}{\cosh t}$$

and curvature

$$\kappa(t) = \tfrac{|T'|}{|X'|}(t) = \tfrac{|e_1 \sinh t + e_3|}{\cosh t \tanh t} = \tfrac{1}{\sinh t}.$$

Remark (Frenet equations). If $T' \neq 0$ we may use *unit normal fields*

$$N := \tfrac{T'}{|T'|} \text{ and } B := T \times N$$

to *frame* the curve $X : I \to \mathcal{E}^3$ by the map $(T, N, B) : I \to \mathrm{SO}(3)$; then

$$\tfrac{d}{ds}(T, N, B) = \tfrac{1}{|X'|}\tfrac{d}{dt}(T, N, B) = (T, N, B)\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

with the curvature κ and *torsion* τ of the curve X. The "best fitting" *osculating circle* has radius $\frac{1}{\kappa}$ and centre $X + N\,\frac{1}{\kappa}$. Note that

$$N' + X'\kappa = B\,|X'|\,\tau \text{ and } \kappa = -\tfrac{\langle N', X'\rangle}{\langle X', X'\rangle}.$$

Def. *The* second fundamental form *of a surface X is the symmetric bilinear form*

$$-\langle dN, dX\rangle \ \textit{with}\ N := \tfrac{X_u \times X_v}{|X_u \times X_v|};$$

the normal curvature *of a curve $t \mapsto X(u(t), v(t))$ on the surface is*

$$\kappa := -\tfrac{\langle N', X'\rangle}{\langle X', X'\rangle},$$

the curve is

- asymptotic line *if $N' \perp X'$, i.e., $\kappa \equiv 0$;*
- curvature line *if* $N' + X'\kappa \equiv 0$ (Rodrigues' equation).

Example. The pseudosphere $X(x, t)$ has unit normal field

$$N(x, t) = (e_1 \cos x + e_2 \sin x)\tanh t + e_3 \tfrac{1}{\cosh t},$$

hence the parameter lines are curvature lines,

$$N_x - X_x \sinh t = N_t + X_t \tfrac{1}{\sinh t} = 0;$$

since

$$\langle N_x \pm N_t, X_x \pm X_t\rangle = E \sinh t - G\tfrac{1}{\sinh t} = 0$$

the parameter lines of its reparametrization $\tilde{X}(u,v) = X(u+v,u-v)$ are asymptotic lines.

Remark. Analogous to the first fundamental form ds^2 of $X(u,v)$ we write

$$-\langle dN, dX\rangle = e\,du^2 + 2f\,dudv + g\,dv^2$$

with $e = \langle N, X_{uu}\rangle$, $f = \langle N, X_{uv}\rangle$ and $g = \langle N, X_{vv}\rangle$; if the parameter lines of $X(u,v)$ are asymptotic and form a Chebyshev net,

$$e = g = 0 \text{ and } E = G = 1,$$

we find that

$$X_{uu} = Y_1 \tfrac{F_u}{\sqrt{1-F^2}}, \quad X_{vv} = Y_2 \tfrac{F_v}{\sqrt{1-F^2}}, \quad X_{uv} = N\,f;$$
$$N_u = Y_1 \tfrac{-f}{\sqrt{1-F^2}}, \quad N_v = Y_2 \tfrac{-f}{\sqrt{1-F^2}},$$

with the unit tangent vector fields

$$Y_1 := \tfrac{-X_u F + X_v}{\sqrt{1-F^2}} \perp X_u \text{ and } Y_2 := \tfrac{X_u - X_v F}{\sqrt{1-F^2}} \perp X_v.$$

Using $Y_{1v} = Y_{2u} = -N\frac{Ff}{\sqrt{1-F^2}}$ the *integrability conditions* yield the

- *Codazzi equations* $\frac{f}{\sqrt{1-F^2}} \equiv c = const$, from $(N_u)_v = (N_v)_u$, hence the surface has constant *Gauss curvature*

$$K := \tfrac{eg-f^2}{EG-F^2} = -c^2;$$

- *Gauss equation* $0 = (\frac{F_u}{\sqrt{1-F^2}})_v + \frac{f^2}{\sqrt{1-F^2}}$, from $(X_{uu})_v = (X_{uv})_u$; with $F = \cos\vartheta$, hence $f = c\sin\vartheta$, the Gauss equation becomes the *sine-Gordon equation*

$$0 = \vartheta_{uv} - c^2 \sin\vartheta.$$

Def & Thm. *A surface X is* pseudospherical, *i.e., has negative constant Gauss curvature $K \equiv -c^2$, iff it admits (re-)parametrization by an asymptotic Chebyshev net.*

Remark. By a rescaling one can always obtain $K = -1$.

Example. The pseudosphere is a pseudospherical surface.

2 Intuitive discretization

An intuitive approach to discretize (parametrized) curves or surfaces often fails quickly or becomes cumbersome — though some approaches proved successful. We discuss some basic notions, and issues/problems.

2.1 Curves

A discrete curve is thought of as a polygon — though edges need not be line segments but may be realized as circular arcs (bi-arc curves) or by polynomial maps (splines). Thus we only encode combinatorial information of edges:

Def. *A* discrete curve *is a (regular) map* $X : \mathbb{Z} \supset V \to \mathcal{E}^3$.

Def & Rem. For adjacent $i, j = i \pm 1 \in V$ we define the (discrete) *derivative*

$$dX_{ij} := X_j - X_i.$$

$dX_{ij} \neq 0$ for all edges $(ij) \in E \subset \mathbb{Z}^2$ is *necessary* for "regularity", but we will wish for more.

Remark. More clarifying: define X on the vertex set V of a 1-dimensional *cellular complex* $Z = (V, E)$.

Tangents. Tangents can be defined on edges or at vertices of the curve:

- $[X_i X_j]$ for each edge $(ij) \in E$, or
- the tangent line at X_i of the circumcircle of X_{i-1}, X_i, X_{i+1}; if the curve is "parametrized by arc length", $|dX|^2 \equiv 1$, then taking average
$$\delta X_i := \tfrac{dX_{i-1,i} + dX_{i,i+1}}{2} = \tfrac{X_{i+1} - X_{i-1}}{2}$$
yields a direction vector.

For the vertex tangents to be defined we need the circumcircle, that is, *regularity*:

$$dX_{i-1,i}, dX_{i,i+1}, \delta X_i \neq 0.$$

Curvature. Taking this circumcircle as the *osculating circle* of X at X_i, the law of sines yields for its radius r_i and $\varphi_i = \angle(dX_{i-1,i}, dX_{i,i+1})$

$$r_i = \tfrac{|\delta X_i|}{\sin(\pi - \varphi_i)} \text{ hence } \kappa_i = \tfrac{1}{r_i} = \tfrac{\sin \varphi_i}{|\delta X_i|}.$$

For an arc length parametrized curve, an alternative classical definition that uses the circle touching the edges in their midpoints was

$$\kappa_i' = 2\tan\frac{\varphi_i}{2}.$$

Both curvatures are defined at the vertices; and "edge curvature" has been defined for planar curves, using three consecutive edges.

Frenet equations. The Frenet equations describe the change of a frame F along a (Frenet) curve, hence two options arise:

- if the frame is vertex-based then transport is along edges,
- if the frame is edge-based transport should be across vertices.

A vertex-based frame can be constructed easily, but we have no edge-based curvature; for an edge-based frame a (principal) normal is missing.

In fact, no satisfactory Frenet theory is known to this lecturer...

2.2 Surfaces

A discrete surface is thought of as a spacial grid — again, edges and faces are only encoded in a combinatorial way; in fact, there is no sensible a-priori way to "embed" (generally non-planar) faces.

Def. *A* discrete surface *is a (regular) map* $X : \mathbb{Z}^2 \supset V \to \mathcal{E}^3$.

Remark. Again, using a *quadrilateral cell complex* $Z = (V, E, F)$, or "*quadgraph*", as a domain helps to clarify notions — and allows to consider topology.

Also, "regularity" is less clear than in the smooth setting, as before.

Tangent plane. For any edge $(ij) \in E$ one may consider, as before,

$$dX_{ij} = X_j - X_i;$$

however, the *partial derivatives* dX_{ij} at a vertex $i \in V$ are generally *not* coplanar, hence do *not* defined a tangent plane.

Similarly, for a face $(ijkl) \in F$ the vertices $X_i, X_j, X_k, X_l \in \mathcal{E}^3$ do *not* need be coplanar, hence the faces of a surface do generally *not* define a tangent plane. One approach is to consider "*face partial derivatives*"

$$\partial_1 X := \frac{dX_{ij} + dX_{lk}}{2} \text{ and } \partial_2 X := \frac{dX_{il} + dX_{jk}}{2}.$$

In integrable discretization of special (parametrized) surfaces these issues often resolve themselves:

- discrete *asymptotic nets* have planar vertex stars;
- discrete *conjugate nets* have planar facets.

Curvature. *Circular nets* (or *discrete curvature line nets*) admit a consistent assignment of vertex normals, hence vertex tangent planes. For such a discrete *Legendre map*

$$(X, N) : V \to \mathcal{E}^3 \times S^2$$

face-based Gauss and mean curvatures can be defined via (vectorial) mixed areas

$$A(X, X)_{(ijkl)} := \tfrac{1}{2}\delta X_{ik} \times \delta X_{jl},$$

using *Steiner's formula* for the parallel surfaces $X^t = X + N\,t$,

$$A(X^t, X^t) = (1 - 2tH + t^2 K)\, A(X, X)$$

or equivalently

$$H = -\tfrac{A(X,N)}{A(X,X)}, \;\; K = \tfrac{A(N,N)}{A(X,X)}.$$

This approach is closely related to integrable discretization.

3 Transformations & Permutability

Transformations and their permutability can be considered as the core of "integrable discretization": a thorough understanding of those transformations is required for any class of curves or surfaces to be discretized. We discuss the Bäcklund transformation of pseudospherical surfaces as an example.

3.1 Bäcklund transformation

Recall. A pseudospherical surface, with Gauss curvature $K \equiv -1^2$, may be parametrized by an asymptotic Chebyshev net $(u,v) \mapsto X(u,v)$,

$$E = G = 1, \;\; F = \cos\vartheta \;\text{ and }\; e = g = 0, \;\; f = \sin\vartheta.$$

We seek to produce a new surface X' from X, of the same kind, and with

$$X' - X \perp N, N' \;\text{ and }\; |X' - X| \equiv const.$$

In particular, we ask that the lines $[XX']$ form a W-congruence.

Derivation. $(X_u, Y_1, N) : U \to \mathrm{SO}(3)$ with $Y_1 = \frac{-X_u F + X_v}{\sqrt{1-F^2}}$ frames X conveniently; the structure equations read

$$(X_u, Y_1, N)_u = (X_u, Y_1, N)\begin{pmatrix} 0 & * & 0 \\ -\vartheta_u & 0 & -1 \\ 0 & * & 0 \end{pmatrix},$$

$$(X_u, Y_1, N)_v = (X_u, Y_1, N)\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ \sin\vartheta & -\cos\vartheta & 0 \end{pmatrix}.$$

Now the condition for X' with $X' - X \perp N$ and $|X' - X| \equiv c$,

$$X' = X + X_u c\cos\varphi + Y_1 c\sin\varphi,$$

to form a Chebyshev net, $E' = G' \equiv 1$, computes to

$$\begin{aligned} 1 &= 1 + (c\,(\varphi - \vartheta)_u - \sin\varphi)^2 - (1 - c^2)\sin^2\varphi \\ &= 1 + (c\,\varphi_v - \sin(\varphi - \vartheta))^2 - (1 - c^2)\sin^2(\varphi - \vartheta) \end{aligned}$$

or, taking suitable roots and with $\varphi = \frac{\vartheta' + \vartheta}{2}$,

$$\begin{aligned} (\tfrac{\vartheta' - \vartheta}{2})_u &= \tfrac{1 \pm \sqrt{1-c^2}}{c}\sin\tfrac{\vartheta' + \vartheta}{2} &= b\sin\tfrac{\vartheta' + \vartheta}{2}, \\ (\tfrac{\vartheta' + \vartheta}{2})_v &= \tfrac{1 \mp \sqrt{1-c^2}}{c}\sin\tfrac{\vartheta' - \vartheta}{2} &= \tfrac{1}{b}\sin\tfrac{\vartheta' - \vartheta}{2}. \end{aligned} \qquad (*)$$

The sine-Gordon equation $\vartheta_{uv} = \sin\vartheta$ is the integrability condition of the system $(*)$, hence it is *completely integrable*.

Def & Thm. *If $(u,v) \mapsto X(u,v)$ is an asymptotic Chebyshev net with angle ϑ then any solution of $(*)$ defines a* Bäcklund transform
$$X' = X + \tfrac{c}{\sin\vartheta}\,\{X_u \sin\tfrac{\vartheta-\vartheta'}{2} + X_v \sin\tfrac{\vartheta+\vartheta'}{2}\}$$
of X, an asymptotic Chebyshev net at constant distance $|X'-X| \equiv c$ with common tangent lines $[XX']$.

<u>*Remark*</u>. The asymptotic angle of X' is ϑ', satisfying $\vartheta'_{uv} = \sin\vartheta'$; and the tangent planes of X and X' intersect at a constant angle
$$\langle N, N'\rangle \equiv (1-bc) = \mp\sqrt{1-c^2}.$$

<u>*Remark*</u>. The Bäcklund transformation is symmetric.

<u>*Example*</u>. For $\vartheta = 2\arctan\sinh t$ with $t(u,v) = u-v$ of the pseudosphere
$$X = O + \tfrac{e_1\cos x + e_2 \sin x}{\cosh t} + e_3(t - \tanh t), \;\; x(u,v) = u+v,$$
we have $\sin\frac{\vartheta}{2} = \tanh t$ and $\cos\frac{\vartheta}{2} = \frac{1}{\cosh t} = \frac{\vartheta_u}{2} = -\frac{\vartheta_v}{2}$ that reduce $(*)$ to
$$\begin{aligned}(\tfrac{\vartheta'}{2})_u &= \tfrac{1}{\cosh t}\{1 + b\,(\sin\tfrac{\vartheta'}{2} + \sinh t\cos\tfrac{\vartheta'}{2})\},\\ (\tfrac{\vartheta'}{2})_v &= \tfrac{1}{\cosh t}\{1 + \tfrac{1}{b}(\sin\tfrac{\vartheta'}{2} - \sinh t\cos\tfrac{\vartheta'}{2})\};\end{aligned}$$
then, for $b = c = -1$, the trivial sine-Gordon solution $\vartheta' \equiv \pi$ yields the axis as a degenerate Bäcklund transform:
$$\begin{aligned}X' &= X + \tfrac{c}{\sin\vartheta}\,\{X_u \sin\tfrac{\vartheta-\vartheta'}{2} + X_v\sin\tfrac{\vartheta+\vartheta'}{2}\}\\ &= X + \tfrac{2c}{\sin\vartheta}\,\{X_x \sin\tfrac{\vartheta}{2}\cos\tfrac{\vartheta'}{2} - X_t\cos\tfrac{\vartheta}{2}\sin\tfrac{\vartheta'}{2}\} \quad = \quad O + e_3 t.\end{aligned}$$

<u>3.2 Bianchi permutability</u>

Crucial for the discretization scheme presented here is the following:

Bianchi permutability theorem. *Given two b_i-Bäcklund transforms X_i, $i = 1, 2$, of X with $b_1 \neq b_2$, there is a unique surface X' that is*

- *b_2-Bäcklund transform of X_1 and*
- *b_1-Bäcklund transform of X_2, at the same time.*

Assuming existence of X' the system $(*)$ for the transformations yields
$$\begin{aligned}0 &= (\tfrac{\vartheta'-\vartheta_1}{2})_u + (\tfrac{\vartheta_1-\vartheta}{2})_u - (\tfrac{\vartheta'-\vartheta_2}{2})_u - (\tfrac{\vartheta_2-\vartheta}{2})_u\\ &= b_2\sin\tfrac{\vartheta'+\vartheta_1}{2} + b_1\sin\tfrac{\vartheta_1+\vartheta}{2} - b_1\sin\tfrac{\vartheta'+\vartheta_2}{2} - b_2\sin\tfrac{\vartheta_2+\vartheta}{2},\end{aligned}$$

hence
$$\tan\tfrac{\vartheta'-\vartheta}{4} = \tfrac{b_2+b_1}{b_2-b_1}\tan\tfrac{\vartheta_2-\vartheta_1}{4}. \qquad (**)$$
Thus, if X' exists then it is uniquely and *algebraically* determined by, say,
$$X' = X_1 + \tfrac{c_2}{\sin\vartheta_1}\{X_{1u}\sin\tfrac{\vartheta_1-\vartheta'}{2} + X_{1v}\sin\tfrac{\vartheta_1+\vartheta'}{2}\},$$
where ϑ' is given by $(**)$ and $c_2 = \frac{2b_2}{1+b_2^2}$.

To prove existence, one verifies that ϑ', hence X' thus defined, yields a simultaneous Bäcklund transform.

Remark. Opposite "edges" in a *Bianchi quadrilateral* (X, X_1, X', X_2) have equal (constant) lengths,
$$|X' - X_2| = |X_1 - X| \equiv c_1 \text{ and } |X' - X_1| = |X_2 - X| \equiv c_2,$$
similar to a reparametrization $(u, v) = (c_1\tilde{u}, c_2\tilde{v})$ of a Chebyshev net,
$$du^2 + 2F\,dudv + dv^2 = c_1^2 d\tilde{u}^2 + 2c_1c_2F\,d\tilde{u}d\tilde{v} + c_2^2 d\tilde{v}^2.$$
Note that a reciprocal rescaling, $(u, v) = (b\tilde{u}, \frac{1}{b}\tilde{v})$, is a symmetry of the sine-Gordon equation and eliminates/changes the parameter in $(*)$.

3.3 Other transformations

Similar transformations are attached to a number of other surface classes.

Further background: Every surface admits (locally, away from umbilics) a (re-)parametrization $X(x, y)$ by curvature lines,
$$N_x + X_x\kappa_1 = N_y + X_y\kappa_2 = 0;$$
κ_i are its *principal curvatures*, its Gauss and *mean curvature* are
$$K = \kappa_1\kappa_2 \text{ resp } H := \tfrac{\kappa_1+\kappa_2}{2}.$$
By *Bonnet's theorem* any surface X of constant *positive* Gauss curvature $K = c^2$ has two parallel surfaces of constant mean curvature $H^\pm = \pm\frac{1}{2c}$,
$$X^\pm = X \pm \tfrac{1}{c}\,N.$$
A *linear Weingarten surface* is a surface with a non-trivial relation
$$0 = a\,K + 2b\,H + c;$$
for parallel surfaces $X^t = X + N\,t$ this type of condition is preserved.

These notions allow to formulate a number of examples:

<u>*Spherical surfaces*</u>. $K \equiv +c^2$, admit (complex) *Bäcklund transformations*; two complex conjugate Bäcklund transformations produce a real transformation that also satisfy Bianchi permutability, by 4D-consistency of the Bäcklund transformation.

<u>*Linear Weingarten surfaces*</u>. Employing parallel surfaces, the Bäcklund transformations for (pseudo-)spherical surfaces are extended to surfaces with

$$0 = a\,K + 2b\,H + c, \text{ where } b^2 - ac, c \neq 0.$$

In particular, constant mean curvature (cmc) $H \neq 0$ surfaces admit Bäcklund transformations.

<u>*Isothermic surfaces*</u>. Surfaces with conformal curvature line coordinates,

$$N_x + X_x\kappa_1 = N_y + X_y\kappa_2 = 0 \text{ and } ds^2 = E\,(dx^2 + dy^2),$$

admit *Darboux transformations*, that also depend on a *spectral parameter* and satisfy a Bianchi permutability theorem.

The real (double) Bäcklund transformations of cmc surfaces appears as an *integrable reduction*.

<u>*Ω-surfaces*</u>. Surfaces that envelop (a pair of) isothermic sphere congruences admit Darboux transformations, induced by those of the enveloped sphere congruences.

The real (double) Bäcklund transformations of linear Weingarten surfaces and the Darboux transformations of isothermic and Guichard surfaces occur as integrable reductions.

<u>*Principal nets*</u>. Any surface X admits Ribaucour transformations X', where X and X' envelop a common sphere congruence so that curvature lines of both surfaces correspond.

Its Bianchi permutability theorem differs from those of Bäcklund-Darboux transformations: the fourth surface is not determined algebraically but by an integration, the constant of integration yields a 1-parameter *Demoulin family* of solutions whose points lie on common circles.

Any of the aforementioned Darboux transformations is a reduction.

4 Integrable discretization

Using the Bäcklund transformation of pseudospherical surfaces (Sect 3) we discuss a discretization for these surfaces, parametrized by an asymptotic Chebyshev net (Sect 1).

4.1 Discrete pseudospherical nets

Idea. Starting from a pseudospherical surface $X_o : U \to \mathcal{E}^3$ the Bäcklund transformation can be used to generate a "net of surfaces" $(X_i)_{i\in\mathbb{Z}^2}$ of pseudospherical surfaces $X_i : U \to \mathcal{E}^3$, by repeatedly employing Bianchi permutability.

$$\mathbb{Z}^2 \ni i \mapsto X_i(u,v) \in \mathcal{E}^3,$$

with fixed $(u,v) \in U$, then defines a discrete pseudospherical surface:

Def. *A* pseudospherical net *is a discrete surface $X : V \to \mathcal{E}^3$ so that*

- *every* vertex star $S_i = \{X_i\} \cup \{X_i + dX_{ij} \,|\, (ij) \in E\}$ *is planar, that is, X is an* asymptotic net, *and*
- *opposite edges of faces have the same lengths, for $(ijkl) \in F$*

$$|dX_{ij}| = |dX_{kl}| \text{ and } |dX_{il}| = |dX_{jk}|,$$

that is, X is a Chebyshev net.

Tangent plane. By definition any asymptotic net, hence any pseudospherical net, comes with vertex-based tangent planes, of its vertex stars. In particular, "a line is tangent to an asymptotic net X at X_i" is a sensible statement.

Def. *Two pseudospherical nets $X, X' : V \to \mathcal{E}^3$ with the same edge lengths,*

$$\forall (ij) \in E : |dX'_{ij}| = |dX_{ij}|,$$

are related by a Bäcklund transformation *if*

- *$[XX']_i$ is tangent to X and X' at X_i resp X'_i for every $i \in V$, that is, X and X' are related by a* Weingarten transformation, *and*

- $|X' - X| \equiv c = const.$

<u>*Remark*</u>. As in the smooth case, a pseudospherical net admits a 2-parameter family of Bäcklund transforms. The existence of a transform can be interpreted as "3D-consistency" of smooth transformations.

<u>*Remark*</u>. A (sequence of) Bäcklund transform(s) "extends" a discrete pseudospherical net into a new dimension: the "vertical" quadrilaterals (X_i, X_j, X_j', X_i') exhibit the same characteristics as those of either net.

Thus "multi-dimensional" pseudospherical nets can be constructed using (discrete) Bianchi permutability:

Bianchi permutability theorem. *Given two c_i-Bäcklund transforms X_i, $i = 1, 2$, of X with $c_1 \neq c_2$, there is a unique surface X' that is*

- *c_2-Bäcklund transform of X_1 and*
- *c_1-Bäcklund transform of X_2, at the same time.*

<u>*Remark*</u>. The proposed discretization of pseudospherical surfaces is not only well adapted to their transformation theory: for example, smooth as well as discrete pseudospherical surfaces X are characterized by the Lorentz-harmonicity of suitable Gauss maps N in S^2,

$$N_{uv} \parallel N \text{ resp } \delta N_{jl} \parallel \delta N_{ik};$$

further, X can be reconstructed from N by the *Lelieuvre representation formula*

$$\left.\begin{aligned} X_u &= -N \times N_u \\ X_v &= N \times N_v \end{aligned}\right\} \text{resp } dX_{ij} = N_{ij} \times dN_{ij}.$$

This representation is used in [Bobenko/Suris (2008) Sect 5.7] to obtain an approximation result for pseudospherical surfaces.

<u>4.2 Discretization principle</u>

Integrable discretization is an efficient (and nearly algorithmic) method to *discretize a theory* (not just its objects). In this way, an independent discrete theory is obtained, which has several benefits:

- highly efficient and stable numerical algorithms for applications;
- sheds light on the mathematical mechanisms behind algorithms;
- provides insight into the integrable nature of the smooth theory being discretized.

While integrable discretization yields an independent theory, various approximation results have been established that link the smooth and discrete theories.

We now summarize the "discretization algorithm".

Prerequisites. The presented integrable discretization procedure requires a good understanding of the smooth theory; in particular:

- a Darboux-Bäcklund type transformation that depend on a (spectral) parameter;
- a Bianchi permutability theorem, that provides an algebro-geometric relation across higher dimensional cells.

Remark. Typically, a "lift" to a higher geometry allows to describe the transformations by parallel sections of a parameter-dependent connection, that is, the differential equations of the transformation can be linearized.

Procedure. The orbit of a single point under repeated transformations yields the "model" of a discrete net:

- the algebraic/geometric relations of Bianchi permutability yield a characterization/definition of a discrete net;
- the higher-dimensional consistency of the smooth transformations yields existence/permutability of the discrete transformations.

Remark. A "discrete integrable theory" is established through the discrete transformations; as in the smooth case, the introduction of a loop of (discrete) flat connections may be used to linearize the defining equations.

Semi-discrete theories. In a similar way, semi-discrete theories may be established: for example,

- a surface is formed by a sequence of (transformations of) curves,
- a transformation is obtained from a net of permuting transforms;

- their permutability is obtained from the higher dimensional consistency of the transformations of curves.

Thus we obtain the following scheme for a theory of 2-dimensional objects:

	definition	transformation	permutability
surface	classical	transformation	permutability
semi-discrete surface	curve: transformation	curve: quad permutab	curve: cube permutab
discrete surface	point: quad permutab	point: cube permutab	point: 4D-permutab

Epilogue

We discussed one instance of integrable discretization, for pseudospherical surfaces, in order to demonstrate the discretization scheme/algorithm.

The second part of the lectures by Gudrun Szewieczek will discuss another, in some sense complementary, instance of integrable discretization, for isothermic surfaces: this is a more general class of surfaces and also admits an interesting *integrable reduction*, or "*symmetry breaking*".

To get better acquainted with the discussed integrable discretization scheme, it will be useful to go beyond these lectures though, and to work on further topics based on relevant research papers ...