Differential Geometry

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Manifesto

Differential geometry is an area of mathematics that — as the title suggests — combines geometry with methods from calculus/analysis, most notably, differentiation (and integration). During the 20th century, geometry and analysis have also swopped roles in this relationship, giving rise to the closely related area of "global analysis".

Besides the fact that differential geometry is a beautiful field in mathematics it is a key tool in various applications: in the natural sciences, most notably, in physics — for example, when considering a moving particle or planet, or when studying the shape of thin plates — and also in engineering or architecture, where more complicated shapes need to be modelled — for example, when designing the shape of a car or a building.

This intimate relation of differential geometry to the natural sciences and other applications is also reflected in its history: for example, Newton's approach to calculus was motivated by consideration of the motion of a particle in space; in fact, analysis, (differential) geometry and applications in physics or engineering were hardly distinguished at this time. Similarly, Gauss draws a connection between his geodetic work in Hannover and his work in differential geometry that, in turn, provided the foundation for Riemann's generalization to higher dimensions and hence for Einstein's general relativity theory. Note the link to the original meaning of the word

$$\gamma\epsilon\omega + \mu\epsilon\tau\rho\acute{\iota}\alpha \;\simeq\; \begin{cases} \gamma\eta & = \text{ earth,} \\ \mu\epsilon\tau\rho\acute{\omega} & = \text{ measure.} \end{cases}$$

An application of the methods from calculus/analysis requires the investigated geometric objects to "live" in a space where differentiation can be employed, e.g., a Euclidean space. Further, the investigated objects must admit differentiation, i.e., need to be "smooth" in a certain sense.

Most of the key concepts of differential geometry can already be fully grasped (and easily pictured) in the context of curves and surfaces in a Euclidean 3-space. To avoid technical difficulties at the beginning we describe these curves and surfaces as (images of) maps that we assume to be sufficiently smooth (i.e., arbitrarily often differentiable): a curve may be thought of as the path of a particle/point moving in time, and a surface as the shape created by a 2-parametric motion of a point. Certain deficiencies of this approach — for example, that a sphere does not qualify as a surface (cf Sect 2.1), or a cumbersome formulation of the existence and uniqueness theorem for geodesics (cf Sect 3.2) — will hint at the necessity for a better foundation of the theory, such as the notion of a submanifold — which in turn leads to consequences that may be undesirable, such as the prohibition of self-intersections.

Thus the first two chapters are devoted to a discussion of the basics of curves and surfaces in Euclidean 3-space: while taking a parametric approach we shall divert from classical texts in that we shall focus on the key concepts only — on connections (on vector bundles) and on curvature(s), with a view to generalizations to submanifolds. As a consequence, beautiful diversions from the main theme, such as the four vertex theorem for planar curves or the Gauss-Bonnet theorem for surfaces, will be omitted.

The third chapter, devoted to curves on surfaces, does then not only discuss special curves on (parametrized) surfaces — such as geodesics or curvature lines — but also discusses the exponential map of surfaces and special parametrizations, as well as their use in deriving properties of surfaces, such as Minding's theorem.

These first three chapters constitute the core of this text.

In the fourth chapter, classes of surfaces that are defined by curvature properties are investigated: developable surfaces, minimal surfaces and, more generally, linear Weingarten surfaces. We arrive at some classification theorems, completely describing a class of geometric objects. For example, we present a classification theorem for rotational linear Weingarten surfaces that provides explicit parametrizations in terms of Jacobi elliptic functions. This may provide a glimpse of what research in differential geometry is about.

In the fifth and final chapter, we then discuss the concept of submanifolds in a Euclidean space: on the one hand, this yields an approach to deal with some of the aforementioned deficiencies of discussions in Chaps 1-3; on the other hand, it clarifies some of the basic concepts of differential geometry more clearly. To elaborate the relation between the classical approach to differential geometry presented in the first chapters and the modern vector bundle approach more clearly, some of the classical material has already been phrased in a vector bundle friendly way; and some of the classical material is taken up again in a vector bundle incarnation in the final chapter.

The first three chapters of these notes grew out of a 10-week lecture course on curves and surfaces, delivered for several years at the University of Bath: the material was covered in ca 20 50' lectures, thus in 1000'.

The last two chapters were developed for two different units, a BSc level unit in our teacher training programme resp an MSc level unit for mathematics students. Typically, selected material from one of these chapters was delivered during the last 3-4 weeks of a 14 week lecture course at TU Wien, thus in ca 500' total.

Disclaimer. These lecture notes are not a textbook. In particular, they are not meant for self-study, as can for example be detected from the lack of figures that illustrate the topics/objects described in the text.

Also, parts of these notes were written late at night and have only swiflty (or not at all!) been proof read, hence may contain misprints as well as mathematical errors. Therefore I expressively welcome blunder alerts, see the copyright note for my current address.

References

Differential geometry is a classical subject, covered by an uncountable number of textbooks, of varying quality. The present lecture notes were based on a variety of sources, most notably, on notes from lectures of the author's teachers, *D Ferus* and *U Pinkall*. In particular, some of U Pinkall's ingenious ideas have found their way into the present text, always striving for a fresh view on a classical topic, and some of D Ferus' beautiful and concise lines of argument may be discovered. However, if some approach seem distorted, or if errors sneaked into the exposition, then the present author takes the blame for inadequately presenting the original ideas or arguments.

As usual, students are advised to consult additional resources for the material covered in the present text, as well as for the many beautiful topics that did not make it into this text.

- M do Carmo: *Differential geometry of curves and surfaces*; Prentice-Hall, Englewood Cliffs (1976) [German version by Vieweg]

  This is *the* classical reference for (classical) differential geometry, giving the topic a modern treatment that does away with coordinate representations that obscure the geometry.

- N J Hicks: *Notes on differential geometry*; Van Norstrand, New York (1965)

  At least three or four different ways of computation and reasoning are common in differential geometry, making the communication between geometers from "different schools" difficult at times — this text describes and compares these ways in a variety of topics in differential geometry.

- S Kobayashi, K Nomizu: *Foundations of differential geometry*; Wiley Classics, New Jersey (1996)

  This is an advanced text(book) that has been rather influential in the development of the field, providing a modern approach to differential geometry based on connections.

- B O'Neill: *Semi-Riemannian geometry*; Acad Press, London (1983)

  As the subtitle "With applications to relativity" indicates, this text is geared towards the mathematical background for theoretical physics; besides dealing in detail with signature metrics it also provides excellent introductions to, for example, Lie groups and symmetric spaces. The book is written in a modern and very approachable style.

- M Spivak: *A comprehensive introduction to differential geometry*; Publish or Perish, Berkeley (1979)

  As the title suggests, this is a *comprehensive* introduction, that covers a large variety of topics: several thousand pages of beautiful differential geometry, sometimes though it is not very easy to detect which of the voluminous five volumes to consult.

- K Strubecker: *Differentialgeometrie*; deGruyter, Berlin (1964-1969)

  This is a nice introduction to classical differential geometry, using a classical approach. Despite its small format, the three booklets contain a wealth of information as well as rather original ideas, beyond what "common" textbooks offer.

- D J Struik: *Lectures on Classical Differential Geometry*; 2^{nd} Ed: Dover, New York (1988)

  This is a classic, that the first version of the present text, back in 2006, was based on.

1 Curves

Curves provide an entry point to differential geometry, where most of the key concepts of the theory can be understood without the added difficulty that arises from the appearance of "integrability conditions" for differential equations on higher dimensional domains. We shall focus on the basic concepts of differential geometry: metric/arc length — which could be considered as analysis — and shape/curvature — which is at the core of (differential) geometry.

Thus, focusing on the core of the theory, we will omit many interesting and beautiful topics, most notably issues of the global theory of planar curves, e.g., the four and six vertex theorems or an investigation of the winding of curves around a point.

1.1 Parametrization & Arc length

We shall discuss the geometry of curves in a *Euclidean ambient space* $\mathcal{E}$, over a Hilbert space $(V, \langle .,. \rangle)$, where differentiation can be employed. Therefor, to introduce the key concepts, it suffices to work in a 3-*dimensional* ambient geometry $\mathcal{E}^3$, over the standard Euclidean vector space $V = \mathbb{R}^3$. We may thus describe a *curve* in various ways:

- as the path $(t_0, t_1) \ni t \mapsto X(t) \in \mathcal{E}^3$ of a point moving in time;
- as the solution of an equation $F(x_1, x_2, x_3) = 0 \in \mathbb{R}^2$ for the (affine, or cartesian) coordinates x_i of a point $X = O + \sum_{i=1}^{3} e_i x_i \in \mathcal{E}^3$.

For a start, the first of these descriptions will be more convenient:

Def. *A* (parametrized) curve *is a map* $X : \mathbb{R} \supset I \to \mathcal{E}^3$ *on an open interval* $I \subset \mathbb{R}$ *that is* regular, *i.e.,*

$$\forall t \in I : X'(t) \neq 0;$$

We also say: X *is a* parametrization *of the* curve $C = X(I) \subset \mathcal{E}^3$.

Agreement. In this course, all maps will be C^∞ (unless stated otherwise).

Problem 1. Find parametrizations for the conic sections

$$C = \{O + e_1 x + e_2 y + e_3 z \,|\, x^2 + y^2 = z^2, \;\; x \cos\alpha + z \sin\alpha = d\},$$

$\alpha \in [0, \frac{\pi}{2}]$ and $d \neq 0$. [Hint: distinguish $\alpha < \frac{\pi}{4}$, $\alpha = \frac{\pi}{4}$ and $\alpha > \frac{\pi}{4}$.]

Problem 2. Prove that $t \mapsto X(t)$ is a straight line if $X'(t)$ and $X''(t)$ are linearly dependent for all t.

Example. A *cirular helix* with radius $r > 0$ and pitch $h \in \mathbb{R}$ is the curve

$$\mathbb{R} \ni t \mapsto X(t) := O + e_1 r \cos t + e_2 r \sin t + e_3 h t \in \mathcal{E}^3,$$

where $(O; e_1, e_2, e_3)$ denotes a cartesian reference system.

Note: if $h \neq 0$ then $X(\mathbb{R})$ is the solution of the equation

$$(x_1 - r\cos\tfrac{x_3}{h}, x_2 - r\sin\tfrac{x_3}{h}) = (0, 0).$$

Example. A *Neile parabola* $C = \{O + e_1 x + e_2 y \in \mathcal{E}^2 \mid y^2 = x^3\}$ is *not* a curve: there is no (regular) parametrization X with $C = X(I)$.

Def. *A* reparametrization *of a parametrized curve* $I \ni t \mapsto X(t) \in \mathcal{E}^3$ *is a new parametrized curve*

$$\tilde{X} = X \circ t : \tilde{I} \to \mathcal{E}^3, \ \ \textit{where} \ \ t : \tilde{I} \to I \ \ \textit{is onto and regular.}$$

Rem. Regularity of t guarantees that $\tilde{X}$ is regular: by chain rule

$$\forall s \in \tilde{I} : \tilde{X}'(s) = X'(t(s)) \cdot t'(s) \neq 0 \ \text{ since } \ \forall s \in \tilde{I} : t'(s) \neq 0.$$

Motivation. For the path $t \mapsto X(t)$ of a point/particle moving in time

- the vector $X'(t) \in \mathbb{R}^3$ is its *velocity* a time t, and
- the number $|X'(t)| \in \mathbb{R}$ its *speed* at time t;

The path of the particle resp its distance travelled from $X(o)$ can be recovered by integration:

$$X(t) = X(o) + \int_o^t X'(t)\,dt \ \text{ and } \ s(t) = \int_o^t |X'(t)|\,dt.$$

Def. *The* arc length *of a curve* $X : I \to \mathcal{E}^3$, *measured from* $X(o)$ *for some* $o \in I$, *is*

$$s(t) := \int_o^t |X'(t)|\,dt.$$

Rem. The arc length is indeed the length of the curve between $X(o)$ and $X(t)$, as can be proved by polygonal approximation of the curve.

Hence, the arc length does not depend the parametrization.

<u>*Problem 3*</u>. Use substitution to show that the arc length is invariant under reparametrization of a parametrized curve.

Lemma & Def. *Any curve $t \mapsto X(t)$ can be reparametrized by arc length, i.e., so that it has constant speed* 1*. This is called an* arc length parametrization *of X, and usually denoted by $s \mapsto X(s)$.*

Proof. Observe that $s'(t) = |X'(t)| > 0$ for all t. Hence we can invert s to obtain $t = t(s)$ and let $\tilde{X}(s) := X(t(s))$. Then

$$|\tilde{X}'(s)| = |X'(t)|\, t'(s) = \tfrac{|X'(t)|}{s'(t)} = 1$$

has length 1 (note: $t'(s) = \frac{1}{s'(t)}$ by chain rule). ∎

<u>*Rem*</u>. An arc length parametrization is unique up to choice of an "initial point" $X(o)$ and a sense of direction (orientation) of the curve.

<u>*Example*</u>. A helix $t \mapsto X(t) = O + e_1 r \cos t + e_2 r \sin t + e_3 h t$ has arc length

$$s(t) = \int_0^t \sqrt{r^2 + h^2}\, dt = \sqrt{r^2 + h^2}\, t;$$

hence an arc length (re-)parametrization

$$s \mapsto \tilde{X}(s) = O + e_1 r \cos \tfrac{s}{\sqrt{r^2+h^2}} + e_2 r \sin \tfrac{s}{\sqrt{r^2+h^2}} + e_3\, \tfrac{hs}{\sqrt{r^2+h^2}}.$$

<u>*Rem & Expl*</u>. It is often difficult to determine an arc length parametrization explicitely: an *ellipse* $t \mapsto O + e_1 a \cos t + e_2 b \sin t$ has arc length

$$s(t) = \int_0^t \sqrt{b^2 + (a^2 - b^2)\sin^2 t}\, dt,$$

which is an *elliptic integral*, hence an arc-length reparametrization cannot be given in terms of elementary functions.

<u>*Problem 4*</u>. Consider the curve given implicitely by $(\frac{x}{a})^2 + (\frac{y}{b})^2 + (\frac{z}{c})^2 = 1$ and $a\sqrt{b^2 - c^2}\, z = c\sqrt{a^2 - b^2}\, x$, where $a > b > c$. Compute its arc length and find an arc length (re)parametrization.

1.2 Ribbons & Frames

A (regular) curve $X : I \to \mathcal{E}^3$ has, at every point $X(t)$, a *tangent line* and a *normal plane*,

$$\mathcal{T}(t) = X(t) + [X'(t)] \text{ and } \mathcal{N}(t) = X(t) + \{X'(t)\}^\perp;$$

this corresponds to an orthogonal decomposition

$$\mathbb{R}^3 = [X'(t)] \oplus_\perp \{X'(t)\}^\perp$$

of $\mathbb{R}^3$ into a *tangent* resp *normal space* of the curve:

Def. *The* tangent *and* normal bundles *of a curve $X : I \to \mathcal{E}^3$ are given by the maps*

$$t \mapsto T_tX := [X'(t)] \text{ resp } t \mapsto N_tX := \{X'(t)\}^\perp;$$

a map $Y : I \to \mathbb{R}^3$ is called

- *a* tangent field *along X if $\forall t \in I : Y(t) \in T_tX$, and*
- *a* normal field *along X if $\forall t \in I : Y(t) \in N_tX$.*

<u>*Rem & Def*</u>. Any curve $X : I \to \mathcal{E}^3$ comes with a natural *unit tangent field*

$$T : I \to \mathbb{R}^3, \ t \mapsto T(t) := \tfrac{X'(t)}{|X'(t)|};$$

however, there are plentyful (unit) normal fields:

Def. *A* ribbon *is a pair (X, N), consisting of a curve $X : I \to \mathcal{E}^3$ and a unit normal field $N : I \to \mathbb{R}^3$ along X, i.e., $N \perp T$ and $|N| \equiv 1$.*

<u>*Rem & Def*</u>. Thus a ribbon (X, N) can be thought of as a curve with a sense of "upwards", e.g., useful to model the movement of an airplane.

Further, a sense of "sideways" can then be encoded by the *binormal field*

$$B : I \to \mathbb{R}^3, \ t \mapsto B(t) := T(t) \times N(t).$$

Here we require that $\mathcal{E}^3$ has *dimension* $\dim \mathcal{E}^3 = \dim \mathbb{R}^3 = 3$ and a *volume form* $\det : \mathbb{R}^{3\times 3} \to \mathbb{R}$, for the cross product: the corresponding results *do not* generalize to higher dimensions.

<u>*Motivation*</u>. The "curvature" of a curve $X : I \to \mathcal{E}^3$ can be measured by how fast its tangent line changes, the "torsion" of a ribbon (X, N) by how fast the normal field twists around the curve:

Def & Lemma. *The* (adapted) frame *of a ribbon* $(X, N) : I \to \mathcal{E}^3 \times S^2$ *is the map*

$$F = (T, N, B) : I \to \mathsf{SO}(3);$$

its structure equations *take the form*

$$F' = F\Phi \ \ with \ \ \Phi = |X'| \begin{pmatrix} 0 & -\kappa_n & \kappa_g \\ \kappa_n & 0 & -\tau \\ -\kappa_g & \tau & 0 \end{pmatrix},$$

where

- κ_n *is the* normal curvature,
- κ_g *the* geodesic curvature, *and*
- τ *the* torsion *of the ribbon* (X, N).

Proof. Since $F : I \to \mathsf{SO}(3)$ we have $F^t F \equiv \mathrm{id}$, hence

$$0 = (F^t F)' = (F^t F') + (F^t F')^t = \Phi + \Phi^t,$$

that is, $\Phi : I \to \mathfrak{o}(3)$ is skew symmetric; consequently, there are unique functions $\kappa_n, \kappa_g, \tau : I \to \mathbb{R}$ so that Φ is of the above form. ∎

Rem. The curvatures and torsion are *geometric invariants* of a ribbon, i.e., are independent of the position and parametrization of the ribbon:

- if $(\tilde{X}, \tilde{N}) = (\tilde{O} + A(X - O), AN)$ with $O, \tilde{O} \in \mathcal{E}^3$ and $A \in \mathsf{SO}(3)$ is a Eudclidean motion of (X, N), then $\tilde{F} = AF$, hence $\tilde{\Phi} = \Phi$;
- if $s \mapsto (\tilde{X}, \tilde{N})(s) = (X, N)(t(s))$ denotes an orientation preserving (i.e., $t' > 0$) reparametrization of (X, N), then

  $$\tilde{\Phi}(s) = \Phi(t(s))\, t'(s) \ \text{ and } \ |\tilde{X}'(s)| = |X'(t(s))|\, |t'(s)|,$$

  consequently $\tilde{\kappa}_n(s) = \kappa_n(t(s))$, etc

Problem 5. Let (X, N) be a ribbon and $\tilde{X} = X \circ t$ a reparametrization of X with $t' > 0$; set $\tilde{N} := N \circ t$. Show that $(\tilde{X}, \tilde{N})$ is a ribbon with $\tilde{\kappa}_n = \kappa_n \circ t$, $\tilde{\kappa}_g = \kappa_g \circ t$ and $\tilde{\tau} = \tau \circ t$.

Rem & Def. If a ribbon $(\tilde{X}, \tilde{N})$ is obtained from (X, N) by a *normal rotation*, $(\tilde{X}, \tilde{N}) = (X, N \cos\varphi + B \sin\varphi)$ with $\varphi : I \to \mathbb{R}$, then

$$\begin{pmatrix} \tilde{\kappa}_n \\ \tilde{\kappa}_g \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \kappa_n \\ \kappa_g \end{pmatrix} \ \text{ and } \ \tilde{\tau} = \tau + \tfrac{\varphi'}{|X'|}.$$

Problem 6. Determine how the curvatures and torsion of a ribbon are changed by a normal rotation of the ribbon.

Examples.

(1) *Circular helix.* Consider the ribbon given by the unit normal field

$$N(t) = -(e_1 \cos t + e_2 \sin t)$$

along the circular helix $X(t) = O + e_1 r \cos t + e_2 r \sin t + e_3 ht$; thus we obtain the frame $F = (T, N, B) : \mathbb{R} \to \mathrm{SO}(3)$ with

$$T(t) = (-e_1 r \sin t + e_2 r \cos t + e_3 h)\frac{1}{\sqrt{r^2+h^2}}$$

and

$$B(t) = (e_1 h \sin t - e_2 h \cos t + e_3 r)\frac{1}{\sqrt{r^2+h^2}};$$

hence the structure equations read

$$\begin{array}{rclccc} T' & = & & N\,\frac{r}{\sqrt{r^2+h^2}} & & \\ N' & = & T\,\frac{-r}{\sqrt{r^2+h^2}} & + & & B\,\frac{h}{\sqrt{r^2+h^2}} \\ B' & = & & N\,\frac{-h}{\sqrt{r^2+h^2}} & & \end{array}$$

which yields, with $|X'| \equiv \sqrt{r^2+h^2}$,

$$\kappa_n = \frac{r}{r^2+h^2}, \;\; \kappa_g \equiv 0, \;\; \tau = \frac{h}{r^2+h^2}.$$

(2) *Spherical curve.* Let $s \mapsto X(s) \in \mathcal{E}^3$ be an arc length parametrized spherical curve, i.e., with the centre $O \in \mathcal{E}^3$ and radius $r > 0$ of the target sphere

$$|X - O|^2 \equiv r^2 \;\text{ and }\; |X'|^2 \equiv 1.$$

Observe that

$$\langle X', X - O\rangle = \tfrac{1}{2}(|X - O|^2)' \equiv 0,$$

showing that $N := (X - O)\frac{1}{r}$ yields a unit normal field along X; hence we compute

$$\begin{array}{rcrcl} \kappa_n & = & -\langle T, N'\rangle & \equiv & -\frac{1}{r}; \\ \kappa_g & = & -\langle B, T'\rangle & = & \frac{1}{r}\det(X - O, X', X''); \\ \tau & = & \langle N', B\rangle & = & \frac{1}{r^2}\langle X', X' \times (X - O)\rangle \;\;\equiv\;\; 0. \end{array}$$

Problem 7. Let X parametrize a straight line, $X' \times X'' \equiv 0$, and let F denote any adapted frame for X. Show that $\kappa_n = \kappa_g = 0$. Find a unit normal field N so that $\tau \equiv 1$.

Problem 8. Prove that an arc length parametrized curve $s \mapsto X(s)$ is planar if and only if it has a unit normal field so that $\kappa_g = \tau \equiv 0$.

Rem. Note that $\kappa_g \equiv 0$ in the first example, whereas $\tau \equiv 0$ in the second; these two conditions characterize two prominent classes of ribbons that we will discuss in more detail later.

Def. *A ribbon (X, N) is called*

- *an* asymptotic ribbon *if $\kappa_n \equiv 0$,*
- *a* geodesic ribbon *if $\kappa_g \equiv 0$,*
- *a* curvature ribbon *if $\tau \equiv 0$.*

Fundamental theorem for ribbons. *Given three functions*

$$\kappa_n, \kappa_g, \tau : I \to \mathbb{R}, \;\; s \mapsto \kappa_n(s), \kappa_g(s), \tau(s),$$

there is an arc length parametrized curve X and a unit normal field N along X so that κ_n, κ_g and τ are the normal and geodesic curvatures and the torsion of the ribbon (X, N), respectively.
Moreover, this ribbon (X, N) is unique up to Euclidean motion.

Proof. Fix $o \in I$ and $F_o \in \mathrm{SO}(3)$; by the Picard-Lindelöf Theorem the initial value problem

$$F' = F\Phi, \;\; F(o) = F_o, \;\; \text{with } \Phi := \begin{pmatrix} 0 & -\kappa_n & \kappa_g \\ \kappa_n & 0 & -\tau \\ -\kappa_g & \tau & 0 \end{pmatrix}, \qquad (*)$$

has a unique solution $F = (T, N, B) : I \to \mathbb{R}^{3\times 3}$; furthermore

- $(FF^t)' = F(\Phi + \Phi^t)F^t = 0$, hence $F : I \to \mathrm{O}(3)$ as $F_o \in \mathrm{O}(3)$;
- $\det F : I \to \{\pm 1\}$ is continuous, hence $\det F \equiv 1$ since $\det F_o = 1$.

Consequently, the solution $F = (T, N, B)$ of $(*)$ takes values in $\mathrm{SO}(3)$.

In particular, $|T| \equiv 1$, hence integration yields an arc length parametrized curve

$$X := O + \int_o^s T(s)ds;$$

clearly, the ribbon (X, N) has curvatures and torsion κ_n, κ_g and τ.

The uniqueness statement follows from the uniqueness of F after a choice of F_o and that of X after a choice of $O = X(o)$. ∎

Problem 9. Let $F, \tilde{F} : I \to \mathrm{SO}(3)$ be two solutions of the structure equations $F' = F\Phi$; prove that $\tilde{F} = GF$ for some $G \in \mathrm{SO}(3)$.

1.3 Normal connection & Parallel transport

We shall now go on to study certain special normal fields resp ribbons for space curves, as defined above: we start with normal fields that yield curvature ribbons, i.e., ribbons with $\tau \equiv 0$.

The derivative of a normal field $N : I \to \mathbb{R}^3$ along a curve $X : I \to \mathcal{E}^3$ naturally splits into tangent and normal parts,

$$N' = T\,\langle T, N' \rangle + (N' - T\,\langle T, N' \rangle) \in TX \oplus_\perp NX;$$

the tangential part is related to the curvature κ_n, our next mission is to investigate the normal part more closely:

Def. *A normal field $N : I \to \mathbb{R}^3$ along $X : I \to \mathcal{E}^3$ is called* parallel *if*

$$\nabla^\perp N := (N')^\perp = N' - T\,\langle T, N' \rangle \equiv 0,$$

where $\nabla^\perp$ denotes the normal connection *along X.*

Note. In this definition, we do *not* assume $|N| \equiv 1$.

Lemma. *The normal connection $\nabla^\perp$ of a curve $X : I \to \mathcal{E}^3$ is* metric, *i.e.,*

$$\langle N_1, N_2 \rangle' = \langle \nabla^\perp N_1, N_2 \rangle + \langle N_1, \nabla^\perp N_2 \rangle;$$

parallel normal fields have constant length and make constant angles.

Proof. First we prove that $\nabla^\perp$ is metric, i.e., satisfies *Leibniz' rule*:

$$\langle \nabla^\perp N_1, N_2 \rangle + \langle N_1, \nabla^\perp N_2 \rangle = \langle N_1', N_2 \rangle + \langle N_1, N_2' \rangle = \langle N_1, N_2 \rangle'.$$

Consequently, if N_1 and N_2 are parallel then $\langle N_1, N_2 \rangle' \equiv 0$

In particular, $(|N|^2)' = 2\langle N, \nabla^\perp N \rangle = 0$ for a parallel normal field N, showing that N has constant length; and the angle α of two normal fields N_1 and N_2

$$\alpha = \arccos \frac{\langle N_1, N_2 \rangle}{|N_1|\,|N_2|} \equiv const$$

as soon as N_1 and N_2 are both parallel. ∎

Problem 10. Prove that any two parallel normal fields of a curve are related by a constant normal rotation.

Rem. If (X,N) is a curvature ribbon, $\tau \equiv 0$, then N is a (unit) parallel normal field:

$$\nabla^\perp N = (N')^\perp = B\,|X'|\tau = 0;$$

conversely, if N is a parallel normal field along X then $(X, \frac{N}{|N|})$ is a curvature ribbon, by the same computation.

Rem. If N is a parallel normal field along X then so is $B := T \times N$.

Problem 11. Prove that a normal field that is obtained by a constant normal rotation from a parallel normal field is parallel.

Rem. If a ribbon (X,N) is obtained from $(X,\tilde N)$ by a normal rotation, i.e., $(X,N) = (X, \tilde N \cos\varphi + \tilde B \sin\varphi)$ with $\varphi : I \to \mathbb{R}$, then

$$\tau = \tilde\tau + \frac{\varphi'}{|X'|};$$

consequently, a curvature ribbon resp unit parallel normal field is obtained with

$$\varphi := \varphi_o - \int_o \tau ds, \text{ where } ds = |X'|\,dt$$

denotes the arc length element — the constant φ_o of integration accounts for *constant* normal rotations. As *constant* scales of parallel normal fields are parallel, we obtain the following:

Lemma. *Let $X : I \to \mathcal{E}^3$ be a curve, $o \in I$ and $N_o \in N_oX$; then there is a unique parallel normal field $N : I \to \mathbb{R}^3$ along X with $N(o) = N_o$.*

Problem 12. Prove the existence and uniqueness of parallel normal fields for curves $X : I \to \mathcal{E}$ in Euclidean spaces of arbitrary dimension.

Example. For the radial normal field $N = -(e_1 \cos t + e_2 \sin t)$ along a circular helix $t \mapsto X(t) = O + e_1 r \cos t + e_2 r \sin t + e_3 ht$ we have

$$\tau = \frac{h}{r^2+h^2};$$

hence

$$\tilde N = N \cos\varphi + B \sin\varphi \text{ with } \varphi(t) = -\frac{ht}{\sqrt{r^2+h^2}}$$

yields a parallel unit normal field along X, i.e., a curvature ribbon $(X, \tilde N)$.

Problem 13. Compute explicitely, and sketch, a unit parallel normal field along a circular helix.

Cor & Def. *Parallel normal fields along $X : I \to \mathcal{E}^3$ yield a linear isometry from the normal space N_oX at $X(o)$ to the normal space N_tX at $X(t)$. This isometry is called* parallel transport *along X.*

<u>*Rem*</u>. This explains the term "connection" for $\nabla^\perp$: it provides a way to identify normal planes of a curve at different points.

Proof. Fix some $N_o \in N_oX$; by the preceding lemma there is a *unique* parallel normal field N along X with $N(o) = N_o$; thus parallel normal fields define a *map*

$$\pi : N_oX \to N_tX.$$

As the equation $\nabla^\perp N = 0$ is linear in N, constant linear combinations of parallel normal fields are parallel ("superposition principle"); hence π is linear. As parallel normal fields have constant length and make constant angles, π is an isometry. ∎

<u>*Problem 14*</u>. Show that a curve takes values in a sphere or a plane if and only if the curvatures κ_n and κ_g of a parallel frame satisfy the equation of a line in the plane.
How can the radius of the sphere be read off from this equation?

<u>1.4 Frenet curves</u>

We conclude by discussing the "classical curve theory" of the 18th and 19th century: this is characterized by the condition $\kappa_g \equiv 0$.

Recall that a normal rotation $(\tilde{X}, \tilde{N}) = (X, N\cos\varphi + B\sin\varphi)$ of a ribbon (X, N) results in the same rotation of the curvatures,

$$\begin{pmatrix} \tilde{\kappa}_n \\ \tilde{\kappa}_g \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \kappa_n \\ \kappa_g \end{pmatrix};$$

in particular, $\tilde{\kappa}_n = -\kappa_g$ and $\tilde{\kappa}_g = \kappa_n$ for the ribbon $(X, \tilde{N}) = (X, B)$. Thus the geometry of geodesic ($\kappa_g \equiv 0$) and of asymptotic ($\kappa_n \equiv 0$) ribbons will be rather similar, though different in interpretation, as illustrated by the motion of an air plane during taxi and during the flight: besides forward or backward forces (caused by change of speed), a passenger experiences the forces caused by change of direction as sideways forces during taxi, but as up- or downward forces during flight — this is achieved by "twist" (torsion) of the plane during flight.

Def. *A curve* $X : I \to \mathcal{E}^3$ *is called a* Frenet curve *if*

$$\forall t \in I : (X' \times X'')(t) \neq 0.$$

Rem. The Frenet condition is invariant under reparametrization.

Problem 15. Convince yourself that the Frenet condition is invariant under reparametrization. How to generalize it to higher dimensions?

Lemma & Def. *If* $X : I \to \mathcal{E}^3$ *is a Frenet curve then* $\forall t \in I : T'(t) \neq 0$ *and*

$$t \mapsto N(t) := \tfrac{T'(t)}{|T'(t)|}$$

defines a unit normal field of X*: this is the* principal normal field *of* X*.*

Proof. By the Frenet condition

$$0 \neq X' \times X'' = X' \times (|X'|T)' = X' \times |X'|T'$$

Further, $0 = (|T|^2)' = 2\langle T, T'\rangle$, showing that $T' \perp T$, so that N defines a unit normal field of X. ∎

Rem. If X is a Frenet curve then a ribbon (X, N) is a geodesic ribbon if and only if N is, up to sign, the principal normal field of X.

Problem 16. Let $X : I \to \mathcal{E}^3$ be a Frenet curve. Prove that (X, N) is a geodesic ribbon if and only if $\pm N$ is the principal normal field of X.

Lemma & Def. *If* X *is a Frenet curve with principal normal field* N*, then the structure equations of its* Frenet frame $F = (T, N, B)$ *take the form of the* Frenet-Serret equations,

$$F' = F\,\Phi \;\text{ with }\; \Phi = |X'| \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$$

with the curvature $\kappa > 0$ *and the* torsion τ *of the curve* X*.*

Rem. Thus, for a Frenet frame, $\kappa := \kappa_n > 0$ and $\kappa_g \equiv 0$.

Proof. $\kappa_g = \frac{\langle T', B\rangle}{|X'|} = \frac{\langle N|T'|, B\rangle}{|X'|} = 0$ and $\kappa_n = \frac{\langle T', N\rangle}{|X'|} = \frac{|T'|}{|X'|} > 0$. ∎

Example. A *circular helix* $X(t) = O + e_1 r\cos t + e_2 r\sin t + e_3 ht$ is a Frenet curve with principal normal field $N(t) = -(e_1\cos t + e_2\sin t)$, and with curvature $\kappa = \frac{r}{r^2+h^2} > 0$ and torsion $\tau = \frac{h}{r^2+h^2}$.

Problem 17. Let $s \mapsto X(s)$ be an arc-length parametrized Frenet curve and define its *Darboux vector field* by $D := \tau T + \kappa B$. Prove that the Frenet equations can be written as

$$T' = D \times T, \quad N' = D \times N, \quad B' = D \times B.$$

Problem 18. Express curvature κ and torsion τ of a Frenet curve in terms of κ_n and κ_g of a parallel frame, and vice versa.

Rem. Curvature and torsion of a Frenet curve are given by

$$\kappa = \frac{|X'\times X''|}{|X'|^3} \text{ and } \tau = \frac{\det(X',X'',X''')}{|X'\times X''|^2}.$$

In particular, they can be uniquely determined in terms of the curve alone (without reference to a choice of normal field or frame).

Problem 19. Let $X : I \to \mathcal{E}^3$ be a Frenet curve; prove the formulas

$$\kappa = \frac{|X'\times X''|}{|X'|^3} \quad \text{and} \quad \tau = \frac{\det(X',X'',X''')}{|X'\times X''|^2}.$$

Conclude that κ and τ are invariant under Euclidean motions of X.
[Hint: recall that κ and τ are invariant under reparametrization.]

For Frenet curves, our earlier Fundamental theorem for ribbons specializes to a central theorem of classical curve theory:

Fundamental theorem for Frenet curves. *Given two functions*

$$\kappa, \tau : I \to \mathbb{R} \text{ with } \forall s \in I : \kappa(s) > 0,$$

there is an arc-length parametrized Frenet curve $X : I \to \mathcal{E}^3$ with curvature and torsion κ and τ, respectively.
Moreover, this curve X is unique up to Euclidean motion.

Proof. By the fundamental theorem for ribbons there is a ribbon (X, N) with $|X'|^2 \equiv 1$, $\kappa_n = \kappa$, $\kappa_g \equiv 0$ and torsion τ; this ribbon is unique up to Euclidean motion. By the structure equations $T' = N\kappa \neq 0$, hence

- X is Frenet, $X' \times X'' = T \times T' = B\kappa \neq 0$, and
- N is its principal normal field, $N = T'\frac{1}{\kappa} = \frac{T'}{|T'|}$. ∎

Rem. There is a similar, simpler statement for planar curves $X : I \to \mathcal{E}^2$, where only one function $s \mapsto \kappa(s)$ appears.

Problem 20. Formulate a Fundamental theorem for curves $X : I \to \mathcal{E}^2$; prove it without using the Picard-Lindelöf Theorem.

Example. Let $\kappa > 0$ and $\tau \in \mathbb{R}$ be two numbers; by the Fundamental Theorem for Frenet curves, there is a unique (up to Euclidean motion) arc length parametrized Frenet curve $X : I \to \mathcal{E}^3$ with curvature κ and torsion τ. On the other hand, we know that the circular helix

$$s \mapsto X(s) = O + e_1 r \cos \tfrac{s}{\sqrt{r^2+h^2}} + e_2 r \sin \tfrac{s}{\sqrt{r^2+h^2}} + e_3 \tfrac{hs}{\sqrt{r^2+h^2}},$$

where

$$r = \tfrac{\kappa}{\kappa^2+\tau^2} \text{ and } h = \tfrac{\tau}{\kappa^2+\tau^2},$$

is a curve with the given curvature and torsion. Thus every curve with constant curvature and torsion is a circular helix:

Thm (Classification of circular helices). *A Frenet curve is a circular helix if and only if it has constant curvature and torsion.*

2 Surfaces

After discussing the core concepts of connection and curvature for curves we now turn to the more complex topic of surfaces: the core concepts remain the same, but the presence of "integrability conditions" for the involved differential equations enriches the theory — or, otherwise said, makes it more complicated. Most notably, a fundamental theorem for surfaces will be considerably more complex than the corresponding theorem(s) for curves/ribbons.

Again, we shall restrict ourselves to the core concepts, with a view to generalization to higher dimensions; consequently, we will omit some beautiful topics, such as the Gauss-Bonnet theorem for closed surfaces.

2.1 Parametrization & Metric

As for curves, there is a variety of ways to describe surfaces in a Euclidean space $\mathcal{E}^3$; again, it will be convenient to adopt a parametric description:

Def. *A* (parametrized) surface *is a map* $X : \mathbb{R}^2 \supset M \to \mathcal{E}^3$ *of an open connected domain* $M \subset \mathbb{R}^2$ *into a Euclidean space* $\mathcal{E}^3$ *that is* regular, *i.e.,*

$$\forall (u,v) \in M : d_{(u,v)}X : \mathbb{R}^2 \to \mathbb{R}^3 \ \textit{injects.}$$

We also say: X *is a* parametrization *of the* surface $X(M) \subset \mathcal{E}^3$.

<u>*Rem*</u>. Here $d_{(u,v)}X : \mathbb{R}^2 \to \mathbb{R}^3$ is the derivative of X at $(u,v) \in M$,

$$\begin{aligned} X(u+x, v+y) &= X(u,v) + d_{(u,v)}X \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + o(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)) \\ &= X(u,v) + X_u(u,v)\, x + X_v(u,v)\, y + o(x,y); \end{aligned}$$

thus we may identify $dX \simeq (X_u, X_v)$ with the pair of partial derivatives or, after a choice of basis of $\mathbb{R}^3$, with the *Jacobian matrix* of X.

Hence $d_{(u,v)}X$ injects iff $(X_u, X_v)(u,v)$ is linearly independent or, as $\dim \mathbb{R}^3 = 3$, equivalently $(X_u \times X_v)(u,v) \neq 0$.

<u>*Example*</u>. A *helicoid* $X : \mathbb{R}^2 \to \mathcal{E}^3$ is the (ruled) surface

$$\mathbb{R}^2 \ni (r,v) \mapsto X(r,v) := O + e_1 r \cos v + e_2 r \sin v + e_3 v \in \mathcal{E}^3.$$

Problem 1. Let $(O; e_1, e_2, e_3)$ be a cartesian reference system of $\mathcal{E}^3$ and let $0 < r < R$; prove that the *torus* $T^2 \subset \mathcal{E}^3$ is a surface, where

$$T^2 := \{O + e_1x_1 + e_2x_2 + e_3x_3 \,|\, (\sqrt{x_1^2 + x_2^2} - R)^2 + x_3^2 = r^2\}.$$

Example. A common "parametrization" of the 2-sphere $S^2 \subset \mathcal{E}^3$ is given by

$$X(u, v) := O + e_1 \cos u \cos v + e_2 \cos u \sin v + e_3 \sin u;$$

however, X ceases to be regular for $\cos u = 0$ and $\sin u = \pm 1$, i.e., at the "poles" of the sphere. This problem is symptomatic and cannot be resolved: *there is no parametrization of the whole sphere at once*.

This weakness of our definition of a surface can (later) be resolved by the notion of a *submanifold*.

Problem 2. Let $a \geq b \geq c > 0$; show that the (twice punctured) ellipsoid

$$E^2 = \{O + e_1x + e_2y + e_3z \,|\, (\tfrac{x}{a})^2 + (\tfrac{y}{b})^2 + (\tfrac{z}{c})^2 = 1, |z| < c\}$$

is a surface by finding a regular (prove it) parametrization.

Motivation. At any point $X(u, v)$ of a parametrized surface $X : M \to \mathcal{E}^3$ its derivative $d_{(u,v)}X : \mathbb{R}^2 \to \mathbb{R}^3$ can be used to identify *tangent vectors*

$$Y = X_u(u, v)\, x + X_v(u, v)\, y = d_{(u,v)}X \begin{pmatrix} x \\ y \end{pmatrix} \text{ with } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2;$$

to compute the length of, and angle between, tangent vectors it is then convenient to "pull back" the inner product of $\mathbb{R}^3$:

Def & Lemma. *The* (induced) metric *or* first fundamental form *of a parametrized surface $X : M \to \mathcal{E}^3$ is defined by*

$$\mathrm{I} := \langle dX, dX \rangle;$$

for $(u, v) \in M$, it yields a positive definite symmetric bilinear form

$$\mathbb{R}^2 \times \mathbb{R}^2 \ni (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) \mapsto \langle d_{(u,v)}X \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, d_{(u,v)}X \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \rangle \in \mathbb{R}.$$

Proof. Clearly, $\mathrm{I}|_{(u,v)} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a symmetric bilinear form. Further

$$\mathrm{I}|_{(u,v)}(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}) = |d_{(u,v)}X \begin{pmatrix} x \\ y \end{pmatrix}|^2 > 0$$

for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$ since $d_{(u,v)}X : \mathbb{R}^2 \to \mathbb{R}^3$ injects. ∎

Notation & Rem. The first fundamental form is often written in terms of its Gram matrix, as

$$\mathrm{I} = E\,du^2 + 2F\,du\,dv + G\,dv^2 \text{ or } \mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

with $E := |X_u|^2$, $F := \langle X_u, X_v \rangle$ and $G := |X_v|^2$. Then, at $(u,v) \in M$,

$$\mathrm{I}|_{(u,v)}\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}^t \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Examples.

(1) A *cylinder* $(u,v) \mapsto X(u,v) := O + e_1 x(u) + e_2 y(u) + e_3 v$ has induced metric
$$\mathrm{I} = (x'^2 + y'^2)\,du^2 + dv^2;$$
in particular, if its *profile curve* $u \mapsto O + e_1 x(u) + e_2 y(u)$ is arc-length parametrized, then X is *isometric* (length preserving),
$$\mathrm{I} = du^2 + dv^2.$$

(2) The *helicoid* $(r,v) \mapsto X(r,v) = O + e_1 r\cos v + e_2 r \sin v + e_3 v$ has induced metric
$$\mathrm{I}|_{(r,v)} = dr^2 + (1 + r^2)\,dv^2;$$
using a reparametrization $r = r(u) = \sinh u$ (satisfying $1 + r^2 = r'^2$) we obtain
$$\mathrm{I}|_{(u,v)} = \cosh^2(u)\,(du^2 + dv^2),$$
that is, $(u,v) \mapsto X(r(u), v)$ is *conformal* (angle preserving).

Problem 3. Compute the induced metric of the *catenoid*

$$(u,v) \mapsto X(u,v) := O + e_1 \cosh u \cos v + e_2 \cosh u \sin v + e_3 u.$$

Def. *A parametrized surface $X : M \to \mathcal{E}^3$ is called*

- conformal *if $E = G$ and $F = 0$;*
- isometric *if $E = G = 1$ and $F = 0$.*

Problem 4. Find a conformal parametrization $X : \mathbb{R}^2 \to \mathcal{E}^3$ of the unit sphere with its north pole removed [Hint: stereographic projection].

Rem. A parametrization is conformal iff it preserves angles, i.e., iff the angle of any two tangent vectors of the surface can be measured in $\mathbb{R}^2$.

Problem 5. Prove: a parametrization is conformal iff it preserves angles.

Motivation. We know (Sect 1.1): any curve can be (re-)parametrized by arc-length, i.e., isometrically. For surfaces, isometric parametrizations are very special, an do *not* normally exist (not even locally) — we shall see later what the obstructions are. In contrast to this:

Thm. *Any surface admits locally a conformal (re-)parametrization.*

Rem. This theorem is the key to treat (real) surfaces as complex curves: a viewpoint that has far reaching consequences in surface theory.

Rem. A proof of this theorem is beyond this text; a beautiful proof uses techniques from Complex (and Functional) Analysis.

To understand the statement: "*locally*" means that, for any $(u, v) \in M$, the domain M can be reduced to a neighbourhood of (u, v) so that the statement holds; and a "*reparametrization*" is defined as for curves:

Def. *A* reparametrization *of a parametrized surface $X : M \to \mathcal{E}^3$ is a new parametrized surface*

$$\tilde{X} = X \circ (u, v) : \tilde{M} \to \mathcal{E}^3 \ \text{ with a diffeomorphism } (u, v) : \tilde{M} \to M,$$

i.e., a smooth bijection with smooth inverse $(u, v)^{-1} : M \to \tilde{M}$.

Rem. If $(x, y) \mapsto \tilde{X}(x, y) = X(u(x, y), v(x, y)) \in \mathcal{E}^3$ then, by chain rule,

$$\tilde{X}_x \times \tilde{X}_y = ((X_u \times X_v) \circ (u, v)) \cdot \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix};$$

thus a reparametrization of a parametrized surface is regular.

2.2 Gauss map & Shape operator

A surface $X : M \to \mathcal{E}^3$ has, at every point $X(u, v)$, a *tangent plane* and a *normal line*,

$$\begin{aligned} \mathcal{T}(u, v) &= X(u, v) + [\{X_u(u, v), X_v(u, v)\}] \text{ and} \\ \mathcal{N}(u, v) &= X(u, v) + [(X_u \times X_v)(u, v)], \end{aligned}$$

which corresponds to the orthogonal decomposition

$$\mathbb{R}^3 = [\{X_u, X_v\}](u, v) \oplus_\perp [X_u \times X_v](u, v)$$

of $\mathbb{R}^3$ into a *tangent* resp *normal space* of the surface:

Def. *The* tangent *and* normal bundles *of a surface* $X : M \to \mathcal{E}^3$ *are given by the maps*

$$\begin{aligned}(u,v) \mapsto T_{(u,v)}X &:= [\{X_u, X_v\}](u,v) \quad \textit{resp} \\ (u,v) \mapsto N_{(u,v)}X &:= [X_u \times X_v](u,v);\end{aligned}$$

a map $Y : M \to \mathbb{R}^3$ *is called*

- *a* tangent field *along* X *if* $\forall (u,v) \in M : Y(u,v) \in T_{(u,v)}X$, *and*
- *a* normal field *along* X *if* $\forall (u,v) \in M : Y(u,v) \in N_{(u,v)}X$.

The Gauss map *of* X *is the unit normal field* $N := \frac{X_u \times X_v}{|X_u \times X_v|} : M \to \mathbb{R}^3$.

Example. Consider a *surface of revolution*

$$(u,v) \mapsto X(u,v) := O + e_1 r(u) \cos v + e_2 r(u) \sin v + e_3 h(u);$$

each *profile curve* (*meridian*) $v \equiv const$ is the orthogonal intersection of the surface with the plane $x \sin v = y \cos v$; hence we obtain the Gauss map N by rotating the unit tangent field of the meridians by 90°:

$$(u,v) \mapsto N(u,v) = \frac{-(e_1 \cos v + e_2 \sin v)\, h'(u) + e_3 r'(u)}{\sqrt{(r'^2 + h'^2)(u)}}.$$

Rem. The Gauss map of a parametrized surface is a geometric object: after a Euclidean motion, $\tilde{X} = \tilde{O} + A(X - O)$ with $A \in SO(3)$, we obtain

$$\tilde{N} = \frac{(AX_u) \times (AX_v)}{|(AX_u) \times (AX_v)|} = AN.$$

that is, the Gauss map rotates with the surface. A reflection will change the sign, as may a reparametrization do (e.g., swap of parameters).

This reflects that a unit vector $N(u,v) \in N_{(u,v)}X$ is unique up to sign.

Rem. Orientability issues do not arise in our setting: the Gauss map of a *parametrized* surface is well defined — a *non-orientable surface*, e.g., a *Möbius strip*, may be described by a doubly covering parametrization.

Problem 6. Let $r > 0$ and define a parametrization of a Möbius strip by

$$X(u,v) := O + (e_1 \cos 2u + e_2 \sin 2u)(r + v \cos u) + e_3 v \sin u.$$

Prove that $X(u + \pi, 0) = X(u, 0)$ but $N(u + \pi, 0) = -N(u, 0)$.

Motivation. The normal curvature κ_n of a ribbon (X, N) is given by the equation

$$0 = N'^T + T\,|X'|\kappa_n = N'^T + X'\kappa_n,$$

where $t \mapsto N'^T(t) \in T_tX$ denotes the tangential part of the derivative.

In a similar way, the curvature or "shape" of a surface $X : M \to \mathcal{E}^3$ with Gauss map $N : M \to \mathbb{R}^3$ can be described:

Lemma & Def. *The Gauss map N of a surface X differentiates into the tangent bundle,*

$$\forall (u,v) \in M : d_{(u,v)}N : \mathbb{R}^2 \to T_{(u,v)}X.$$

Hence we define the shape operator *of X at $(u,v) \in M$ by*

$$\mathrm{S}|_{(u,v)} := -d_{(u,v)}N \circ (d_{(u,v)}X)^{-1} \in \mathsf{End}(T_{(u,v)}X).$$

Proof. Firstly, N differentiates into the tangent bundle $TX = \{N\}^\perp$ since

$$1 \equiv |N|^2 \quad\Rightarrow\quad 0 = d(|N|^2) = 2\langle N, dN\rangle.$$

Since $d_{(u,v)}X : \mathbb{R}^2 \to \mathbb{R}^3$ injects for $(u,v) \in M$, it yields an isomorphism

$$d_{(u,v)}X : \mathbb{R}^2 \to T_{(u,v)}X \subset \mathbb{R}^3$$

that can be inverted to obtain a linear map $(d_{(u,v)}X)^{-1} : T_{(u,v)}X \to \mathbb{R}^2$; composition with the linear map $-d_{(u,v)}N : \mathbb{R}^2 \to T_{(u,v)}X$ yields an endomorphism $\mathrm{S}|_{(u,v)} \in \mathsf{End}(T_{(u,v)}X)$, showing that S is well defined. ∎

Rem. The map $(u,v) \mapsto \mathrm{S}|_{(u,v)}$ yields an *endomorphism field* along X; S is also called the *Weingarten tensor field* of the surface X;

Rem. As $(X_u(u,v), X_v(u,v))$ yields a basis of $T_{(u,v)}X$ for $(u,v) \in M$, we may determine the shape operator by its values on the basis vectors,

$$\mathrm{S}(X_u) = \mathrm{S}(dX(\begin{pmatrix}1\\0\end{pmatrix})) = -dN(\begin{pmatrix}1\\0\end{pmatrix}) = -N_u \text{ and } \mathrm{S}(X_v) = -N_v.$$

Lemma. $\mathrm{S}|_{(u,v)} \in \mathsf{End}(T_{(u,v)}X)$ *is symmetric for each* $(u,v) \in M$.

Proof. We verify symmetry on the basis $(X_u(u,v), X_v(u,v))$ of $T_{(u,v)}X$: as $N \perp X_u, X_v$ we obtain

$$0 = \langle X_u, N\rangle_v = \langle X_{uv}, N\rangle + \langle X_u, N_v\rangle = \langle X_{uv}, N\rangle - \langle X_u, \mathrm{S}X_v\rangle,$$

$$0 = \langle X_v, N\rangle_u = \langle X_{vu}, N\rangle + \langle X_v, N_u\rangle = \langle X_{vu}, N\rangle - \langle X_v, \mathrm{S}X_u\rangle;$$

hence $\langle X_u, \mathrm{S}X_v\rangle = \langle \mathrm{S}X_u, X_v\rangle$ since $X_{uv} = X_{vu}$. ∎

Problem 7. Determine the Gauss map and shape operator of the helicoid.

Motivation. As the normal curvature κ_n of a ribbon, the shape operator S may be defined by the equation

$$0 = dN + \mathrm{S} \circ dX = dN^T + \mathrm{S} \circ dX,$$

thus suggesting that it encodes the curvature(s) of a surface:

Def. *Let* S *denote the shape operator of* $X : M \to \mathcal{E}^3$*; then:*

- $H := \frac{1}{2} \operatorname{tr} \mathrm{S}$ *is the* mean curvature *of* X*;*
- $K := \det \mathrm{S}$ *is the* Gauss curvature *of* X*; and*
- *the eigenvalues* $\kappa^{\pm} = H \pm \sqrt{H^2 - K}$ *and eigendirections of* S *are the* principal curvatures *resp* curvature directions *of the surface* X*.*

Rem. Note that $H = \frac{1}{2}(\kappa^+ + \kappa^-)$, hence "mean curvature".

Example. For a surface of revolution with arc-length parametrized profile curve,

$$X(u,v) = O + e_1 r(u) \cos v + e_2 r(u) \sin v + e_3 h(u)$$

with $r'^2 + h'^2 \equiv 1$, we obtain the Gauss map (see above)

$$N(u,v) = -e_1 h'(u) \cos v - e_2 h'(u) \sin v + e_3 r'(u);$$

now $r'r'' + h'h'' = 0$ yields $h'(r'h'' - r''h') + r'' = r'(r'h'' - r''h') - h'' = 0$ so that

$$N_u + X_u(r'h'' - r''h') = N_v + X_v \tfrac{h'}{r} = 0.$$

Thus X_u and X_v yield curvature directions with principal curvatures

$$\kappa^+ = r'h'' - r''h' \text{ and } \kappa^- = \tfrac{h'}{r}$$

and, with $r'r'' + h'h'' = 0$ again, the Gauss curvature of X simplifies to

$$K = -\tfrac{r''}{r}.$$

Problem 8. Compute mean and principal curvatures of the helicoid.

Problem 9. Compute the Gauss curvatures of the helicoid and catenoid.

Rem. The shape operator and curvatures are geometric objects:

- if $\tilde{X} = X \circ (u,v)$ is reparametrization of X and $\tilde{N} = N \circ (u,v)$ then
$$\tilde{\mathrm{S}} = -(d_{(u,v)}N \circ d(u,v)) \circ (d_{(u,v)}X \circ d(u,v))^{-1} = \mathrm{S}|_{(u,v)},$$

hence, in particular, $\tilde{H} = H \circ (u, v)$, $\tilde{K} = K \circ (u, v)$, etc

- if $\tilde{X} = \tilde{O} + A(X - O)$ with $A \in \mathrm{SO}(3)$ is a Euclidean motion of X then
$$\tilde{\mathrm{S}} = -(A \circ dN) \circ (A \circ dX)^{-1} = A \circ \mathrm{S} \circ A^{-1},$$
hence the curvatures remain unchanged but the curvature directions "rotate with the surface".

Def. *A point $X(u, v)$ of a surface $X : M \to \mathcal{E}^3$ is called*

- umbilic *if $\kappa^+(u, v) = \kappa^-(u, v)$, i.e., if $(H^2 - K)(u, v) = 0$;*
- flat point *if $\kappa^+(u, v) = \kappa^-(u, v) = 0$.*

Rem. A point $X(u, v)$ is an umbilic or a flat point if and only if
$$\mathrm{S}|_{(u,v)} = H(u, v)\,\mathrm{id}_{T_{(u,v)}X} \text{ resp } \mathrm{S}|_{(u,v)} = 0.$$

Example. Suppose $(u, v) \mapsto X(u, v)$ takes values in a fixed plane,
$$\pi = \{X \in \mathcal{E}^3 \,|\, \langle X - O, n\rangle = d\}, \text{ i.e., } \langle dX, n\rangle \equiv 0;$$
then $N \equiv \pm n$, hence $\mathrm{S} \equiv 0$ and every point of X is a flat point.

Conversely, if every point is a flat point, $\mathrm{S} \equiv 0$, then $N \equiv const$ and X takes values in a fixed plane as, with some origin $O \in \mathcal{E}^3$,
$$0 = \langle dX, N\rangle = d\langle X - O, N\rangle.$$

Problem 10. Prove that all points of a sphere of radius $r > 0$ are umbilics and compute its Gauss curvature.

Matrix representation. Using (X_u, X_v) as a tangential basis field, the shape operator S can be written as a matrix:
$$(\mathrm{S}X_u, \mathrm{S}X_v) = -(N_u, N_v) = (X_u, X_v) \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},$$
hence taking inner products with X_u and X_v yields
$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} -\langle X_u, N_u\rangle & -\langle X_u, N_v\rangle \\ -\langle X_v, N_u\rangle & -\langle X_v, N_v\rangle \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},$$
that is, the (symmetric) Gram matrix of the *second fundamtental form*:

Def & Lemma. *The* second fundamental form *of a surface* $X : M \to \mathcal{E}^3$ *is defined by*
$$\mathrm{I\!I} := -\langle dX, dN\rangle;$$
for each $(u,v) \in M$, *it yields a symmetric bilinear form*
$$\mathbb{R}^2 \times \mathbb{R}^2 \ni \left(\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)\right) \mapsto -\langle d_{(u,v)}X \left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), d_{(u,v)}N \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)\rangle \in \mathbb{R}.$$

Proof. Clearly, $\mathrm{I\!I}|_{(u,v)}$ is a bilinear form on $\mathbb{R}^2$. Further,
$$-\langle X_u, N_v\rangle = \langle X_{uv}, N\rangle = \langle X_{vu}, N\rangle = -\langle X_v, N_u\rangle,$$
showing that $\mathrm{I\!I}$ is symmetric. ■

Problem 11. Determine the second fundamental form of the helicoid.

Problem 12. Investigate how the first and second fundamental forms change under Euclidean motion and under reparametrization.

Rem. Thus, given the first fundamental form I, the shape operator S can be computed from the second fundamental form $\mathrm{I\!I}$, and vice versa.

Warning. Even though S is a symmetric endomorphism, its matrix
$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \frac{1}{EG-F^2} \begin{pmatrix} Ge - Ff & Gf - Fg \\ Ef - Fe & Eg - Ff \end{pmatrix}$$
is usually *not symmetric* (as (X_u, X_v) is not orthonormal, in general).

Rem. Note that the Gauss curvature $K = \frac{eg-f^2}{EG-F^2}$.

2.3 Covariant differentiation & Curvature tensor

Similarly to the normal connection of a curve we define a "connection" for tangent vector fields along a surface:

Def. *The* covariant derivative *of a tangent field* $Y : M \to \mathbb{R}^3$ *along a surface* $X : M \to \mathcal{E}^3$ *is the tangential part of its derivative,*
$$\nabla Y := (dY)^T = dY - N\,\langle dY, N\rangle,$$
where ∇ *denotes the* Levi-Civita connection *along* X.

Rem. Note that $\langle X_{uu}, N\rangle = -\langle X_u, N_u\rangle = e$, etc, consequently
$$\begin{array}{lll} \nabla_{\frac{\partial}{\partial u}} X_u = X_{uu} - Ne & \text{and} & \left.\begin{array}{l} \nabla_{\frac{\partial}{\partial v}} X_u \\ \nabla_{\frac{\partial}{\partial u}} X_v \end{array}\right\} = X_{uv} - Nf. \\ \nabla_{\frac{\partial}{\partial v}} X_v = X_{vv} - Ng & \text{and} & \end{array}$$

Lemma. *The Levi-Civita connection satisfies the Leibniz rule,*

$$\nabla(Yx) = (\nabla Y)\,x + Y\,dx \ \text{ for any function } x : M \to \mathbb{R},$$

and is metric,

$$d\langle Y, Z\rangle = \langle \nabla Y, Z\rangle + \langle Y, \nabla Z\rangle.$$

Proof. The Leibniz rule for scalar factors: by the usual product rule

$$d(Yx) = (dY)\,x + Y\,dx \ \text{ hence } \ \nabla(Yx) = (dY)^T x + Y\,dx;$$

∇ is metric: if Y, Z are tangent fields then $\langle dY, Z\rangle = \langle \nabla Y, Z\rangle$, hence

$$d\langle Y, Z\rangle = \langle dY, Z\rangle + \langle Y, dZ\rangle = \langle \nabla Y, Z\rangle + \langle Y, \nabla Z\rangle$$

by the usual Leibniz rule for the inner product. ∎

Matrix representation. As the covariant derivative of a tangent field is tangential we obtain, in particular, for the basis fields X_u and X_v,

$$\left.\begin{array}{l} \nabla_{\frac{\partial}{\partial u}}(X_u, X_v) = (X_u, X_v)\,\Gamma_1 \\ \nabla_{\frac{\partial}{\partial v}}(X_u, X_v) = (X_u, X_v)\,\Gamma_2 \end{array}\right\} \ \text{with } \ \Gamma_i = \begin{pmatrix} \Gamma^1_{i1} & \Gamma^1_{i2} \\ \Gamma^2_{i1} & \Gamma^2_{i2} \end{pmatrix}; \qquad (*)$$

thus, for a general tangent field $Y = X_u x + X_v y = dX \binom{x}{y}$,

$$\nabla_{\frac{\partial}{\partial u}} Y = dX((\tfrac{\partial}{\partial u} + \Gamma_1)\tbinom{x}{y}) \ \text{ and } \ \nabla_{\frac{\partial}{\partial v}} Y = dX((\tfrac{\partial}{\partial v} + \Gamma_2)\tbinom{x}{y}) \quad (**)$$

or, otherwise said,

$$\nabla_{\frac{\partial}{\partial u}} \circ dX = dX \circ (\tfrac{\partial}{\partial u} + \Gamma_1) \ \text{ and } \ \nabla_{\frac{\partial}{\partial v}} \circ dX = dX \circ (\tfrac{\partial}{\partial v} + \Gamma_2).$$

Note that, for $\binom{x}{y} = \binom{1}{0}$ and $\binom{x}{y} = \binom{0}{1}$, we recover $(*)$ from $(**)$.

Warning. $\nabla_{\frac{\partial}{\partial u}}$ and $\nabla_{\frac{\partial}{\partial v}}$ are *differential operators* (not endomorphisms), even though we may use matrices to describe them!

Def & Lemma. Γ^k_{ij} *are called the* Christoffel symbols *of X; they are symmetric,*

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Proof. $\nabla_{\frac{\partial}{\partial u}} X_v = (X_{vu})^T = (X_{uv})^T = \nabla_{\frac{\partial}{\partial v}} X_u$ yields symmetry. ∎

Koszul's formulas. *With matrices* $\mathrm{I} := \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ *and* $\mathrm{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *we have*
$$\tfrac{1}{2}\mathrm{I}_u - \tfrac{E_v - F_u}{2}\mathrm{J} = \mathrm{I}\Gamma_1 \ \ and \ \ \tfrac{1}{2}\mathrm{I}_v + \tfrac{G_u - F_v}{2}\mathrm{J} = \mathrm{I}\Gamma_2.$$

Proof. Multiplication of $\nabla_{\frac{\partial}{\partial u}} X_u = X_u\Gamma^1_{11} + X_v\Gamma^2_{11}$ with X_u and X_v yields
$$\begin{array}{rcccl} E\Gamma^1_{11} + F\Gamma^2_{11} & = & \langle X_u, X_{uu}\rangle & = & \frac{1}{2}E_u, \\ F\Gamma^1_{11} + G\Gamma^2_{11} & = & \langle X_v, X_{uu}\rangle & = & F_u - \langle X_u, X_{vu}\rangle \\ & & & = & F_u - \frac{1}{2}E_v; \end{array}$$
the other equations are obtained in a similar way. ∎

Cor. *The Christoffel symbols* Γ^k_{ij} *depend on the induced metric* I *only.*

Example. For an isometric parametrization, $E = G \equiv 1$ and $F \equiv 0$, Koszul's formulas yield $\Gamma^k_{ij} = 0$.

Problem 13. Compute the Christoffel symbols of a conformally parametrized surface.

Using the Levi-Civita connection we introduce a new kind of "curvature", the curvature tensor of a surface:

Def. *For a tangent field* $Y : M \to \mathbb{R}^3$ *along a surface* $X : M \to \mathcal{E}^3$ *we define*
$$\mathrm{R}Y := \nabla_{\frac{\partial}{\partial u}} \nabla_{\frac{\partial}{\partial v}} Y - \nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial u}} Y;$$
R *is called the* curvature tensor *of* X.

Rem. This is a simplified version of the "true" curvature tensor, which is sufficient in our setting of 2-dimensional surfaces though.

Lemma. R *is a skew symmetric* tensor *of the tangent bundle* TX, *that is,* $\mathrm{R}|_{(u,v)} \in \mathsf{End}(T_{(u,v)}X)$ *is skew symmetric for each* $(u,v) \in M$, *and*
$$\mathrm{R}(Yx) = (\mathrm{R}Y)\,x \ \ \text{for any function} \ \ x : M \to \mathbb{R}.$$

Proof. Clearly $\mathrm{R}|_{(u,v)} \in \mathsf{End}(T_{(u,v)}X)$ for each $(u,v) \in M$; to see skew symmetry observe that
$$\tfrac{1}{2}(|Y|^2)_{vu} = \langle Y, \nabla_{\frac{\partial}{\partial u}} \nabla_{\frac{\partial}{\partial v}} Y\rangle + \langle \nabla_{\frac{\partial}{\partial u}} Y, \nabla_{\frac{\partial}{\partial v}} Y\rangle,$$

hence

$$\langle Y, \mathrm{R}Y\rangle = \tfrac{1}{2}(|Y|^2)_{vu} - \tfrac{1}{2}(|Y|^2)_{uv} = 0.$$

To see that R is a tensor, $\mathrm{R}(Yx) = (\mathrm{R}Y)\,x$, is a straightforward computation using the Leibniz rule for the Levi-Civita connection. ■

Problem 14. Verify that the curvature tensor of a surface is a tensor.

Matrix representation. To determine the matrix representation of the curvature tensor in terms of the basis field (X_u, X_v) first recall that

$$\nabla_{\frac{\partial}{\partial u}} \circ dX = dX \circ (\tfrac{\partial}{\partial u} + \Gamma_1) \text{ and } \nabla_{\frac{\partial}{\partial v}} \circ dX = dX \circ (\tfrac{\partial}{\partial v} + \Gamma_2);$$

consequently, for $X_u = dX\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $X_v = dX\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$,

$$\begin{aligned}\mathrm{R}X_u &= dX\big(((\tfrac{\partial}{\partial u} + \Gamma_1)(\tfrac{\partial}{\partial v} + \Gamma_2) - (\tfrac{\partial}{\partial v} + \Gamma_2)(\tfrac{\partial}{\partial u} + \Gamma_1))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\big)\\ &= dX\big((\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2])\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\big) \text{ and}\\ \mathrm{R}X_v &= dX\big((\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2])\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\big),\end{aligned}$$

where $[\Gamma_1, \Gamma_2] := \Gamma_1\Gamma_2 - \Gamma_2\Gamma_1$ denotes the *commutator* of matrices; thus

$$(\mathrm{R}X_u, \mathrm{R}X_v) = (X_u, X_v)(\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]).$$

On the other hand, taking inner products with X_u and X_v and using the skew symmetry of R, we learn that there is a function $\varrho : M \to \mathbb{R}$ so that

$$\left(\begin{smallmatrix}0 & -\varrho\\ \varrho & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}E & F\\ F & G\end{smallmatrix}\right)(\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]).$$

Note that ϱ depends on the induced metric I only.

Problem 15. Show that $\mathrm{R}Y = \frac{1}{r^2}Y \times (X_u \times X_v)$ if $X : M \to \mathcal{E}^3$ takes values in a sphere of radius $r > 0$, i.e., if $|X - Z|^2 \equiv r^2$ for some $Z \in \mathcal{E}^3$. [Hint: compute ∇Y directly from the definition, using $N = \pm\frac{1}{r}X$.]

2.4 The Gauss-Codazzi equations

As for ribbons, an *adapted (moving) frame* $F = (X_u, X_v, N)$ may be used to investigate the geometry of a parametrized surface $X : M \to \mathcal{E}^3$; its structure equations, encoding the "shape" of a surface, can then be expressed in terms of notions from the previous sections: the induced metric, the shape operator/second fundamental form and covariant derivative:

Def & Lemma. *The* (adapted) frame *of a surface $X : M \to \mathcal{E}^3$ is the map*

$$F = (X_u, X_v, N) : M \to \mathsf{Gl}(\mathbb{R}^3);$$

its structure equations *take the form*

$$\left.\begin{array}{l} F_u = F\Phi \\ F_v = F\Psi \end{array}\right\} \text{ with } \Phi = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & -s_{11} \\ \Gamma^2_{11} & \Gamma^2_{12} & -s_{21} \\ e & f & 0 \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} \Gamma^1_{21} & \Gamma^1_{22} & -s_{12} \\ \Gamma^2_{21} & \Gamma^2_{22} & -s_{22} \\ f & g & 0 \end{pmatrix}.$$

These are the Gauss-Weingarten equations *of the surface $X : M \to \mathcal{E}^3$.*

<u>*Warning*</u>. Note that, in general, $F : M \not\to \mathsf{SO}(3)$.

<u>*Remark/Proof*</u>. The claim follows directly from the definitions of the Christoffel symbols Γ^k_{ij}, the (components s_{ij} of the) shape operator S, and the (coefficients of the) second fundamental form $\mathrm{I\!I}$. Note that, for example,

$$\langle X_{uu}, N\rangle = -\langle X_u, N_u\rangle = e.$$

Classically, the Gauss-Weingarten equations are written without matrices:

$$\begin{array}{rcccl} X_{uu} & = & \nabla_{\frac{\partial}{\partial u}} X_u + Ne & = & X_u\Gamma^1_{11} + X_v\Gamma^2_{11} + Ne, \\ X_{vu} & = & \nabla_{\frac{\partial}{\partial u}} X_v + Nf & = & X_u\Gamma^1_{12} + X_v\Gamma^2_{12} + Nf, \\ X_{vv} & = & \nabla_{\frac{\partial}{\partial v}} X_v + Ng & = & X_u\Gamma^1_{22} + X_v\Gamma^2_{22} + Ng; \end{array}$$

and

$$\begin{array}{rcccl} N_u & = & -\mathrm{S}X_u & = & -(X_u s_{11} + X_v s_{21}), \\ N_v & = & -\mathrm{S}X_v & = & -(X_u s_{12} + X_v s_{22}), \end{array}$$

where

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \frac{1}{EG-F^2} \begin{pmatrix} Ge - Ff & Gf - Fg \\ Ef - Fe & Eg - Ff \end{pmatrix}.$$

Integrability of the adapted frame, $F_{uv} = F_{vu}$, then yields two central equations of surface theory:

Gauss-Codazzi equations. *For a parametrized surface $X : M \to \mathcal{E}^3$*

(G) $\langle X_u, \mathrm{R}X_v\rangle = K\,(EG - F^2)$ — Gauss equation,

(C) $(\nabla_{\frac{\partial}{\partial u}} \mathrm{S})\, X_v = (\nabla_{\frac{\partial}{\partial v}} \mathrm{S})\, X_u$ — Codazzi equation,

where $\nabla\mathrm{S}$ denotes the covariant derivative of the shape operator, *for a tangent field Y,*

$$(\nabla\mathrm{S})Y := \nabla(\mathrm{S}Y) - \mathrm{S}(\nabla Y).$$

Rem. The *partial covariant derivatives* $\nabla_{\frac{\partial}{\partial u}}\mathrm{S}$ and $\nabla_{\frac{\partial}{\partial v}}\mathrm{S}$ of the shape operator are tensor fields on TX, e.g., $\nabla_{\frac{\partial}{\partial u}}\mathrm{S}|_{(u,v)} \in \mathrm{End}(T_{(u,v)}X)$ and

$$\nabla_{\frac{\partial}{\partial u}}\mathrm{S}(Yx) = \nabla_{\frac{\partial}{\partial u}}\mathrm{S}(Y)\,x \text{ for any function } x : M \to \mathbb{R}.$$

Problem 16. Verify that the partial covariant derivatives of the shape operator yield tensor fields on the tangent bundle.

Proof. First compute, with $f = -\langle N_v, X_u\rangle$ and $e = -\langle N_u, X_u\rangle$ of the second fundamental form,

$$\begin{array}{rcll}
(X_u)_{vu} & = & (\nabla_{\frac{\partial}{\partial v}} X_u + Nf)_u & \\
& = & (\nabla_{\frac{\partial}{\partial u}}\nabla_{\frac{\partial}{\partial v}} X_u + N_u f) & + \quad N\,(f_u + \langle N, (\nabla_{\frac{\partial}{\partial v}} X_u)_u\rangle) \\
& = & (\nabla_{\frac{\partial}{\partial u}}\nabla_{\frac{\partial}{\partial v}} X_u + N_u f) & + \quad N\,(f_u - \langle N_u, \nabla_{\frac{\partial}{\partial v}} X_u\rangle), \\
(X_u)_{uv} & = & (\nabla_{\frac{\partial}{\partial v}}\nabla_{\frac{\partial}{\partial u}} X_u + N_v e) & + \quad N\,(e_v - \langle N_v, \nabla_{\frac{\partial}{\partial u}} X_u\rangle
\end{array}$$

to deduce

$$\begin{array}{rcl}
0 & = & (\mathrm{R}X_u - N_v e + N_u f) \\
& + & N\,((f_u + \langle N_v, \nabla_{\frac{\partial}{\partial u}} X_u\rangle) - (e_v + \langle N_u, \nabla_{\frac{\partial}{\partial v}} X_u\rangle)) \\
& = & (\mathrm{R}X_u - N_v e + N_u f) - N\,\langle \nabla_{\frac{\partial}{\partial u}} N_v - \nabla_{\frac{\partial}{\partial v}} N_u, X_u\rangle.
\end{array}$$

Hence, taking inner products with X_u and X_v yields

$$\langle \mathrm{R}X_u, X_u\rangle = 0 \text{ and } \langle \mathrm{R}X_u, X_v\rangle + (eg - f^2) = 0,$$

where the first equation is trivial by the skew symmetry of R and the second yields the Gauss equation. The normal part, on the other hand, yields

$$0 = \langle \nabla_{\frac{\partial}{\partial u}}\mathrm{S}X_v - \nabla_{\frac{\partial}{\partial v}}\mathrm{S}X_u, X_u\rangle.$$

In a similar way, $(X_v)_{uv} = (X_v)_{vu}$ yields

$$0 = (\mathrm{R}X_v + N_u g - N_v f) - N\,\langle \nabla_{\frac{\partial}{\partial u}} N_v - \nabla_{\frac{\partial}{\partial v}} N_u, X_v\rangle;$$

the tangential part only reproduces the Gauss equation, and the normal part yields

$$0 = \langle \nabla_{\frac{\partial}{\partial u}}\mathrm{S}X_v - \nabla_{\frac{\partial}{\partial v}}\mathrm{S}X_u, X_v\rangle.$$

Thus $\nabla_{\frac{\partial}{\partial u}}\mathrm{S}X_v = \nabla_{\frac{\partial}{\partial v}}\mathrm{S}X_u$ and, using $\nabla_{\frac{\partial}{\partial u}} X_v = \nabla_{\frac{\partial}{\partial v}} X_u$, we obtain the Codazzi equation. ∎

Rem. $F_{uv} = F_{vu}$ yields no further equations beyond the Gauss-Codazzi equations: $N_{uv} = N_{vu}$ is equivalent to the Codazzi equation.

Problem 17. Let N denote that Gauss map of a surface $X : M \to \mathcal{E}^3$; prove that $N_{uv} = N_{vu}$ is equivalent to the Codazzi equation.

Matrix representation. With the matrix representation of the curvature tensor,

$$(\mathrm{R}X_u, \mathrm{R}X_v) = (X_u, X_v)(\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]),$$

the Gauss equation reads

$$K(EG - F^2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} (\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\varrho,$$

or, by the skew symmetry of the curvature tensor,

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} (\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]) = -K(EG - F^2)\,\mathrm{J} \text{ with } \mathrm{J} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Codazzi equation was verified using

$$\begin{aligned} 0 &= f_u + \langle N_v, \nabla_{\frac{\partial}{\partial u}} X_u \rangle - e_v - \langle N_u, \nabla_{\frac{\partial}{\partial v}} X_u \rangle \\ &= f_u - (f, g)\Gamma_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - e_v + (e, f)\Gamma_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \end{aligned}$$

a similar equation is readily derived for $g_u - f_v$ to obtain

$$(f, g)_u - (f, g)\Gamma_1 = (e, f)_v - (e, f)\Gamma_2$$

as a version of the Codazzi equation in terms of the second fundamental form. In terms of the columns σ_i of the matrix of the shape operator,

$$0 = N_u + (X_u, X_v)\sigma_1 = N_v + (X_u, X_v)\sigma_2,$$

the Codazzi equation can be expressed as

$$\sigma_{2u} + \Gamma_1\sigma_2 = \sigma_{1v} + \Gamma_2\sigma_1.$$

Problem 18. Let Σ denote the matrix of the shape operator; prove that the partial covariant derivatives, $\nabla_{\frac{\partial}{\partial u}} \mathrm{S}$ and $\nabla_{\frac{\partial}{\partial v}} \mathrm{S}$, have matrix representations

$$\Sigma_u + [\Gamma_1, \Sigma] \text{ and } \Sigma_v + [\Gamma_2, \Sigma],$$

and derive an a expression of the Codazzi equation in terms of these.

Rem. In orthogonal coordinates, $\mathrm{I} = E\,du^2 + G\,dv^2$, the Gauss equation reads

$$0 = K\sqrt{EG} + \left(\tfrac{(\sqrt{G})_u}{\sqrt{E}}\right)_u + \left(\tfrac{(\sqrt{E})_v}{\sqrt{G}}\right)_v;$$

in particular, for a conformal parametrization, $0 = 2EK + \Delta \ln E$.

Problem 19. Verify the Gauss equation for orthogonal coordinates.

As a consequence of the Gauss-Codazzi equations we obtain one of the most prominent theorems in surface theory:

Gauss' theorema egregium. *K depends on I only.*

Proof. By the Gauss equation $K = -\frac{\varrho}{EG-F^2}$, where ϱ only depends on the induced metric. ∎

Cor. *If a surface admits an isometric (re-)parametrization then, necessarily, its Gauss curvature vanishes, $K \equiv 0$.*

Proof. For an isometric parametrization all $\Gamma_{ij}^k = 0$, hence $\mathrm{R} \equiv 0$ and, consequently, $K \equiv 0$. As the Gauss curvature is a geometric invariant of a surface, $\tilde{K} = K \circ (u,v)$ for a reparametrization $\tilde{X} = X \circ (u,v)$, we necessarily have $K \equiv 0$ as soon as a surface admits an isometric (re-)parametrization. ∎

Example. If $X : M \to \mathcal{E}^3$ takes values in a sphere of radius $r > 0$ then $K \equiv \frac{1}{r^2} \neq 0$, hence X does not admit (local) isometric parametrizations.

As another consequence of the Gauss-Codazzi equations a classification of surfaces all of whose points are umbilics is obtained.

Example. If $X : M \to \mathcal{E}^3$ takes values in a sphere or a plane in $\mathcal{E}^3$ then all of its points are umbilics.

Def. *A surface is called* totally umbilic *if every point is an umbilic.*

Thm. *Any totally umbilic surface is (part of) a plane or a sphere.*

Proof. If $X : M \to \mathcal{E}^3$ is totally umbilic then, for all $(u,v) \in M$,

$$\mathrm{S}_{(u,v)} = H(u,v)\,\mathrm{id}_{T_{(u,v)}X}$$

and the Codazzi equation reads

$$0 = (\nabla_{\frac{\partial}{\partial u}} \mathrm{S})X_v - (\nabla_{\frac{\partial}{\partial v}} \mathrm{S})X_u = H_u X_v - H_v X_u.$$

Hence $H_u = H_v \equiv 0$ so that $H \equiv const$.

If $H \equiv 0$ then $N \equiv const$ and the surface is part of a plane.

If $H \equiv const \neq 0$ then $Z := X + \frac{1}{H}N \equiv const$ and $|X - Z|^2 \equiv \frac{1}{H^2}$, showing that X takes values in a sphere of radius $\frac{1}{|H|}$ centred at Z. ∎

Problem 20. Suppose that $X : M \to \mathcal{E}^3$ parametrizes a surfaces so that X_u and X_v yield curvature directions, $0 = N_u + \kappa^+ X_u = N_v + \kappa^- X_v$. Prove: the Codazzi equation(s) reads

$$0 = \kappa^+_v + \tfrac{E_v}{2E}(\kappa^+ - \kappa^-) = \kappa^-_u - \tfrac{G_u}{2G}(\kappa^+ - \kappa^-).$$

Recall. The Gauss-Codazzi equations are the compatibility conditions of the Gauss-Weingarten equations of a surface $X : M \to \mathcal{E}^3$ with Gauss map N,

$$F_u = F\Phi, \;\; F_v = F\Psi \;\text{ for }\; F = (X_u, X_v, N), \tag{$*$}$$

where Φ and Ψ only depend on the first and second fundamental forms I and $\mathrm{I\!I}$ of X: the Gauss-Codazzi equations are equivalent to

$$0 = F^{-1}(F_{vu} - F_{uv}) = \Psi_u - \Phi_v + [\Phi, \Psi]. \tag{$**$}$$

Thus we obtain the following theorem, usually attributed to O Bonnet:

Fundamental theorem for Surfaces. *Given symmetric bilinear forms*

$$\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \;\text{ and }\; \mathrm{I\!I} = \begin{pmatrix} e & f \\ f & g \end{pmatrix},$$

with I *positive definite and satisfying the Gauss-Codazzi equations, there is (locally) a surface* X *with* I *and* $\mathrm{I\!I}$ *as its first and second fundamental forms.*

Moreover, this surface is unique up to Euclidean motion.

Rem. In contrast to the fundamental theorems for curves, here we need to require the Gauss-Codazzi equations as necessary and, locally, sufficient *compatibility conditions* for the existence of a surface X.

Proof. To formulate the Gauss-Weingarten equations ($*$) we use I and $\mathrm{I\!I}$ to determine the matrices $\Sigma = \mathrm{I}^{-1}\mathrm{I\!I}$ of the shape operator and Γ_i, by Koszul's formulas

$$\mathrm{I}\Gamma_1 = \tfrac{1}{2}\mathrm{I}_u - \tfrac{E_v - F_u}{2}\,\mathrm{J} \;\text{ and }\; \mathrm{I}\Gamma_2 = \tfrac{1}{2}\mathrm{I}_v + \tfrac{G_u - F_v}{2}\,\mathrm{J}.$$

The integrability conditions $(**)$ become the Gauss-Codazzi equations

(G) $\mathrm{I}\,(\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]) = -(eg - f^2)\,\mathrm{J}$ (*Gauss equation*), and

(C) $(f, g)_u - (f, g)\,\Gamma_1 = (e, f)_v - (e, f)\,\Gamma_2$ (*Codazzi equation*).

Thus, by the Maurer-Cartan lemma, the Gauss-Weingarten equations $(*)$ admit a local solution, $(u, v) \mapsto F(u, v) = (F_1, F_2, F_3)(u, v) \in \mathrm{Gl}(3)$, that is unique up to post-composition by a constant $A \in \mathrm{Gl}(3)$.

Since $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $\mathrm{I\!I}$ is symmetric, $F_{1v} = F_{2u}$ by the Gauss-Weingarten equations; hence, by the Poincaré lemma, locally

$$X_u = F_1 \text{ and } X_v = F_2.$$

with a map $(u, v) \mapsto X(u, v) \in \mathcal{E}^3$, that is unique up to translation.

We seek: X has first fundamental form I and F_3 is a unit normal field. As

$$((F^t)^{-1} \left(\begin{smallmatrix} \mathrm{I} & 0 \\ 0 & 1 \end{smallmatrix}\right) F^{-1})_u = (F^t)^{-1} \left(\begin{smallmatrix} \mathrm{I}_u - \mathrm{I}\Gamma_1 - \Gamma_1^t \mathrm{I} & 0 \\ 0 & 0 \end{smallmatrix}\right) F^{-1} = 0$$

by Koszul's formulas, and similarly for the v-deritative, we learn that

$$(F^t)^{-1} \left(\begin{smallmatrix} \mathrm{I} & 0 \\ 0 & 1 \end{smallmatrix}\right) F^{-1} \equiv const, \text{ hence } F^t F = \left(\begin{smallmatrix} \mathrm{I} & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

as soon as we fix the constant of integration, $A \in \mathrm{Gl}(3)$, so that F satisfies this equation at an initial point; this choice fixes F up to $A \in \mathrm{O}(3)$ as

$$F^t F = \tilde{F}^t \tilde{F} = F^t A^t A F \;\; \Rightarrow \;\; A^t A = \mathrm{id}_{\mathbb{R}^3}.$$

We seek: $N := F_3$ is the Gauss map of X. By the above choice F_3 is already a unit normal field and $\det F = \pm\sqrt{EG - F^2}$ does nowhere vanish, hence all we ask is

$$\langle X_u \times X_v, N \rangle = \det(X_u, X_v, N) = \det F > 0,$$

which can be achieved by possibly post-composing F with a reflection; this further choice fixes F up to post-composition with $A \in SO(3)$.

Finally, we seek: X has second fundamental form $\mathrm{I\!I}$. This follows directly from the construction of Φ and Ψ and the above choices as, for example,

$$\langle X_{uu}, N \rangle = \langle F_{1u}, F_3 \rangle = \langle F_1 \Gamma^1_{11} + F_2 \Gamma^2_{11} + F_3 e, F_3 \rangle = e.$$

Moreover, after the above choices, X is unique up to Euclidean motion, $X \mapsto \tilde{O} + A(X - O)$ with $A \in SO(3)$ and $O, \tilde{O} \in \mathcal{E}^3$. ∎

3 Curves on surfaces

We now turn to curves on surfaces, where our notions of special ribbons (asymptotic, geodesic, curvature wibbons) will become useful beyond what we discussed in the first section. The key observation is that a curve on a surface comes with a natural unit normal field, given by the Gauss map of the surface.

However, the analysis of curves on surfaces has wider geometric implications, when considering parametrizations so that all "parameter lines" have special geometric properties. For example, we will see that the vanishing of the Gauss curvature is not only a necessary condition for the existence of local isometric (re-)parametrizations (theorema egregium), but also a sufficient criterion (Minding's theorem below).

3.1 Natural ribbon & Special lines on surfaces

If $X : \mathbb{R}^2 \supset M \to \mathcal{E}^3$ is a (parametrized) surface then the composition

$$I \ni t \mapsto X(u(t), v(t)) \in \mathcal{E}^3$$

with a map $(u, v) : I \to M$ defines a *curve on the surface* $X(M) \subset \mathcal{E}^3$ as soon as $X \circ (u, v)$ is regular, that is, as soon as (dropping arguments)

$$\forall t \in I : (X_u u' + X_v v')(t) \neq 0 \quad \Leftrightarrow \quad \forall t \in I : \binom{u'}{v'}(t) \neq 0$$

since $d_{(u,v)}X : \mathbb{R}^2 \to \mathbb{R}^3$ injects for every $(u, v) \in M$.

Problem 1. Carefully compute, including all arguments, the unit tangent field of a curve $X \circ (u, v) : I \to \mathcal{E}^3$ on a surface at some $t \in I$.

Expl & Def. The *parameter lines* of a surface $X : M \to \mathcal{E}^3$ are the curves

$$t \mapsto X(u + t, v) \text{ and } t \mapsto X(u, v + t).$$

Rem & Def. If $t \mapsto X(u(t), v(t))$ is a curve on a surface $X : M \to \mathcal{E}^3$ then

$$T_t(X \circ (u, v)) \subset T_{(u(t),v(t))}X$$

or, otherwise said, its unit tangent field T is always tangential to the surface:

$$T = \frac{X_u u' + X_v v'}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}};$$

thus the Gauss map N of the surface X yields a unit normal field

$$I \ni t \mapsto N(u(t), v(t)) \in \mathbb{R}^3$$

along the curve, hence defines the *natural ribbon* for the curve; the corresponding frame is sometimes called its *Darboux frame*.

Def. *A curve $t \mapsto X(u(t), v(t))$ on a surface $X : M \to \mathcal{E}^3$ is called*

- *a* curvature line *if its natural ribbon is a curvature ribbon, $\tau \equiv 0$;*
- *an* asymptotic line *if its natural ribbon is asymptotic, i.e., $\kappa_n \equiv 0$;*
- *a* pre-geodesic line *if its natural ribbon is geodesic, i.e., $\kappa_g \equiv 0$.*

Rem. Thus a curve is curvature line on a surface if and only if the Gauss map of the surface is parallel along the curve.

As parallel normal fields make constant angles we obtain:

Joachimsthal's theorem. *Suppose that two surfaces intersect along a curve and that the curve is a curvature line for one of the two surfaces; then it is a curvature line for the other surface also if and only if the two surfaces intersect at a constant angle.*

Problem 2. Prove Joachimsthal's Theorem.

Rodrigues' equation. *$t \mapsto X(u(t), v(t))$ is a curvature line if and only if*

$$0 = (dN + \kappa\, dX)\binom{u'}{v'}.$$

Proof. The structure equations for the natural ribbon yield

$$\nabla^{\perp}(N \circ (u, v)) = (N_u u' + N_v v') + (X_u u' + X_v v')\,\kappa_n,$$

hence $t \mapsto (X, N)(u(t), v(t))$ is a curvature ribbon iff Rodrigues' equation holds. ∎

Rem. Thus $t \mapsto X(u(t), v(t))$ is a curvature line iff $dX(\begin{smallmatrix} u' \\ v' \end{smallmatrix})$ is a curvature direction at every point of the curve, as

$$(dN + \kappa\, dX)\binom{u'}{v'} = (-\mathrm{S} + \kappa\, \mathrm{id}_{TX})(dX\binom{u'}{v'}).$$

Example. For a surface X of revolution with Gauss map N (cf Sect 2.2),

$$\begin{aligned} X(u,v) &= O + e_1 r(u) \cos v + e_2 r(u) \sin v + e_3 h(u), \\ N(u,v) &= -e_1 h'(u) \cos v - e_2 h'(u) \sin v + e_3 r'(u), \end{aligned}$$

we deduced

$$N_u \parallel X_u \text{ and } N_v \parallel X_v;$$

hence the parameter lines of the surface are curvature lines, by Rodrigues' equation. Alternatively, this follows directly from Joachimsthal's theorem.

Def & Thm. $X : \mathbb{R}^2 \supset M \to \mathcal{E}^3$ *is a* curvature line parametrization, *if all parameter lines are curvature lines. Any surface admits locally, away from umbilics, a curvature line (re-)parametrization.*

Problem 3. Find a curvature line reparametrization for the helicoid.

Rem. For X a curvature line parametrization (X_u, X_v) diagonalizes the shape operator,

$$\mathrm{S}X_u \parallel X_u \text{ and } \mathrm{S}X_v \parallel X_v;$$

hence, as S is symmetric, we learn that $X_u \perp X_v$ and $N_u = -\mathrm{S}X_u \perp X_v$ or, otherwise said,

$$F = f = 0$$

for the mixed coefficients of the fundamental forms of X,

$$\mathrm{I} = E\, du^2 + 2F\, dudv + G\, dv^2 \text{ and } \mathrm{II} = e\, du^2 + 2f\, dudv + g\, dv^2.$$

Conversely, if $F = f = 0$, then X is a curvature line parametrization, as follows from the matrix representations of the shape operator.

Problem 4. Prove: $F = f = 0$ characterizes curvature line parametrizations.

Lemma. *The normal curvature for a curve* $t \mapsto X(u(t), v(t))$ *on a surface is given by*

$$\kappa_n = \frac{\mathrm{II}(\binom{u'}{v'}, \binom{u'}{v'})}{\mathrm{I}(\binom{u'}{v'}, \binom{u'}{v'})}.$$

Proof. The normal curvature of a ribbon (X, N) is given by

$$\kappa_n = \tfrac{1}{|X'|} \langle T', N \rangle = \tfrac{1}{|X'|^2} \langle X'', N \rangle = -\tfrac{\langle X', N' \rangle}{\langle X', X' \rangle};$$

applying the chain rule to the natural ribbon yields the claim. ∎

Rem & Def. The normal curvature κ_n (of the natural ribbon) for a curve on a surface depends only on the tangent direction (and *not* on u'' or v''). Thus we also term it the "*normal curvature κ_n of a tangent direction*".

As an immediate consequence we obtain:

Euler's theorem. *The normal curvatures κ_n at a point of a surface satisfy*

$$\kappa_n(\vartheta) = \kappa^+ \cos^2 \vartheta + \kappa^- \sin^2 \vartheta,$$

where $\kappa^\pm$ are the principal curvatures and ϑ is the angle between the tangent direction of $\kappa_n(\vartheta)$ and the curvature direction of κ^+.

Problem 5. Prove Euler's Theorem. [Hint: fix $(u, v) \in M$ and use a basis (e_1, e_2) of $\mathbb{R}^2$ that is orthonormal for $\mathrm{I}|_{(u,v)}$ and diagonalizes $\mathrm{II}|_{(u,v)}$.]

Cor. *The principal curvatures can be characterized as the extremal values of the normal curvatures at a point of a surface.*

As another application of the above lemma we obtain the differential equation for asymptotic lines:

Cor. $t \mapsto X(u(t), v(t))$ *is an asymptotic line of* $(u, v) \mapsto X(u, v)$ *if and only if*

$$e\, u'^2 + 2f\, u'v' + g\, v'^2 \equiv 0.$$

Example. *Circular helices as asymptotic lines on the helicoid*:

$$\mathrm{II}|_{(u,v)} = -2du\, dv$$

for the helicoid $X(u,v) = O + e_1 \sinh u \cos v + e_2 \sinh u \sin v + e_3 v$, hence the parameter lines of X are asymptotic lines, in particular, the helices

$$t \mapsto X(u,t) = O + e_1 r \cos t + e_2 r \sin t + e_3 t \text{ with } r = \sinh u.$$

Problem 6. Fix a point $X(u,v)$ on a parametrized surface. Prove that an asymptotic line can pass through $X(u,v)$ in two, one or no directions, depending on the sign of the Gauss curvature $K(u,v)$.

3.2 Geodesics & Exponential map

Geometrically, geodesics can be thought of as the shortest possible paths on a surface between any two points (at least locally); equivalently, they can be characterized as the "straight lines" in the surface, i.e., those which are not curved: $\kappa_g \equiv 0$. This is what we call "pre-geodesic lines".

From a physics point of view, one may think of a geodesic as the path of a particle on a surface that no forces are acting upon, i.e., that is not accelerated (inside the surface):

Def. *The* covariant derivative *of a tangent field $Y : I \to \mathbb{R}^3$ along a curve $t \mapsto X(u(t), v(t))$ on a surface $X : M \to \mathcal{E}^3$ is the tangential part of its derivative,*

$$\tfrac{D}{dt} Y := Y' - N\,\langle N, Y' \rangle.$$

A geodesic *is an acceleration-free curve $t \mapsto C(t) = X(u(t), v(t))$ on a surface,*

$$\tfrac{D}{dt} C' \equiv 0.$$

Rem. Thus $t \mapsto C(t)$ is geodesic iff its velocity C' is *parallel* (w.r.t. $\frac{D}{dt}$).

Example. *Circular helices as geodesics of a circular cylinder*:

$$t \mapsto C(t) = O + e_1 r \cos t + e_2 r \sin t + e_3 h t = X(ht, t)$$

is a geodesic on the circular cylinder of radius $r > 0$,

$$(u,v) \mapsto X(u,v) = O + e_1 r \cos v + e_2 r \sin v + e_3 u,$$

since

$$C''(t) \perp X_u(ht,t), X_v(ht,t) \;\;\Rightarrow\;\; \tfrac{D}{dt} C'(t) \equiv 0.$$

Thm. *Geodesics are the constant speed pre-geodesic lines.*

Proof. Firstly, any geodesic C has constant speed, by Leibniz' rule:

$$\tfrac{1}{2}(|C'|^2)' = \langle C', \tfrac{D}{dt}C'\rangle \equiv 0.$$

Secondly, assume that $|C'| \equiv const$ for a curve, $C(t) = X(u(t), v(t))$; then

$$C'' \frac{1}{|C'|^2} = T' \frac{1}{|C'|} = N\kappa_n - B\kappa_g \parallel N \;\;\Leftrightarrow\;\; \kappa_g \equiv 0$$

by the structure equations of the Darboux frame $(T, N, B) : I \to \mathrm{SO}(3)$ of the curve C. ∎

Clairaut's theorem. *For a geodesic on a surface of revolution the product*

$$r \sin\theta \equiv const,$$

where $r = r(s)$ is the distance from the axis and $\theta = \theta(s)$ is the angle that the geodesic makes with the profile curves.

Proof. Let $C(s) = O + e_1 r(s)\cos v(s) + e_2 r(s)\sin v(s) + e_3 h(s)$ be a geodesic on a surface of revolution, wlog., arc length parametrized, set

$$C_t(s) := O + A(t)(C(s) - O) \text{ and } Y(s) := \frac{\partial}{\partial t}\Big|_{t=0} C_t(s),$$

where

$$A(t)(e_1, e_2, e_3) = (e_1, e_2, e_3)\begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe that $Y(s) = (-e_1 \sin v(s) + e_2 \cos v(s))\, r(s)$ so that

$$r\sin\theta = r\cos(\tfrac{\pi}{2} - \theta) = \langle C', Y\rangle = \langle \tfrac{\partial}{\partial s}C_t, \tfrac{\partial}{\partial t}C_t\rangle|_{t=0}.$$

Then, as each C_t is an arc length parametrized geodesic, we compute

$$\frac{\partial}{\partial s}\langle \frac{\partial}{\partial s}C_t, \frac{\partial}{\partial t}C_t\rangle = \langle \underbrace{\frac{\partial}{\partial s}\frac{\partial}{\partial s}C_t}_{\parallel N}, \underbrace{\frac{\partial}{\partial t}C_t}_{\perp N}\rangle + \frac{1}{2}\frac{\partial}{\partial t}\underbrace{\langle \frac{\partial}{\partial s}C_t, \frac{\partial}{\partial s}C_t\rangle}_{\equiv const} \equiv 0,$$

hence $r\sin\theta \equiv const$. ∎

Rem. This proof generalizes to obtain similar theorems for geodesics on any surface that is invariant under a 1-parameter family of isometries, e.g., under screw motions or under translations (cylinders).

Rem & Expl. Clairaut's theorem provides a *necessary* condition for a geodesic, not a *sufficient* condition: for example, consider the straight line

$$t \mapsto C(t) = O + e_1 + (e_2 + e_3)\,t = X(u(t), v(t))$$

as a geodesic on the 1-sheeted hyperboloid parametrized by

$$(u, v) \mapsto X(u, v) := O + e_1 \cosh u \cos v + e_2 \cosh u \sin v + e_3 \sinh u;$$

hence

$$r \sin\theta = \langle \tfrac{C'}{|C'|}, Y\rangle = \tfrac{\cosh u \cos v}{\sqrt{2}} \equiv \tfrac{1}{\sqrt{2}}.$$

On the other hand, every *circle of latitude* $t \mapsto X(u, t)$ satisfies Clairaut's condition, $r \sin\theta \equiv \cosh u$, but is in general not a geodesic.

Problem 7. Let $(u, v) \mapsto X(u, v) = (r(u) \cos v, r(u) \sin v, h(u))$ be a surface of revolution. Prove that:

(a) if a circle of latitude $t \mapsto X(u, t)$ is geodesic then $r'(u) = 0$;

(b) if $r'^2 + h'^2 \equiv 1$ then the profile curves $t \mapsto X(t, v)$ are geodesic.

Motivation. It seems obvious that an initial point and velocity determine the path of an acceleration-free particle uniquely. To substantiate this intuition we derive the differential equations of a geodesic: thus consider a tanget field $t \mapsto Y(t) = X_u(u(t), v(t))\, x(t) + X_v(u(t), v(t))\, y(t)$ along a curve $t \mapsto C(t) = X(u(t), v(t))$ on a surface $X : M \to \mathcal{E}^3$ to compute

$$\begin{aligned}
\tfrac{D}{dt} Y &= X_u x' + ((\nabla_{\frac{\partial}{\partial u}} X_u)\, u' + (\nabla_{\frac{\partial}{\partial v}} X_u)\, v')\, x \\
&+ X_v y' + ((\nabla_{\frac{\partial}{\partial u}} X_v)\, u' + (\nabla_{\frac{\partial}{\partial v}} X_v)\, v')\, y \\
&= X_u\, (x' + (\Gamma^1_{11} u' + \Gamma^1_{21} v')\, x + (\Gamma^1_{12} u' + \Gamma^1_{22} v')\, y) \\
&+ X_v\, (y' + (\Gamma^2_{11} u' + \Gamma^2_{21} v')\, x + (\Gamma^2_{12} u' + \Gamma^2_{22} v')\, y);
\end{aligned}$$

in particular, for $Y = C'$, that is, $x = u'$ and $y = v'$,

$$\begin{aligned}
\tfrac{D}{dt} C' &= X_u\, (u'' + \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2) \\
&+ X_v\, (v'' + \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2).
\end{aligned}$$

Thus we obtain:

Geodesic equations. *A curve* $t \mapsto C(t) = X(u(t), v(t))$ *is a geodesic iff*

$$\begin{array}{rcl} 0 &=& u'' + \Gamma^1_{11}u'^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}v'^2, \\ 0 &=& v'' + \Gamma^2_{11}u'^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}v'^2. \end{array} \qquad (*)$$

<u>*Rem*</u>. Similarly, using the structure equations for the Darboux frame of a curve $t \mapsto X(u(t), v(t))$, the geodesic curvature can be computed:

$$\kappa_g = \frac{\sqrt{EG-F^2}}{\sqrt{Eu'^2+2Fu'v'+Gv'^2}^3} \det \begin{pmatrix} u' & u''+\Gamma^1_{11}u'^2+2\Gamma^1_{12}u'v'+\Gamma^1_{22}v'^2 \\ v' & v''+\Gamma^2_{11}u'^2+2\Gamma^2_{12}u'v'+\Gamma^2_{22}v'^2 \end{pmatrix}.$$

Cor. *Geodesics can be determined from the induced metric* I *alone.*

<u>*Example*</u>. Geodesics on a circular cylinder are straight lines after developing into a plane: the generators of the cylinder and circular helices.

Cor. *Given* $(u_o, v_o) \in M$ *and* $Y = d_{(u_o,v_o)}X(\binom{x_o}{y_o}) \in T_{(u_o,v_o)}X$, *there is a unique (maximal) geodesic* $t \mapsto C_Y(t) = X(u(t), v(t))$ *on the surface* $X : M \to \mathcal{E}^3$ *with*

$$(u(0), v(0)) = (u_o, v_o) \text{ and } (u'(0), v'(0)) = (x_o, y_o). \qquad (**)$$

<u>*Rem*</u>. The initial condition $(**)$ says that an initial point and tangent direction are given on the surface; if $X(u_o, v_o)$ is a double point of the surface then $(**)$ specifies the leaf of the surface that C_Y "lives" on.

Proof. With $(w_1, w_2, w_3, w_4) = (u, v, u', v')$ the geodesic equations $(*)$ yield a system of ODEs of the form $w' = f(w)$, where f is differentiable:

$$\begin{array}{rcl} w_1' &=& w_3, \\ w_2' &=& w_4, \\ w_3' &=& -\Gamma^1_{11}(w_1, w_2)\, w_3^2 - 2\Gamma^1_{12}(w_1, w_2)\, w_3w_4 - \Gamma^1_{22}(w_1, w_2)\, w_4^2, \\ w_4' &=& -\Gamma^2_{11}(w_1, w_2)\, w_3^2 - 2\Gamma^2_{12}(w_1, w_2)\, w_3w_4 - \Gamma^2_{22}(w_1, w_2)\, w_4^2. \end{array}$$

Hence, the sought geodesic is obtained from a solution of the initial value problem

$$w' = f(w), \quad w(0) = (u_o, v_o, x_o, y_o),$$

and the claim follows from the Picard-Lindelöf Thm. ∎

Problem 8. Find all geodesics of a plane in $\mathcal{E}^3$ with a given initial point.

Problem 9. Find all geodesics on a unit sphere with given initial point and velocity. [Hint: do not parametrize the sphere.]

Lemma. *$C_{Ys}(t) = C_Y(st)$ for $s \in (0,1)$.*

Proof. If $C_Y : I \to \mathcal{E}^3$ is a geodesic satisfying the initial conditions $(**)$ then

$$\tfrac{d}{dt}C_Y(st) = C_Y'(st)\,s \text{ and } \tfrac{D}{dt}\tfrac{d}{dt}C_Y(st) = 0,$$

hence coincides on I with the unique geodesic C_{Ys}. ∎

Rem. By the smooth dependence of solutions C_Y of the initial value problem $(*)$ on the initial condition, we obtain a smooth map

$$\mathbb{R} \times T_{(u_o,v_o)}X \ni (t,Y) \mapsto C_Y(t) \in \mathcal{E}^3,$$

defined on an open neighbourhood $I \times U$ of $(0,0) \in \mathbb{R} \times T_{(u_o,v_o)}X$ with star-shaped U and, wlog., $I \supset [0,1]$, by the above lemma:

Lemma & Def. *Given a point $X(u_o,v_o)$ on a surface $X : M \to \mathcal{E}^3$*

$$Y \mapsto \exp(Y) := C_Y(1)$$

defines a smooth map on an open neighbourhood U of $0 \in T_{(u_0,v_0)}X$ with

$$d_0 \exp = \mathrm{id}_{T_{(u_o,v_o)}X}.$$

exp *is called the* exponential map *of $X : M \to \mathcal{E}^3$ at $X(u_o,v_o)$.*

Proof. Taking differentiability of exp as granted, we compute directional derivatives:

$$d_0 \exp(Y) = \tfrac{d}{dt}|_{t=0} \exp(Yt) = \tfrac{d}{dt}|_{t=0} C_Y(t) = Y$$

for $Y \in T_{(u_o,v_o)}X$. ∎

Rem. Thus $\exp : T_{(u_o,v_o)}X \supset U \to X(M)$ yields a local diffeomorphism and, in particular, a (re-)parametrization for X around $X(u_o,v_o)$.

3.3 Geodesic polar coordinates & Minding's theorem

By the previous section, the exponential map may be used to produce a reparametrization of a surface $X : M \to \mathcal{E}^3$ around any point $X(u_o, v_o)$: in particular, polar coordinates (r, ϑ) around $0 \in T_{(u_o,v_o)}X$ can be molded onto the surface

Def. *A (re-)parametrization of a surface by* geodesic polar coordinates (r, ϑ) *around a point* $X(0,0)$ *of the surface is given by the map*

$$(r, \vartheta) \mapsto X(r, \vartheta) := \exp(e_1 r \cos\vartheta + e_2 r \sin\vartheta),$$

where (e_1, e_2) *is an orthonormal basis of the tangent space* $T_{(0,0)}X$.

Rem. Note that a parametrization by geodesic polar coordinates (r, ϑ) is *not* regular at $r = 0$; however, it is regular on $(0, \varepsilon) \times \mathbb{R}$ for some $\varepsilon > 0$.

Problem 10. Determine a parametrization by geodesic polar coordinates around a point of a sphere $S^2 \subset \mathcal{E}^3$ with centre $O \in \mathcal{E}^3$ and radius $R > 0$. Compute its induced metric.

Lemma. *In geodesic polar coordinates* (r, ϑ) *the induced metric*

$$\mathrm{I} = dr^2 + G\,d\vartheta^2 \ \text{ with } \ \sqrt{G}|_{r=0} = 0 \ \text{ and } \ \tfrac{\partial\sqrt{G}}{\partial r}|_{r=0} = 1.$$

Proof. Denote $Y := e_1 \cos\vartheta + e_2 \sin\vartheta$ and observe that, for fixed ϑ,

$$r \mapsto X(r, \vartheta) = C_{Yr}(1) = C_Y(r)$$

is an arc length parametrized geodesic with $C_Y(0) = X(0,0)$.

$\underline{E = 1}$. C_Y is arc-length parametrized, hence $E = |X_r|^2 = |C'_Y|^2 = 1$.

$\underline{F = 0}$. $X_\vartheta|_{r=0} = 0$, hence $F|_{r=0} = \langle X_r, X_\vartheta\rangle|_{r=0} = 0$; moreover

$$F_r = \langle X_{rr}, X_\vartheta\rangle + \langle X_r, X_{r\vartheta}\rangle = \langle X_{rr}, X_\vartheta\rangle + \tfrac{1}{2}E_\vartheta = 0$$

since $X_{rr} = \frac{D}{dr}C'_Y + N...$; hence $r \mapsto F(r, \vartheta) \equiv 0$ for any fixed ϑ.

$\underline{\sqrt{G}|_{r=0} = 0}$. $X_\vartheta|_{r=0} = 0$, hence $G|_{r=0} = |X_\vartheta|^2\Big|_{r=0} = 0$.

$\underline{(\sqrt{G})_r|_{r=0} = 1}$. Using $X_\vartheta|_{r=0} = 0$ and $|\frac{d}{d\vartheta}X_r|_{r=0}|^2 = |\frac{d}{d\vartheta}Y|^2 = 1$, we get

$$\tfrac{1}{2}G_{rr}\Big|_{r=0} = (\langle X_{rr\vartheta}, X_\vartheta\rangle + \langle X_{r\vartheta}, X_{r\vartheta}\rangle)|_{r=0} = 1,$$

hence
$$\tfrac{G}{r^2} \sim \tfrac{G_r}{2r} \sim \tfrac{G_{rr}}{2} \to 1 \text{ as } r \searrow 0,$$
showing that $\sqrt{G}$ has a (one-sided) derivative $(\sqrt{G})_r|_{r=0} = 1$. ∎

Rem. In geodesic polar coordinates (r, ϑ) the Gauss equation specializes to
$$0 = K\sqrt{G} + (\sqrt{G})_{rr}.$$

Problem 11. Prove: $K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$ in geodesic polar coordinates (r, ϑ).

Cor. *Geodesics are locally the shortest curves between two points.*

Proof. We work in geodesic polar coordinates: let $t \mapsto X(r(t), \vartheta(t))$ be a curve between two points $X(0,0)$ and $X(r(1), \vartheta(1))$; then its length
$$\int_0^1 \sqrt{r'^2 + G(r,\vartheta)\vartheta'^2}\, dt \geq \int_0^1 r'\,dt = r(1)$$
with equality iff $\vartheta' \equiv 0$ and $r' > 0$, that is, iff $t \mapsto X(r(t), \vartheta(t))$ is a radial geodesic, up to reparametrization by the regular function r. ∎

Minding's theorem. *Any two surfaces with the same constant Gauss curvature are locally isometric, i.e., there are local parametrizations X_1 and X_2 so that their induced metrics coincide, $\mathrm{I}_1 = \mathrm{I}_2$.*

Rem. By Gauss theorema egregium: two isometric surfaces do necessarily have the same Gauss curvature; by Minding's theorem: for surfaces of *constant Gauss curvatures* this is also sufficient.

Proof. For a parametrization by geodesic polar coordinates (r, ϑ),
$$\mathrm{I} = dr^2 + G\,d\vartheta^2 \text{ with } \sqrt{G}|_{r=0} = 0 \text{ and } (\sqrt{G})_r|_{r=0} = 1$$
and, by the Gauss equation,
$$K = -\tfrac{(\sqrt{G})_{rr}}{\sqrt{G}}.$$
Thus, if $K \equiv const$ then G satisfies, for fixed ϑ, the initial value problem
$$0 = (\sqrt{G})_{rr} + K\sqrt{G} \text{ with } 0 = \sqrt{G}|_{r=0} \text{ and } 1 = (\sqrt{G})_r|_{r=0},$$

which has a unique solution

$$\sqrt{G(r,\vartheta)} = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K} r) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K} r) & \text{if } K < 0. \end{cases}$$

Hence the metric I is uniquely determined by K and parametrization by geodesic polar coordinates shows that any two surfaces with the same constant Gauss curvature are locally isometric. ∎

4 Special surfaces

As an application of the developed general theory we now turn to some (classes of) "examples" — just as we were able to determine *all* (Frenet) curves with constant curvature and torsion or *all* totally umbilic surfaces, i.e., surfaces with $H^2 - K = 0$, we will now turn to *classify* (describe) more general classes of surfaces satisfying certain curvature conditions.

4.1 Developable surfaces

The cylinder over an arbitrary planar curve, with generators orthogonal to the plane of its profile curve, can be "developed" into a (parameter) plane, that is, admits an isometric parametrization. By Gauss' theorema egregium and by Minding's theorem, a necessary and sufficient condition for a surface to admit such (local) isometric parameters is the vanishing of its Gauss curvature, $K \equiv 0$. This describes the "intrinsic geometry" of such a surface: the length and angle measurements inside a surface of Gauss curvature $K \equiv 0$ is that of a (flat) plane. In this section we investigate the "extrinsic geometry" of such developable surfaces: how they can be curved in space, or their "shape" in space.

Def. *A surface is* developable *if its Gauss curvature vanishes,* $K \equiv 0$.

Example. Consider a *(general) cone*, over a *profile curve* $C : I \to \mathcal{E}^3$ on a unit sphere in $\mathcal{E}^3$ with centre $Z \in \mathcal{E}^3$,

$$X : I \times \mathbb{R} \to \mathcal{E}^3, \ (u, v) \mapsto X(u, v) := Z + (C(u) - Z)\, e^v.$$

Every sphere centred at Z intersects the surface orthogonally along a parameter line $u \mapsto X(u, v)$, and every plane containing a tangent line of the curve C at $u \in I$ and the centre Z touches the surface along a *generator* of the cone, i.e., along a parameter curve $v \mapsto X(u, v)$; hence X is a curvature line parametrization, by Joachimsthal's theorem. In particular, the generators of the cone are curvature lines, and the Gauss curvature of the surface vanishes, with one of its principal curvatures:

$$0 = N_v + X_v \cdot 0 \ \Rightarrow \ K \equiv 0.$$

Problem 1. Let $(u,v) \mapsto Z + (C(u) - Z)\, e^v$ be the cone over an arc length parametrized spherical curve $C : I \to S^2(Z)$, i.e., $|C - Z| \equiv 1$. Specify an isometry (development) of the cone into a Euclidean plane.

Mission. We wish to describe *all* developable surfaces.

Codazzi equations. To this end, let $X : M \to \mathcal{E}^3$ denote a curvature line parametrization of a developable surface with Gauss map $N : M \to \mathbb{R}^3$,

$$0 = N_u + X_u\kappa^+ = N_v + X_v\kappa^-, \text{ where } \begin{cases} \kappa^+ = \kappa, \\ \kappa^- \equiv 0, \end{cases}$$

wlog, since $0 = K = \kappa^+\kappa^-$. The Codazzi equations then read

$$\begin{array}{rcccl} 0 &=& \frac{\kappa_v^+}{\kappa^+ - \kappa^-} + \frac{E_v}{2E} &=& (\ln \kappa\sqrt{E})_v, \\ 0 &=& \frac{\kappa_u^-}{\kappa^+ - \kappa^-} - \frac{G_u}{2G} &=& -(\ln \sqrt{G})_u; \end{array}$$

thus we may adjust the parameter v so that $G \equiv 1$ or, equivalently,

$$\mathrm{I} = E du^2 + dv^2 \text{ and } \mathrm{I\!I} = E\kappa\, du^2.$$

Koszul's formulas. Next we determine the Christoffel symbols,

$$\Gamma_1 = \frac{1}{2E}\begin{pmatrix} E_u & E_v \\ -EE_v & 0 \end{pmatrix} \text{ and } \Gamma_2 = \frac{1}{2E}\begin{pmatrix} E_v & 0 \\ 0 & 0 \end{pmatrix}.$$

In particular, we learn that $X_{vv} = X_u\Gamma_{22}^1 + X_v\Gamma_{22}^2 + Ng = 0$, showing that, for any fixed u and assuming wlog that $(u,0) \in M$,

$$X(u,v) = C(u) + Y(u)\, v \text{ with } \begin{cases} C(u) := X(u,0), \\ Y(u) := X_v(u,0); \end{cases}$$

namely, with $Z(v) := X(u,v) - X(u,0) - X_v(u,0)\, v$ we have

$$Z(0) = 0, \;\; Z'(0) = 0 \text{ and } Z'' \equiv 0 \text{ hence } Z \equiv 0.$$

Thus every parameter line $v \mapsto X(u,v)$ is a straight line, so that X is a ruled surface:

Def. *A* ruled surface *consists of a* 1*-parameter family of straight lines.*

Rem & Expl. General cylinders and cones are ruled surfaces.

The helicoid is obtained by a screw motion of a straight line, hence it is ruled — but not developable as its Gauss curvature $K < 0$ (cf Sect 3.1).

<u>*Problem 2*</u>. Parametrize a 1-sheeted hyperboloid in $\mathcal{E}^3$,

$$H = \{O + e_1x + e_2y + e_3z \in \mathcal{E}^3 \,|\, x^2 + y^2 = 1 + z^2\},$$

as a ruled surface and compute its Gauss curvature. Sketch one of the two rulings of the surface.

<u>*Gauss equation*</u>. With the above normalization, $G \equiv 1$, the shape of the induced metric, $\mathrm{I} = E\,du^2 + dv^2$, is the same as for geodesic polar coordinates; accordingly, the Gauss equation reads

$$0 = K\sqrt{E} + (\sqrt{E})_{vv} = (\sqrt{E})_{vv}.$$

Hence

$$(\sqrt{E})(u,v) = (\sqrt{E})(u,0) + (\sqrt{E})_v(u,0)\,v$$

as above and, if $E_v(u,0) \neq 0$, then the induced metric degenerates,

$$E(u,v(u)) = 0, \text{ for } v(u) = -\tfrac{2}{(\ln E)_v(u,0)}.$$

<u>*General cylinder*</u>. First we consider the case $u \mapsto E_v(u,0) \equiv 0$; since

$$X_{uv} = X_u\Gamma^1_{12} + X_v\Gamma^2_{12} + Nf = X_u(\ln\sqrt{E})_v$$

we deduce that $Y = X_v \equiv const$ is a constant (unit) normal vector of the curve $u \mapsto C(u)$, that hence is the planar profile curve of a general cylinder,

$$X(u,v) = C(u) + Y\,v \text{ with } B \equiv const.$$

<u>*General cone*</u>. When $E_v(u,0) \neq 0$ for $(u,0) \in M$ we may consider the map

$$u \mapsto Z(u) := X(u,v(u)) \text{ with } v(u) := -\tfrac{2}{(\ln E)_v(u,0)}; \qquad (*)$$

if $u \mapsto v(u) \equiv r = const$ then, using $X_{uv} = X_u(\ln\sqrt{E})_v$ again, we learn that

$$Z'(u) = X_u(u,0) - X_{vu}(u,0)\,\tfrac{2}{(\ln E)_v(u,0)} = 0.$$

Hence $Z \equiv const$ and C takes values in a sphere with centre Z and radius r, since

$$|C - Z|^2 = |Y\,r|^2 = r^2;$$

thus $u \mapsto C(u)$ is the profile curve of a general cone with vertex Z,

$$X(u,v) = C(u) + Y(u)\,v = Z + Y(u)(v - r).$$

<u>*Tangent developable*</u>. When $E_v(u,0) \neq 0$ and $v'(u) \neq 0$ for $(u,0) \in M$, then

$$Z' = Y\,v' \neq 0; \text{ hence } X(u,v) = Z(u) + Z'(u)\tfrac{v-v(u)}{v'(u)}$$

is a *tangent developable*: a surface generated by the tangents of its (regular) *curve of regression* or *directrix*.

Any such tangent developable is, in fact, developable: its Gauss map

$$N(u,v) = -\tfrac{Z'\times Z''}{|Z'\times Z''|}(u) \text{ satisfies } 0 = N_v = N_v + X_v \cdot 0,$$

hence the parameter lines $v \mapsto X(u,v)$ are curvature lines and the corresponding principal curvature $\kappa^- \equiv 0$ vanishes.

<u>*Problem 3*</u>. Let $(u,v) \mapsto X(u,v) = Z(u) + Z'(u)\,v$ be the tangent developable of an arc length parametrized Frenet curve $Z : I \to \mathcal{E}^3$; determine the curvature lines of X through a point $X(u,v)$, and describe the construction geometrically.

Thm (Classification of developable surfaces). *A surface is developable if and only if it is a composition of general cylinders, cones and tangent developables, that are fitted smoothly along generators.*

<u>*Rem*</u>. Developable surfaces are of interest in design or architecture; Gehry's Guggenheim museum in Bilbao (Spain) is a prominent example, where developable surfaces have played a role in design.

4.2 Minimal surfaces

Minimal surfaces appear, from a physical point of view, as "soap films": dipping a closed wire frame into soap solution the shape formed by the soap will be a minimal surface; mathematically, the problem of existence (and determination) of a surface of minimum area bounded by a closed space curve is known as "Plateau's problem". We will not take this approach, of defining minimal surfaces as surfaces of (locally) minimal area (or, energy), but we use the corresponding Euler-Lagrange equation to define this class of surfaces:

Def. *A surface is* minimal *if its mean curvature vanishes,* $H \equiv 0$.

<u>*Problem 4*</u>. Show that the helicoid is a minimal surface, and determine its asymptotic and curvature lines.

<u>*Mission*</u>. We wish to show that minimal surfaces minimize the surface area
$$A(X) = \int_M W\,du\,dv \text{ where } W^2 = EG - F^2;$$
thus we aim to see that minimal surfaces X are critical points of the area functional $X \mapsto A(X)$. Unfortunately, we are not able to compute the derivative of the area directly — its arguments are (parametrized) surfaces — but "Calculus of Variations" methods come to the rescue: we consider suitable "variations" (curves) in the space of surfaces and take derivative with respect to the variation parameter.

Def. *A* normal variation *of a surface* $X : M \to \mathcal{E}^3$ *with Gauss map* N *is a map*
$$t \mapsto X_t := X + N\,t\mu, \text{ where } \mu \in C^\infty(M);$$
$\delta X := \frac{d}{dt}\Big|_{t=0} X_t$ *is its* variational vector field, *and the corresponding* variation of area *is*
$$\delta A := \frac{d}{dt}\Big|_{t=0} A(X_t).$$

<u>*Cayley-Hamilton theorem*</u>. The next Lemma follows from this theorem:
$$0 = \chi_\lambda(\lambda) = \lambda^n a_n + \ldots + \lambda^1 a_1 + \mathrm{id}_V a_0;$$
for the characteristic polynomial $\chi_\lambda(t) = t^n a_n + \ldots + t a_1 + a_0$ *of an endomorphism* $\lambda \in \mathrm{End}(V)$*;* in particular, if $\dim V = 2$ then
$$0 = \lambda^2 - \lambda \cdot \mathrm{tr}\,\lambda + \mathrm{id}_V \det\lambda.$$
In fact, the theorem is easily verified when $\dim V = 2$ and (e_1, e_2) is a basis of eigenvectors, $\lambda(e_i) = e_i x_i$ and $\chi_\lambda(t) = t^2 - t\,(x_1 + x_2) + x_1 x_2$:
$$\lambda^2(e_i) - \lambda(e_i)(x_1 + x_2) + e_i\,x_1 x_2 = 0 \text{ for } i = 1, 2.$$

Def & Lemma. *With the* third fundamental form $\mathrm{III} := \langle dN, dN\rangle$ *of a surface* X *with Gauss map* N *we have* $0 = \mathrm{III} - 2H\,\mathrm{II} + K\,\mathrm{I}$.

Proof. Using the shape operator, $dN = -\mathrm{S} \circ dX$, and its symmetry
$$\mathrm{III} - 2H\mathrm{II} + K\mathrm{I} = \langle dX, (\mathrm{S}^2 - 2H\,\mathrm{S} + K\,\mathrm{id}) \circ dX\rangle;$$
thus the claim follows directly from the Cayley-Hamilton theorem. ∎

Thm. *X is minimal iff $\delta A = 0$ for every normal variation X_t of X.*

Proof. The metric of a normal variation $X_t = X + N\, t\mu$ of a surface X is
$$\mathrm{I}_t = \mathrm{I} - 2t\mu\mathrm{II} + t^2\mu^2\mathrm{III} + t^2 d\mu^2 = a\,\mathrm{I} - 2tb\,\mathrm{II} + t^2 d\mu^2$$
with $a = (1 - t^2\mu^2 K)$ and $b = \mu(1 - t\mu H)$, where we used the Cayley-Hamilton theorem; note that $a|_{t=0} = 1$, $\frac{da}{dt}|_{t=0} = 0$ and $b|_{t=0} = \mu$. Thus, using curvature line parameters or with $H = \frac{Eg-2Ff+eG}{2(EG-F^2)}$,
$$W_t^2 = E_tG_t - F_t^2 = W^2(a^2 - 4tab\,H + t^2(...))$$
hence, taking t-derivatives and using that $W_t|_{t=0} = W$,
$$2W\,\tfrac{d}{dt}|_{t=0}W_t = W^2(0 - 4\mu\,H) \;\;\Rightarrow\;\; \tfrac{d}{dt}|_{t=0}W_t = -2\mu H\,W,$$
and we obtain as the *first variation of area*
$$\delta A = \int_M \tfrac{d}{dt}|_{t=0}W_t du\,dv = -2\int_M \mu H\,du\,dv. \qquad (*)$$
Consequently, any minimal surface is critical for the area functional.

To see the converse, we choose μ in $(*)$ suitably: $\mu := H$, for example, to obtain
$$\delta A = -2\int_M \mu H\,du\,dv = -2\int_M H^2\,du\,dv;$$
now the integrand $H^2 \geq 0$, hence $\delta A < 0$ if H does not vanish. ∎

Rem. If a surface has infinite area, or if (Plateau's problem) its boundary is fixed, then a better adapted choice of μ is required: here one may let
$$\mu(u,v) := H(u,v)\,e^{-\frac{1}{\varrho^2-((u-u_o)^2+(v-v_o)^2)}}$$
for $(u-u_o)^2 + (v-v_o)^2 < \varrho^2$ and $= 0$ else, where $\varrho > 0$ is appropriately chosen: then the same argument as before shows that H must vanish around any $(u_o, v_o) \in M$.

Rem. For a normal variation through *parallel surfaces* $X_t = X + N\,t$, a similar computation as above, with $\mu \equiv 1$, yields:

Steiner's formula. *For a parallel surface $X_t = X + N\,t$ of X we have*
$$W_t^2 = W^2(1 - 2tH + t^2K);$$
in particular, X_t is regular wherever $1 - 2tH + t^2K \neq 0$.

Using $0 = \mathrm{III} - 2H\mathrm{II} + K\mathrm{I}$ from the Cayley-Hamilton theorem again we further learn

Thm. *The Gauss map of a minimal surface is (weakly) conformal,*

$$\mathrm{III} = -K\,\mathrm{I};$$

and it is regular away from flat points. Conversely, if the Gauss map of an umbilic free surface is conformal then the surface is minimal.

Rem. For a minimal surface $-K\mathrm{I}$ is positive semi-definite, since

$$2H = \operatorname{tr}\mathrm{S} = 0 \quad\Rightarrow\quad K = \det\mathrm{S} \leq 0.$$

Proof. Clearly the Gauss map N of a minimal surface is weakly conformal since $\mathrm{III} = -K\mathrm{I}$ by the Cayley-Hamilton theorem.

To see the converse first note that, away from umbilics, I and II are linearly independent; hence the Cayley-Hamilton theorem,

$$0 = (\mathrm{III} + K\mathrm{I}) - 2H\mathrm{II},$$

implies $H = 0$ when $\mathrm{III} \parallel \mathrm{I}$. ∎

Def & Lemma. *Any minimal surface is* isothermic, *i.e., admits a local conformal curvature line (re-)parametrization away from umbilics.*

Proof. Suppose that $(u,v) \mapsto X(u,v)$ is a curvature line parametrization of a minimal surface with principal curvatures $\kappa^{\pm} = \pm\kappa$, wlog., $\kappa > 0$. Then, by the Codazzi equations:

$$0 = \tfrac{\kappa^+_v}{\kappa^+ - \kappa^-} + \tfrac{E_v}{2E} = \tfrac{1}{2}(\ln \kappa E)_v \;\text{ and }\; 0 = -\tfrac{1}{2}(\ln \kappa G)_u;$$

thus $\tilde{X}(\tilde{u},\tilde{v}) = X(u(\tilde{u}), v(\tilde{v}))$ with $u'^2 = \frac{1}{\kappa E}(u)$ and $v'^2 = \frac{1}{\kappa G}(v)$ yields

$$\tilde{E} = \tilde{G} = \tfrac{1}{\tilde{\kappa}},$$

hence a conformal curvature line reparametrization of X. ∎

Cor (local Weierstrass representation). *If $N : M \to S^2 \subset \mathbb{R}^3$ is conformal, $|N_u|^2 = |N_v|^2 = \kappa$ and $\langle N_u, N_v\rangle = 0$, then*

$$X_u = -N_u \tfrac{1}{\kappa} \text{ and } X_v = N_v \tfrac{1}{\kappa} \tag{$*$}$$

yields a conformal curvature line parametrization $X : M \to \mathcal{E}^3$ of a minimal surface; conversely, every minimal surface arises, locally and away from umbilics, in this way.

Proof. The second claim follows directly from the above lemma: every minimal surface admits locally a conformal curvature line parametrization; Rodrigues' equations then yield $(*)$,

$$X_u = -N_u \tfrac{1}{\kappa} \text{ and } X_v = N_v \tfrac{1}{\kappa} \text{ with } |N_u|^2 = |N_v|^2 = \kappa^2 E = \kappa.$$

To substantiate the first statement first note that

$$N_{uv} = N_u \tfrac{\kappa_v}{2\kappa} + N_v \tfrac{\kappa_u}{2\kappa}$$

since

$$\langle N_{uv}, N_u\rangle = \tfrac{\kappa_v}{2}, \ \langle N_{uv}, N_v\rangle = \tfrac{\kappa_u}{2} \text{ and } \langle N_{uv}, N\rangle = 0;$$

hence the PDE system $(*)$ is integrable by Poincaré's lemma:

$$X_{vu} - X_{uv} = (N_v \tfrac{1}{\kappa})_u + (N_u \tfrac{1}{\kappa})_v = 0.$$

Clearly X so defined is conformal, $|X_u|^2 = |X_v|^2 = \frac{1}{\kappa}$, and its mean curvature

$$H = -\tfrac{\kappa}{2}(\langle X_u, N_u\rangle + \langle X_v, N_v\rangle) = 0$$

vanishes, showing that X is minimal. ■

Rem. Considering $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$ as a Riemann sphere, a conformal map $N : M \to S^2$ can be thought of as a *meromorphic map*: this establishes the relation with the classical *Enneper-Weierstrass representation* for minimal surfaces. In a similar vein:

Thm. *A conformally parametrized surface $(u, v) \mapsto X(u, v)$ is minimal if and only if X is harmonic, i.e., $\Delta X \equiv 0$.*

Proof. If X is conformal, that is, $E = G$ and $F = 0$, then

$$H = \tfrac{1}{2E}(e + g) = \tfrac{1}{2E}\langle N, X_{uu} + X_{vv}\rangle = \tfrac{1}{2E}\langle N, \Delta X\rangle;$$

hence X is minimal as soon as it is harmonic. Conversely, for a conformal parametrization

$$\langle X_u, \Delta X\rangle = \tfrac{1}{2}(E-G)_u + F_v = 0$$

and similarly $\langle X_v, \Delta X\rangle = 0$, proving the converse. ∎

Problem 5. Prove that the *catenoid* X is a conformally parametrized minimal surface, $X(u,v) = O + e_1 \cosh u \,\cos v + e_2 \cosh u \,\sin v + e_3 u$.

Example. A conformally parametrized surface of revolution,

$$X(u,v) = O + e_1 r(u)\cos v + e_2 r(u)\sin v + e_3 h(u)$$

with $r^2 = r'^2 + h'^2$, is minimal if and only if $\Delta X = 0$, that is, if and only if

$$\left.\begin{array}{l} r'' = r \\ h'' = 0 \end{array}\right\} \;\Leftrightarrow\; \left\{\begin{array}{l} r(u) = a\cosh u + b\sinh u \\ h(u) = c\,u + d \end{array}\right.$$

with constants $a, b, c, d, \in \mathbb{R}$, where $a^2 - b^2 = c^2$ by conformality.

If $c = 0$ then $b = \pm a$ and X parametrizes a "horizontal" plane by *conformal polar coordinates* since

$$r(u) = ae^{\pm u} \text{ and } h(u) \equiv d.$$

If $c \neq 0$ then $a = c\cosh u_o$ and $b = \sinh u_o$ for a suitable u_o, by conformality, hence we arrive at

$$r(u) = c\cosh(u+u_o) \text{ and } h(u) = c(u+u_o) + (d - cu_o),$$

that is, X is, up to parameter shift and similarity, the catenoid

$$(u,v) \mapsto X(u,v) := O + e_1 \cosh u \,\cos v + e_2 \cosh u \,\sin v + e_3 u.$$

Summarizing we have obtained a classification result:

Cor (Classification of minimal surfaces of revolution). *A minimal surface of revolution is either (part of) a plane or (part of) a catenoid.*

Rem. As every harmonic function is the real part of a *holomorphic function*, any (conformally parametrized) minimal surface $X : M \to \mathcal{E}^3$ arises from the real part of a *holomorphic null curve*:

$$X(u,v) = O + \operatorname{Re} C(u+iv),$$

where $C : \mathbb{C} \supset M \to \mathbb{C}^3$ is holomorphic, and $|C'|^2 \equiv 0$. Here, holomorphicity of C implies that X is harmonic, and the null condition implies conformality; namely, using $C' = (\mathrm{Re}\, C)_u - (\mathrm{Re}\, C)_v i$, from the Cauchy-Riemann equations, we obtain

$$0 = |C'|^2 = (|X_u|^2 - |X_v|^2) - 2i\langle X_u, X_v\rangle.$$

Note that $|.|^2$ denotes (the quadratic form of) the *bilinear extension* of the inner product on $\mathbb{R}^3$ to $\mathbb{C}^3$. This is the (complex) local Weierstrass representation of a minimal surface.

Problem 6. Show that the *Enneper surface*

$$(u, v) \mapsto O + e_1 \,\mathrm{Re}\, \tfrac{z^3-3z}{3} + e_2 \,\mathrm{Re}\, \tfrac{i(z^3+3z)}{3} + e_3 \,\mathrm{Re}\, z^2$$

is a conformally parametrized minimal surface.

Problem 7. Determine a holomorphic $C : \mathbb{C} \to \mathbb{C}^3$ so that $\mathrm{Re}\, C$ yields the catenoid. Which surface does $\mathrm{Im}\, C$ yield?

Rem. If C is a holomorphic null curve then so is $Ce^{i\alpha}$, for every $\alpha \in \mathbb{R}$. In this way a 1-parameter family $(X^\alpha)_{\alpha\in\mathbb{R}}$ of (conformally parametrized) minimal surfaces can be obtained, where

$$X^\alpha = O + \mathrm{Re}(Ce^{i\alpha}) = O + (\mathrm{Re}\, C)\cos\alpha - (\mathrm{Im}\, C)\sin\alpha.$$

This is the *associated family* of X, its *conjugate minimal surface* is the surface

$$X^* = X^{\pi/2} = O - \mathrm{Im}\, C.$$

We give a real description of this associated family:

Thm & Def. *Let $X : M \to \mathcal{E}^3$ be minimal, with Gauss map N; with its* conjugate surface *X^* the* associated family *$(X^\alpha)_{\alpha\in\mathbb{R}}$ of X is given by*

$$X^\alpha = X \cos\alpha + X^* \sin\alpha, \ \ \text{where } dX^* = N \times dX. \qquad (*)$$

All surfaces X^α are minimal and share the same induced metric and Gauss map,

$$\mathrm{I}^\alpha = \mathrm{I} \ \text{ and } \ N^\alpha = N.$$

Proof. We assume X to be conformal, that is, we have

$$N \times X_u = X_v \ \text{ and } \ N \times X_v = -X_u.$$

Then $X^* = X^{\frac{\pi}{2}}$ is well defined by $(*)$, because X is harmonic:

$$X^*_{uv} - X^*_{vu} = (N \times X_u)_v - (N \times X_v)_u = \Delta X = 0.$$

To see that all surfaces X^α are isometric, note that $\mathrm{I}^* = \mathrm{I}$ and, hence, $\mathrm{I}^\alpha = \mathrm{I}\cos^2\alpha + \mathrm{I}^*\sin^2\alpha = \mathrm{I}$. From $(*)$ it is obvious that all surfaces have parallel tangent planes, $N^\alpha = N$. Finally

$$\Delta X^* = (N \times X_u)_u + (N \times X_v)_v = X_{vu} - X_{uv} = 0$$

so that X^α is (with X and X^*) harmonic by superposition. ∎

Problem 8. Let $(u,v) \mapsto X(u,v)$ be a conformally parametrized minimal surface. Show that its second fundamental form can be written as

$$\mathrm{I\!I} = \mathrm{Re}\{(e - if)(du + idv)^2\}$$

and that $(e^\alpha - if^\alpha) = e^{i\alpha}(e - if)$ for the surfaces X^α of the associated family of X. Conclude that the curvature and asymptotic lines of X are exchanged for the conjugate surface X^*.

Example. The *associated family of the helicoid* is given by

$$X^\alpha(u,v) = \begin{pmatrix} \sinh u \cos v \cos\alpha - \cosh u \sin v \sin\alpha \\ \sinh u \sin v \cos\alpha + \cosh u \cos v \sin\alpha \\ v\cos\alpha + u\sin\alpha \end{pmatrix}.$$

Problem 9. Compute the associated family of the helicoid.

4.3 Linear Weingarten surfaces

We described developable surfaces and minimal surfaces (soap films) by curvature conditions, $K \equiv 0$ resp $H \equiv 0$. Further interesting surface classes may be described by curvature conditions: soap bubbles by the condition $H \equiv const$, surfaces that hyperbolic or elliptic geometries can be modelled on by $K \equiv const \neq 0$ (cf Minding's theorem). We shall study "linear Weingarten surfaces" as a wider class of surfaces to obtain a better understanding of the interrelations and key properties of the aforementioned classes of surfaces.

Def. *A surface is a* linear Weingrten surface *if its Gauss and mean curvatures satisfy a non-trivial affine relation*

$$0 = a\,K + 2b\,H + c. \qquad (W)$$

We call a linear Weingarten surface tubular *if* $\Delta := b^2 - ac = 0$.

<u>*Rem*</u>. If X is a linear Weingarten surfaces with $a \neq 0$ then (W) reads

$$0 = a(a\,K + 2b\,H + c) = (a\kappa^+ + b)(a\kappa^- + b) - \Delta \qquad (W')$$

with the principal curvatures $\kappa^\pm$ of X; i.e., the *linear relation* of mean and Gauss curvatures is a *quadratic relation* for the principal curvatures.

<u>*Examples*</u>. We already discussed:

(1) *developable (flat) surfaces*, $K \equiv 0$, for $(a, b, c) = (1, 0, 0)$;

(2) *minimal surfaces*, $H \equiv 0$, for $(a, b, c) = (0, 1, 0)$.

As generalizations we obtain the following linear Weingarten surfaces:

(3) *constant mean curvature surfaces*, $H \equiv -c$, for $(a, b, c) = (0, 1, c)$;

(4) *constant Gauss curvature surfaces*, $K \equiv -c$, for $(a, b, c) = (1, 0, c)$.

Tubular linear Weingarten surfaces yield a degenerat case:

(5) If X is a *tubular linear Weingarten surface*, $\Delta = b^2 - ac = 0$, then we must have $a \neq 0$ (since $a = 0 \Rightarrow b = c = 0$) and then

$$0 = a(a\,K + 2b\,H + c) = (a\kappa^+ + b)(a\kappa^- + b),$$

showing that one of the principal curvatures is constant, say

$$\kappa^- \equiv -\tfrac{b}{a} =: -\tfrac{1}{\varrho}.$$

Assuming $b \neq 0$ Rodrigues' equations shows that $C := X - N\,\frac{a}{b}$ is a curve,

$$N_v + X_v \kappa^- = 0 \;\;\Rightarrow\;\; (X - N\varrho)_v = 0,$$

hence X parametrizes a *tube* over C, e.g.,

$$X(u, v) = C(u) + N(u)\,\varrho \cos\varphi(v) + B(u)\,\varrho \sin\varphi(v),$$

justifying the terminology "tubular" in the case $\Delta = 0$.

Lemma. *Parallel surfaces* $X_t = X + Nt$ *of a linear Weingarten surface* X *are linear Weingarten, with*

$$a_t = a + 2tb + t^2 c, \;\; b_t = b + tc, \;\; \text{and} \;\; c_t = c.$$

In particular, $\Delta_t = \Delta$ *for any* t.

Proof. For a parallel surface $X_t = X + N\,t$ we have

$$\mathrm{I}^t = \mathrm{I} - 2t\mathrm{II} + t^2\mathrm{III}, \ \ \mathrm{II}^t = \mathrm{II} - t\mathrm{III}, \ \ \mathrm{III}^t = \mathrm{III};$$

hence

$$0 = (1 - 2tH + t^2K)\,\mathrm{III}^t - 2(H - tK)\,\mathrm{II}^t + K\,\mathrm{I}^t$$

and, away from umbilics, the Cayley-Hamilton theorem implies that

$$H^t = \tfrac{H - tK}{1 - 2tH + t^2K} \ \text{ and } \ K^t = \tfrac{K}{1 - 2tH + t^2K}.$$

Therefore

$$(a + 2tb + t^2c)\,K^t + 2(b + tc)\,H^t + c = \tfrac{aK + 2bH + c}{1 - 2tH + t^2K} \qquad (*)$$

which proves the first claim; $\Delta_t = \Delta$ by a simple computation. ∎

Rem. We use $N_t = N$ for any parallel surface X_t of X; note that X and X_t have parallel tangent planes, hence justifying the terminology.

Rem & Expl. Parallel families of linear Weingarten surfaces come in few types, characterized by special "representatives" in the parallel family. Thus, suppose that X is linear Weingarten, $0 = aK + 2bH + c$.

(1) If X is tubular, $\Delta = 0$, then $\forall t \in \mathbb{R} : \Delta_t = 0$ and all parallel linear Weingarten surfaces are tubular as well (as one expects).

(2) If $\Delta \neq 0$ but $c = 0$, then $b \neq 0$ and the parallel surfaces satisfy

$$0 = (a + 2tb)\,K_t + 2b\,H_t;$$

in particular, $t = -\frac{a}{2b}$ yields a minimal surface X_t, i.e., the parallel surfaces of a minimal surface are characterized by $(a, b, c) = (a, 1, 0)$.

(3) If $\Delta \neq 0$ and $c \neq 0$, then we obtain for $t = -\frac{b}{c}$, from $(*)$,

$$0 = -\tfrac{\Delta}{c}\,K_t + c \ \ \Rightarrow \ \ K_t = \tfrac{c^2}{\Delta},$$

hence X_t is a surface of constant Gauss curvature.
If $\Delta > 0$, then $t = -\frac{b \pm \sqrt{\Delta}}{c}$ yields $a_t = 0$, hence

$$0 = \mp 2\sqrt{\Delta}\,H_t + c \ \ \Rightarrow \ \ H_t = \pm\tfrac{c}{2}\tfrac{1}{\sqrt{\Delta}},$$

that is, a parallel pair of constant mean curvature surfaces X_t.
Observe that these two surfaces of constant mean curvature lie symmetrically on either side of their parallel surface of constant Gauss curvature. This is Bonnet's theorem:

Bonnet's theorem. *If X has constant mean curvature $H \neq 0$, then the parallel surface $X + N\frac{1}{2H}$ has constant Gauss curvature $4H^2$. Conversely, if X has constant Gauss curvature $K > 0$, then its two parallel surfaces $X \pm N\frac{1}{\sqrt{K}}$ have constant mean curvature $\pm\frac{1}{2}\sqrt{K}$.*

<u>*Rem*</u>. If X is a surface of constant positive Gauss curvature $K = \frac{1}{t^2}$ the induced metrics of its two parallel surfaces $X \pm N\frac{1}{t}$ are conformally related, by the Cayley-Hamilton theorem:

$$\mathrm{I}_{\pm t} = \mathrm{I} \mp 2t\mathrm{II} + t^2\mathrm{III} = \tfrac{2}{K}(H \mp \sqrt{K})\,\mathrm{II}.$$

In the case of *negative* constant Gauss curvature $K = -\frac{1}{t^2}$ a similar effect occurs for the complex conjugate pair of parallel surfaces $X_{\pm it} = X \pm N\,it$ of "constant mean curvature" $H_{\pm it} = \pm\frac{1}{2}\sqrt{K}$: we obtain

$$\mathrm{I}_{\pm it} = \mathrm{I} \mp 2it\mathrm{II} - t^2\mathrm{III} = \tfrac{2}{K}(H \mp \sqrt{K})\,\mathrm{II}.$$

<u>*Problem 10*</u>. Suppose that $(u,v) \mapsto X(u,v)$ is umbilic free and has constant mean curvature $H \neq 0$. Prove that the parallel surface

$$(u,v) \mapsto X^*(u,v) := X(u,v) + N(u,v)\,\tfrac{1}{H}$$

induces a conformally equivalent metric $\mathrm{I}^* = \frac{H^2-K}{H^2}\,\mathrm{I}$, and has constant mean curvature $H^* = H$.

<u>*Mission*</u>. Seek a classification of linear Weingarten surfaces of revolution. According to the above analysis, and excluding totally umbilic surfaces, we obtain three cases:

(1) $\Delta = c = 0$, so that the surface is a tubular surface of revolution, that is, a circular cylinder or a circular cone;

(2) $\Delta \neq 0 = c$, and the surface is parallel to a catenoid;

(3) $\Delta, c \neq 0$, where the surface is parallel to a surface of revolution with constant Gauss curvature $K = \pm\frac{1}{c^2} \neq 0$.

Thus (3) is the only case tha tstill requires work — here we may, wlog., assume that $K \equiv \varepsilon = \pm 1$, by scaling the surface if necessary.

4.4 Rotational surfaces of constant Gauss curvature

As discussed at the end of the last section, a complete classification of linear Weingarten surfaces of revolution is obtained as soon as we obtain one for surfaces of constant Gauss curvature $K \equiv \varepsilon = \pm 1$. To this end, we will derive a differential for the profile curve; subsequently, this differential equations is solved, using Jacobi elliptic functions.

Thus let $X(u,v) = O + e_1 r(u)\cos v + e_2 r(u)\sin v + e_3 h(u)$ denote a surface of revolution, and compute its curvatures (cf Sect 2.2):

$$\kappa^+ = \frac{\langle X_{uu}, N\rangle}{\langle X_u, X_u\rangle} = \frac{r'h'' - r''h'}{\sqrt{r'^2+h'^2}^3} \text{ and } \kappa^- = \frac{\langle X_{vv}, N\rangle}{\langle X_v, X_v\rangle} = \frac{1}{r^2}\frac{rh'}{\sqrt{r'^2+h'^2}},$$

hence

$$K = \frac{h'(r'h''-r''h')}{(r'^2+h'^2)^2}.$$

Lemma. *If X has constant Gauss curvature $K \equiv \varepsilon = \pm 1$ then r has only isolated critical points and*

$$\exists c \in \mathbb{R} : (c - \varepsilon r^2)(r'^2 + h'^2) = r'^2. \qquad (*)$$

Proof. Clearly $r''(u) \neq 0$ when $r'(u) = 0$, i.e., r has only isolated critical points; away from those critical points

$$K = \varepsilon = -\frac{1}{2rr'}\left(\frac{1}{1+(\frac{h'}{r'})^2}\right)' \;\Leftrightarrow\; (c - \varepsilon r^2)(r'^2 + h'^2) = r'^2$$

with a constant of integration $c \in \mathbb{R}$, where $c \geq \varepsilon r^2$. ∎

Rem. If $K \equiv 0$ then $r'' = h'' \equiv 0$, confirming that the profile curve of the surface is a straight line.

Rem. Any surface X of constant Gauss curvature $K = \varepsilon = \pm 1$ admits curvature line parameters (u,v) so that (away from umbilics)

$$E = \cosh^2\varphi,\;\; G = \sinh^2\varphi \text{ and } e = g = \cosh\varphi\sinh\varphi,$$

or, respectively,

$$E = \cos^2\varphi,\;\; G = \sin^2\varphi \text{ and } e = -g = \cos\varphi\sin\varphi.$$

Namely, with the ansatz $\kappa^+ = \tanh\varphi$ and $\kappa^- = \coth\varphi$, the Codazzi equations

$$0 = \frac{E_v}{2E} + \frac{\kappa^+_v}{\kappa^+ - \kappa^-} = \frac{G_u}{2G} - \frac{\kappa^-_u}{\kappa^+ - \kappa^-}$$

show that
$$(\tfrac{\sqrt{E}}{\cosh\varphi})_v = (\tfrac{\sqrt{G}}{\sinh\varphi})_u = 0$$
so that a suitable change of curvature line parameters leads to the claimed form of the metric. A similar argument works in the case $K = -1$ and, in fact, for linear Weingarten surfaces in general.

<u>*Problem 11*</u>. Let $(u, v) \mapsto X(u, v)$ be a curvature line parametrization of a surface with constant Gauss curvature $K \equiv -1$. Show that there is a curvature line reparametrization, $u = u(\tilde{u})$ and $v = v(\tilde{v})$, so that
$$\tilde{\mathrm{I}} = \cos^2\varphi\, d\tilde{u}^2 + \sin^2\varphi\, d\tilde{v}^2 \;\text{ and }\; \tilde{\mathrm{II}} = \tfrac{1}{2}\sin 2\varphi\,(d\tilde{u}^2 - d\tilde{v}^2)$$
with a suitable function φ.

Lemma. *If X has constant Gauss curvature $K \equiv \varepsilon = \pm 1$ and the profile curve is parametrized so that*
$$E - \varepsilon G = (r'^2 + h'^2) - \varepsilon r^2 = a$$
with some (chosen) $a \in \mathbb{R}$, then there exists $c \in \mathbb{R}$ so that
$$r'^2 = (a + \varepsilon r^2)(c - \varepsilon r^2), \quad \textit{where} \quad \begin{cases} -a \le r^2 \le c & \textit{if } \varepsilon = +1, \\ -c \le r^2 \le a & \textit{if } \varepsilon = -1. \end{cases} \qquad (**)$$
If $\varepsilon = -1$ then, additionally, $c < 1$.

Proof. With $r'^2 + h'^2 = a + \varepsilon r^2$ the elliptic ODE $(**)$ is obtained from the previous lemma; by continuity, it also holds at the critical points of r:
$$r'^2 = (r'^2 + h'^2)(c - \varepsilon r^2) = (a + \varepsilon r^2)(c - \varepsilon r^2).$$
As both factors must be non-negative, we obtain
$$-a \le r^2 \le c \;\text{ or }\; -c \le r^2 \le a$$
for $\varepsilon = +1$ and $\varepsilon = -1$, respectively. Finally, the height function h is then determined up to constant of integration and sign, i.e., shift or reflection of the surface, by
$$h'^2 = a + \varepsilon r^2 - r'^2 = (a + \varepsilon r^2)((1 - c) + \varepsilon r^2).$$
If $\varepsilon = -1$ we hence also have $c < 1$, since $1 - c \ge r^2$ and $a + \varepsilon r^2 \ge 0$. ■

Def. *Let* $p \in [0,1]$ *and* $q = \sqrt{1-p^2}$. *The* Jacobi amplitude function $\varphi = \operatorname{am}_p u$ *is the inverse function of the* incomplete elliptic integral of the first kind,
$$\varphi \mapsto u(\varphi) := \int_0^\varphi \frac{d\theta}{\sqrt{1-p^2\sin^2\theta}}.$$
The Jacobi elliptic functions of pole type n *are*
$$\operatorname{sn}_p u = \sin\varphi, \;\; \operatorname{cn}_p u = \cos\varphi, \;\; \operatorname{dn}_p u = \sqrt{1-p^2\sin^2\varphi}.$$
The incomplete elliptic integral of the second kind *is*
$$\varphi \mapsto E_p(\varphi) := \int_0^\varphi \sqrt{1-p^2\sin^2\theta}\, d\theta.$$
p *and* q *are the* elliptic modulus *and* co-modulus *of these functions.*

<u>*Rem*</u>. For the extremal cases of the elliptic modulus $p \in \{0,1\}$ we have
$$\operatorname{sn}_0 u = \sin u, \;\; \operatorname{cn}_0 u = \cos u \;\text{ and }\; \operatorname{dn}_0 u = 1,$$
and
$$\operatorname{sn}_1 u = \tanh u, \;\; \operatorname{cn}_1 u = \frac{1}{\cosh u} \;\text{ and }\; \operatorname{dn}_1 u = \frac{1}{\cosh u},$$
where the latter formulas for $p = 1$ are verified by comparing derivatives.

Lemma. *For the Jacobi elliptic functions we have: the derivatives*
$$\operatorname{am}_p' = \operatorname{dn}_p, \;\; \operatorname{sn}_p' = \operatorname{cn}_p \operatorname{dn}_p, \;\; \operatorname{cn}_p' = -\operatorname{sn}_p \operatorname{dn}_p, \;\; \operatorname{dn}_p' = -p^2 \operatorname{sn}_p \operatorname{cn}_p;$$
and the Pythagorean laws
$$1 = \operatorname{sn}_p^2 + \operatorname{cn}_p^2 = \operatorname{dn}_p^2 + p^2 \operatorname{sn}_p^2 = \frac{1}{q^2}(\operatorname{dn}_p^2 - p^2 \operatorname{cn}_p^2).$$

Proof. For the Jacobi amplitude we compute the derivative:
$$\varphi' = \frac{1}{u'} = \sqrt{1-p^2\sin^2\varphi} \;\; \Rightarrow \;\; \operatorname{am}_p' u = \operatorname{dn}_p u;$$
the other derivatives follow by chain rule. The Pythagorean rules follow directly from the definitions. ■

Lemma. *The Jacobi elliptic functions satisfy the* elliptic differential equations*:*
$$\begin{aligned}
\operatorname{sn}_p'^2 &= (1-\operatorname{sn}_p^2)(1-p^2\operatorname{sn}_p^2),\\
\operatorname{cn}_p'^2 &= (1-\operatorname{cn}_p^2)(q^2+p^2\operatorname{cn}_p^2),\\
\operatorname{dn}_p'^2 &= -(1-\operatorname{dn}_p^2)(q^2-\operatorname{dn}_p^2).
\end{aligned}$$

Proof. By direct computation, using the Pythagorean laws. ■

Lemma. *For $p \in (0,1)$ the Jacobi elliptic functions have ranges:*

$$\mathsf{am}_p(\mathbb{R}) = \mathbb{R},\ \ \mathsf{sn}_p(\mathbb{R}) = [-1,1],\ \ \mathsf{cn}_p(\mathbb{R}) = [-1,1],\ \ \mathsf{dn}_p(\mathbb{R}) = [\tfrac{1}{q},1].$$

Proof. For $p \in (0,1)$ and $q = \sqrt{1-p^2} \in (0,1)$ we have

$$\forall u \in \mathbb{R} : q \leq \mathsf{am}_p' \, u = \mathsf{dn}_p \, u = \sqrt{1 - p^2 \sin^2 \mathsf{am}_p \, u} \leq 1,$$

hence

$$\forall u \in [0,\infty) : qu \leq \mathsf{am}_p \, u$$

showing that $\mathsf{am}_p(\mathbb{R}) = \mathbb{R}$ since am_p is an odd function. This then implies the other claims. ∎

Lemma. *Let $\alpha < \beta$. The (elliptic) initial value problem*

$$x' = \sqrt{(-\alpha + x^2)(\beta - x^2)},\ \ x(t_o) = x_o \ \text{ with } \ x_o^2 \in [\alpha, \beta]$$

has at most one solution with isolated critical points.

Proof. Assuming that the critical points of x are isolated we compute

$$x'^2 = (-\alpha + x^2)(\beta - x^2) \ \ \Rightarrow \ \ x'' = -2x^3 + (\alpha + \beta)\, x.$$

The Picard-Lindelöf Theorem then shows that any solution of the equivalent first order system

$$\binom{x}{y}' = \binom{y}{-2x^3 + (\alpha+\beta)x},\ \ \binom{x(t_o)}{y(t_o)} = \binom{x_o}{\sqrt{(-\alpha + x_o^2)(\beta - x_o^2)}}$$

is unique. ∎

<u>*Rem.*</u> The assumption on the critical points is critical: the initial value problem

$$x'^2 + x^2 = 1,\ \ x^2(0) = 1$$

has four solutions: $x \equiv \pm 1$ and $x = \pm\cos$. Taking a (positive) root eliminates the sign ambiguity, and requiring the solution to have only isolated singularities singles out $x = \cos$.

With these preparations we now formulate a complete classification of surfaces of revolution with constant Gauss curvature $K = \varepsilon = \pm 1$:

Thm (Tjaden). *Any rotational surface X of constant Gauss curvature $K \equiv \varepsilon = \pm 1$ is, up to reparametrization, given by one of the following profiles, where $p \in (0,1)$*

$$r(u) = p\,\mathsf{cn}_p(u), \quad h(u) = \begin{cases} (E_p \circ \mathsf{am}_p)(u) & \text{if } \varepsilon = +1, \\ (E_p \circ \mathsf{am}_p)(u) - u & \text{if } \varepsilon = -1; \end{cases}$$

$$r(u) = \tfrac{1}{p}\,\mathsf{dn}_p(\tfrac{u}{p}), \quad h(u) = \begin{cases} \tfrac{1}{p}((E_p \circ \mathsf{am}_p)(\tfrac{u}{p}) - q^2\tfrac{u}{p}) & \text{if } \varepsilon = +1, \\ \tfrac{1}{p}((E_p \circ \mathsf{am}_p)(\tfrac{u}{p}) - \tfrac{u}{p}) & \text{if } \varepsilon = -1; \end{cases}$$

$$r(u) = \tfrac{1}{\cosh u}, \quad h(u) = \begin{cases} \tanh u & \text{if } \varepsilon = +1 \text{ (sphere)}, \\ \tanh u - u & \text{if } \varepsilon = -1 \text{ (pseudosphere)}. \end{cases}$$

Proof. We need to show that the given profiles exhaust, up to parameter shift, the (non-trivial) solutions of the ODEs

$$r'^2 = (a + \varepsilon r^2)(c - \varepsilon r^2) \text{ and } h'^2 = (a + \varepsilon r^2)((1-c) + \varepsilon r^2),$$

where $a, c \in \mathbb{R}$ (here c is given and a can be chosen suitably) so that all factors are non-negative, hence necessarily

$$\begin{aligned} c > 0 \quad &\text{if} \quad \varepsilon = +1, \\ a > 0 \text{ and } c < 1 \quad &\text{if} \quad \varepsilon = -1. \end{aligned}$$

Thus we may choose $a := 1 - c$ in any case to obtain

$$r'^2 = ((1-c) + \varepsilon r^2)(c - \varepsilon r^2) \text{ and } h'^2 = ((1-c) + \varepsilon r^2)^2 \qquad (*)$$

with $c > 0$ if $\varepsilon = +1$ and $c < 1$ if $\varepsilon = -1$. Observe that a the elliptic differential equations for r is invariant under the exchange

$$(\varepsilon = +1, c) \leftrightarrow (\varepsilon = -1, a = 1 - c).$$

Seeking solutions r with only isolated critical points, we obtain a unique solution for any given initial condition; r determines h uniquely up to sign and constant of integration (reflection and translation of the surface).

- Case $c, a = 1 - c \in (0,1)$: we let

$$(p, q) := \begin{cases} (\sqrt{c}, \sqrt{1-c}) & \text{if } \varepsilon = +1, \\ (\sqrt{1-c}, \sqrt{c}) & \text{if } \varepsilon = -1, \end{cases}$$

so that $(*)$ reads

$$r'^2 = p^2(1-(\tfrac{r}{p})^2)(q^2+p^2(\tfrac{r}{p})^2) \text{ and } h'^2 = \begin{cases} (q^2+p^2(\tfrac{r}{p})^2)^2, \\ p^4(1-(\tfrac{r}{p})^2)^2; \end{cases}$$

- Case $c>1$, $a=1-c<0$ resp $c<0$, $a=1-c>1$: we let

$$(p,q) := \begin{cases} (\tfrac{1}{\sqrt{c}}, \tfrac{\sqrt{1-c}}{\sqrt{c}}) & \text{if } \varepsilon=+1, \\ (\tfrac{1}{\sqrt{1-c}}, \tfrac{\sqrt{-c}}{\sqrt{1-c}}) & \text{if } \varepsilon=-1, \end{cases}$$

  so that $(*)$ now becomes

$$r'^2 = -\tfrac{1}{p^4}(1-(pr)^2)(q^2-(pr)^2) \text{ and } h'^2 = \begin{cases} \tfrac{1}{p^4}((pr)^2-q^2)^2, \\ \tfrac{1}{p^4}(1-(pr)^2)^2; \end{cases}$$

- Case $c=1$, $a=1-c=0$ resp $c=0$, $a=1-c=1$: here $(*)$ reads

$$r'^2 = r^4(\tfrac{1}{r^2}-1) \text{ and } h'^2 = \begin{cases} r^4, \\ r^4(\tfrac{1}{r^2}-1)^2. \end{cases}$$

Thus, in each case, one of the claimed solutions is obtained. ■

Rem. Surfaces with $K \equiv -1$ yield "realizations" of the hyperbolic non-Euclidean geometry in Euclidean space: reparametrizing the pseudosphere

$$X(u,v) = O + e_1 \tfrac{\cos v}{\cosh u} + e_2 \tfrac{\sin v}{\cosh u} + e_3(\tanh u - u)$$

with $(x,y) = (\cosh u, v)$, as new (curvature line) coordinates yields

$$E = \langle X_u, X_u\rangle\, u'^2 = \tfrac{1}{x^2} \text{ and } G = \langle X_v, X_v\rangle = \tfrac{1}{x^2},$$

thus $(x,y) \mapsto X(x,y)$ is an isometric parametrization of (half of) the Poincaré half plane into $\mathcal{E}^3$. However, *Hilbert's theorem* asserts that there is no isometric immersion of the complete hyperbolic plane into $\mathcal{E}^3$.

Rem & Def. Apart from the sphere, none of the above surfaces is regular: in the cases $r = p\,\mathrm{cn}_p$ the surfaces cross the axis of rotation, hence develop "cone points"; in all cases the zeroes

$$0 = (1-c) + \varepsilon r^2 = r'^2 + h'^2$$

of r' yield circles of non-regularity, e.g., in the case of the pseudosphere the circle

$$1 - r^2(u) = 1 - \tfrac{1}{\cosh^2 u} = 0 \;\Leftrightarrow\; u = 0$$

yields a *cuspidal edge*: near $u = 0$ the profile curve "looks like" a Neile parabola $(1-r)^3 = h^2$ (cf Sect 1.1). Namely, for $u \sim 0$ we have

$$-\frac{3h(u)}{u^3}\cosh u = 3\,\frac{u\cosh u - \sinh u}{u^3} \sim \frac{u\sinh u}{u^2} \sim \cosh u \sim 1$$

and
$$\frac{2(1-r(u))}{u^2}\cosh u = 2\,\frac{\cosh u - 1}{u^2} \sim \frac{\sinh u}{u} \sim \cosh u \sim 1,$$

hence
$$\frac{9h^2}{8(1-r)^3}(u) = \frac{\left(\frac{3h(u)\cosh u}{u^3}\right)^2}{\left(\frac{2(1-r(u))\cosh u}{u^2}\right)^3}\cosh u \sim \cosh u \sim 1.$$

A more detailed analysis of "surfaces with singularities" is of recent interest, but would exceed the scope of these lectures.

5 Manifolds and vector bundles

We have already seen some problems with our definitions of curves and surfaces: for example,

(1) a hyperbola does not qualify as a curve (according to our definition) as it consists of two components, hence cannot be parametrized by a single regular (hence continuous) map defined on an open interval;

(2) the sphere $S^2 \subset \mathcal{E}^3$ does not qualify as a surface since there cannot be a (regular) parametrization of all of S^2 defined on an open connected subset $U \subset \mathbb{R}^2$ (by the "hairy ball theorem").

The notion of a k-dimensional submanifold of $\mathcal{E}$ resolves this problem — at the cost of introducing another restriction, that can in turn be resolved by the notion of an "immersed abstract manifold". At the same time, the following discussions will shed light on the notion of "local" (as opposed to "global"), used previously in this text in an informal way.

The definition of abstract manifolds does create several technical issues that are cumbersome to resolve, hence we only discuss submanifolds here. By the Whitney embedding theorem, this is not a restriction: every k-dimensional (abstract) manifold can be embedded into $\mathbb{R}^{2k}$.

Throughout the first two sections of this chapter we aim to avoid arguments that rely on finite dimensions; the background Hilbert space structure assumed to be omnipresent ensures that (Frechet-)differentiability is a notion available to us.

5.1 Submanifolds in a Euclidean space

There are several equivalent definitions/characterizations of submanifolds in a Euclidean space $\mathcal{E}$: these allow to pass from an implicit representation of, e.g., a curve or a surface to a parametric description, and vice versa. We define a submanifold to be a subset that "*can locally be flattened*":

Def. *$M \subset \mathcal{E}$ is a (k-dimensional)* submanifold *of $\mathcal{E}$ if it is locally diffeomorphic to a (k-dimensional) subspace $\mathcal{S} \leq \mathcal{E}$:*

$$\exists \mathcal{S} \leq \mathcal{E} \forall X \in M \exists \varphi : U \to \varphi(U) : \varphi(M \cap U) = \mathcal{S} \cap \varphi(U),$$

where $\mathcal{S}$ is a (k-dimensional) affine subspace of $\mathcal{E}$ and $\varphi : U \to \varphi(U)$ a diffeomorphism defined on an open neighbourhood $U \subset \mathcal{E}$ of $X \in M$.

Rem. We use the notation $\mathcal{S} \leq \mathcal{E}$ for a structure preseving subset relation, e.g., for affine (Euclidean) subsets of an affine or Euclidean space.

Rem. We require neither $\mathcal{E}$ nor $\mathcal{S}$ to be finite dimensional here.

Example. Any open subset $U \overset{\circ}{\subset} \mathcal{E}$ is a submanifold. This example is as important as it is trivial.

Example. The graph $M = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \,|\, y = g(x)\} \subset \mathbb{R}^n$ of any smooth map $g : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is a k-dimensional submanifold via

$$\varphi : \mathbb{R}^n \to \mathbb{R}^n, \;\; (x, y) \mapsto \varphi(x, y) := (x, y - g(x)).$$

Problem 1. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (x, y - g(x))$, with a differentiable map $g : \mathbb{R} \to \mathbb{R}$. Prove that φ yields a local diffeomorphism around every point $(x, g(x)) \in \mathbb{R}^2$, $x \in \mathbb{R}$.

Alternatively, any submanifold can locally be described by equations:

Lemma. *$M \subset \mathcal{E}$ is submanifold iff it is locally level set of submersions:*

$$\exists W \forall X \in M \exists F : U \to W : M \cap U = F^{-1}(\{0\}),$$

where W is a Banach space and $F : U \to W$ is a submersion defined on an open neighbourhood $U \subset \mathcal{E}$ of X. We have $\dim M = \dim \ker dF$.

Rem. Here, it is sufficient to require $d_X F$ to surject for all $X \in M$: if $d_X F$ surjects then, by the Inertia principle ($X \mapsto d_X F$ is continuous), there is a neighbourhood $\tilde{U} \subset U$ of X so that $d_Y F$ surjects for all $Y \in \tilde{U}$.

Example. The graph $M = \{(x, y) \in \mathbb{R}^n \,|\, y = g(x)\}$ of $g : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is level set of a single submersion

$$F : \mathbb{R}^n \to \mathbb{R}^{n-k}, \;\; (x, y) \mapsto F(x, y) := y - g(x).$$

Yet another alternative characterization is to describe a submanifold by local parametrizations or charts — this approach is paramount for the definition of *abstract manifolds*.

Lemma & Def. *$M \subset \mathcal{E}$ is submanifold iff it can locally be parameterized:*
$$\exists V \forall X \in M \exists f : V \overset{\circ}{\supset} D \to U \overset{\circ}{\subset} \mathcal{E} : M \cap U = f(D),$$
where V is a Banach space, $U \subset \mathcal{E}$ an open neighbourhood of $X \in M$, and $f : D \to \mathcal{E}$ a parametrization *of $M \cap U$, that is,*

- *$f : D \to \mathcal{E}$ is an immersion, so that*
- *$f : D \to M \cap U$ is a homeomorphism (with the induced topology).*

The map $f^{-1} : M \cap U \to V$ is called a chart. *We have* $\dim M = \dim V$.

<u>*Rem*</u>. f being an immersion excludes "kinks" or "sharp edges" of M, injectivity of f prevents self-intersections of M, and continuity of f^{-1} excludes "T-junctions".

<u>*Rem*</u>. We temporarily use f to denote a parametrization, to distinguish it from points $X \in M$.

<u>*Example*</u>. The graph $M = \{(x, y) \in \mathbb{R}^n \,|\, y = g(x)\}$ of $g : \mathbb{R}^k \to \mathbb{R}^{n-k}$ can be parametrized by a single parametrization
$$f : \mathbb{R}^k \to \mathbb{R}^n, \;\; x \mapsto (x, g(x));$$
a (global) chart is given by the projection
$$f^{-1} : M \to \mathbb{R}^k, \;\; (x, y) \mapsto x.$$
This example of a graph yields an approach to prove the lemmas.

Proof. We prove both lemmas.

Let $M \subset \mathcal{E}$ be a submanifold, $X \in M$ and $\varphi : U \to \varphi(U)$ a diffeomorphism with
$$\varphi(M \cap U) = \mathcal{S} \cap \varphi(U).$$
Take $O = \varphi(X) \in \mathcal{S}$ as origin and, with a complementary subspace W of V, write
$$O + V = \mathcal{S} \leq \mathcal{E} = O + (V \oplus W);$$
accordingly $\varphi - O = \mu + F \in V \oplus W$, where

- $F : U \to W$ yields a submersion with $M \cap U = F^{-1}(\{0\})$, and
- $\mu|_M : M \cap U \to V$ is a chart around X, as inverse of the parametrization
$$f := (\varphi - O)^{-1}|_{\mu(U)} : \mu(U) \to \mathcal{E}.$$

We reverse engineer this construction to obtain proofs of the converses.

- Let $F : \mathcal{E} \overset{\circ}{\supset} U \to W$ be a submersion with $M \cap U = F^{-1}(\{0\})$.

  Let $V := \ker d_X F$ and fix a complementary W', $\mathcal{E} = X + (V \oplus W')$; further let $\pi : V \oplus W' \to V$ and $\pi' : V \oplus W' \to W'$ denote the corresponding projections. Then
  $$d_X F|_{W'} : W' \to W$$
  is an isomorphism, hence invertible, and we may define $\varphi : U \to \mathcal{E}$ by
  $$Y \mapsto \varphi(Y) := X + \pi(Y - X) + (d_X F|_{W'}^{-1} \circ F)(Y);$$
  as
  $$d_X \varphi = \pi + d_X F|_{W'}^{-1} \circ d_X F = \pi + \pi' = \mathrm{id}_{V \oplus W'}$$
  is an isomorphism φ is a diffeomorphism, by the Inverse Mapping Thm, after possibly restricting to a smaller U. By construction
  $$\varphi(M \cap U) = (X + V) \cap \varphi(U).$$

- Let $f : V \overset{\circ}{\supset} D \to \mathcal{E}$ be a local parametrization, $M \cap U = f(D)$.

  Suppose, wlog, $0 \in D$ and $X = f(0)$; set $V' := d_0 f(V)$ and fix a complementary subspace W, so that $\mathcal{E} = X + (V' \oplus W)$; as above, let $\pi : V' \oplus W \to V'$ and $\pi' : V' \oplus W \to W$ be the corresponding projections. As
  $$d_0 f : V \to V'$$
  is invertible we may define $\psi : U' \to \mathcal{E}$ on $U' := X + \pi^{-1}(d_0 f(D))$ by
  $$Y \mapsto \psi(Y) := (f \circ d_0 f^{-1} \circ \pi)(Y - X) + \pi'(Y - X);$$
  as
  $$d_X \psi = d_0 f \circ d_0 f^{-1} \circ \pi + \pi' = \pi + \pi' = \mathrm{id}_{V' \oplus W}$$
  is an isomorphism ψ is a diffeomorphism, by the Inverse Mapping Thm, after possibly restricting to a smaller U'. Replacing U by $U \cap \psi(U')$ and adjusting the domains U' of ψ resp D of f accordingly we obtain
  $$M \cap U = f(D) = \psi((X + V') \cap U'),$$
  hence $\varphi := \psi^{-1} : U \to U' = \varphi(U)$ yields the sought-after diffeomorphism.

It is obvious that the dimension claims hold. ∎

Rem. This proof is insensitive to dimension, in particular, it also works in an infinite dimensional setting.

Problem 2. Use the Implicit Mapping Thm to show directly that any implicitely defined submanifold has local parametrizations.

Example. Let $\mathcal{E}$ be a Euclidean space over V, choose an origin $O \in \mathcal{E}$ and let $\beta : V \times V \to \mathbb{R}$ denote a non-trivial symmetric bilinear form; set

$$F : \mathcal{E} \to \mathbb{R}, \ \ X \mapsto \beta(X - O, X - O) - 1.$$

Then the (proper) *central quadric* $Q := F^{-1}(\{0\}) \subset V$ is a submanifold since

$$v \mapsto d_X F(v) = 2\beta(X - O, v)$$

surjects for $X \in Q$ (since $d_X F(X - O) = 2 \neq 0$ and by Inertia).

Such central quadrics come in different flavours, depending on the signature of β: for example, if $V = \mathbb{R}^3$ and β is non-degenerate then Q may be an ellipsoid, a 1-sheeted or a 2-sheeted hyperboloid.

In particular, Q may

- not be connected (2-sheeted hyperboloid) or
- not be coverable by a single parametrization (ellipsoid).

On the other hand, a *cone* $Q = \{X \in \mathcal{E} \,|\, \beta(X - O, X - O) = 0\}$ is, in general, not a submanifold.

Problem 3. Show that $C = \{O + e_1 x_1 + e_2 x_2 \in \mathcal{E}^2 \,|\, x_1^2 = x_2^2\}$ is not a submanifold.

Example. *Gerono's lemniscate* is the curve $G = F^{-1}(\{0\}) \subset \mathcal{E}^2$, where

$$F : \mathcal{E}^2 \to \mathbb{R}, \ \ O + e_1 x_1 + e_2 x_2 = X \mapsto F(X) := x_1^4 - x_1^2 + x_2^2.$$

It is (check) the image of the immersion $t \mapsto O + e_1 \sin t + e_2 \sin t \cos t$, yet not a submanifold of $\mathcal{E}^2$.

Problem 4. Prove that Gerono's lemniscate is not a submanifold.

Rem. If $F = F_1 + F_2 : \mathcal{E} \overset{\circ}{\supset} U \to W = W_1 \oplus W_2$ is a submersion, then so are $F_i : U \to W_i$, $i = 1, 2$; in this way the submanifold

$$M = F^{-1}(\{0\}) = F_1^{-1}(\{0\}) \cap F_2^{-1}(\{0\}) = M_1 \cap M_2$$

appears as the intersection of two (higher dimensional) submanifolds.

Example. Identifying $\mathbb{R}^4 \cong \mathbb{C}^2$, the (square) *Clifford torus*

$$T^2 = \{O + e_1 z + e_2 w \in \mathcal{E}^4 \mid |z|^2 + |w|^2 = 1,\ |z|^2 - |w|^2 = 0\}$$

appears as the intersection of two quadrics: the 3-sphere and a cone.

Problem 5. Prove that the conic sections $C = F^{-1}(\{0\}) \subset \mathcal{E}^3$, where

$$F : \mathcal{E}^3 \to \mathbb{R}^2,\ \ F(O + e_1 x + e_2 y + e_3 z) := \begin{pmatrix} x^2 + y^2 - z^2 \\ x\cos\alpha + z\sin\alpha - d \end{pmatrix}$$

with $\alpha \in \mathbb{R}$ and $d \neq 0$, are 1-dimensional submanifolds of $\mathcal{E}^3$.

5.2 Tangent space & Derivative

Previously, functions, vector fields, etc, *along* a (parametrized) curve or surface were considered as maps of the same parameter(s) as the parametrization. Hence it was clear how to differentiate them. Now, in the setting of submanifolds, these maps will be defined on a submanifold M, in particular, not on an open subset of a linear or affine space. Consequently, the basic notions of analysis need to be revisited.

Def. *The* tangent space *of a (k-dimensional) manifold $M \subset \mathcal{E}$ at a point $X \in M$ is the (k-dimensional, linear) subspace*

$$T_X M := d_x f(V),$$

where $f : V \supset D \to \mathcal{E}$ is a parametrisation of M around $X = f(x)$.

Rem. $T_X M$ is independent of the choice of local parametrisation: if $\tilde{f} : \tilde{D} \to X$ is another local parametrisation around $X = \tilde{f}(\tilde{x})$ then there is a diffeomorphism $g : \tilde{D} \to D$ so that $\tilde{f} = f \circ g$ and $x = g(\tilde{x})$; hence

$$d_{\tilde{x}}\tilde{f}(\tilde{V}) = d_x f(d_{\tilde{x}} g(\tilde{V})) = d_x f(V).$$

Problem 6. Let V be a Euclidean vector space with inner product $\langle .,. \rangle$ and let $x \mapsto x^* := \langle x, . \rangle$ denote the canonical embedding of V into V^*. Prove that the image $M := f(S)$ of the unit sphere $S \subset V$ under the *Veronese embedding*

$$f : S \to \mathsf{Sym}(V),\ \ x \mapsto f(x) := x \cdot x^*$$

a submanifold is and determine its tangent space at $X = f(x) \in M$.

Lemma. *If $M = F^{-1}(\{0\})$ is the level set of a submersion $F : U \to W$ then*

$$T_X M = \ker d_X F \text{ for } X \in M.$$

Proof. For any local parametrization f around $X = f(x)$ we have

$$F \circ f \equiv 0 \ \Rightarrow \ d_X F \circ d_x f \equiv 0 \ \Rightarrow \ T_X M \subset \ker d_X F;$$

then $\dim T_X M = \dim \ker d_X F$ implies $T_X M = \ker d_X F$, if $\dim M < \infty$.

If $\dim M = \infty$ then a more intricate argument following the proof of the characterizations of submanifolds is required. ∎

Example. The tangent space of a central quadric $Q = F^{-1}(\{0\}) \subset \mathcal{E}$ with $F(X) = \beta(X - O, X - O) - 1$ is given by

$$T_X Q = \ker d_X F = \{v \in V \,|\, v \perp_\beta X - O\}.$$

Rem. If $C : (-\varepsilon, \varepsilon) \to M \subset \mathcal{E}$ is a curve (smooth map) with $C(0) = X$ then $C'(0) \in T_X M$ is a *tangent vector*: if $M \cap U = F^{-1}(\{0\})$ then

$$F \circ C \equiv 0 \ \Rightarrow \ d_X F(C'(0)) = 0 \ \Rightarrow \ C'(0) \in \ker d_X F.$$

Conversely, every tangent vector $v \in T_X M$ is of that form, $v = C'(0)$ for a suitable curve C: if $M \cap U = f(D)$ with $X = f(x)$ and $v = d_x f(w)$ take

$$C(t) := f(x + tw), \text{ hence } C'(0) = d_x f(w) = v.$$

Further, two curves $C, \tilde{C} : (-\varepsilon, \varepsilon) \to M$ through $X = \tilde{C}(0) = C(0)$ yield the same tangent vector iff

$$\tilde{C} \sim C :\Leftrightarrow \tilde{C}'(0) = C'(0).$$

Hence

$$T_X M = \{C : (-\varepsilon, \varepsilon) \to M \,|\, C(0) = X\}/_\sim.$$

Problem 7. Describe the tangent space $T_X M$ of a manifold $M \subset \mathcal{E}$ in terms of a flattening diffeomorphism φ with $\varphi(M \cap U) = \mathcal{S} \cap \varphi(U)$.

Now we are prepared to tackle the issue of differentiablity for functions defined on (sub-)manifolds — the familiar notion of "linear approximation" is no longer available since a manifold is not a linear or affine space.

Here, the key idea is to define the derivative so that the chain rule holds:

Def. *A map $g : M \to M'$ between manifolds is* differentiable *at $X \in M$ with* derivative

$$d_X g := d_{x'} f' \circ d_x(f'^{-1} \circ g \circ f) \circ (d_x f)^{-1}$$

if $(f'^{-1} \circ g \circ f) : D \to D'$ is differentiable at $x \in D$, where $f : D \to M$ and $f' : D' \to M'$ are local parametrizations around $X = f(x)$ and around $X' = g(X) = f'(x')$, respectively.

Rem. This definition makes sense as differentiability and derivative of g do not depend on the choice of parametrization: if $\tilde{f} = f \circ \varphi$ and $\tilde{f}' = f' \circ \varphi'$ are other parametrizations around $X = \tilde{f}(\tilde{x})$ and $X' = \tilde{f}'(\tilde{x}')$ then

$$\tilde{f}'^{-1} \circ g \circ \tilde{f} = \varphi'^{-1} \circ (f'^{-1} \circ g \circ f) \circ \varphi$$

is differentiable at $\tilde{x} = \varphi^{-1}(x)$ since φ and φ' are diffeomorphisms and, with $x' = \varphi'(\tilde{x}')$, the chain rule yields

$$d_{\tilde{x}'} \tilde{f}' \circ d_{\tilde{x}}(\tilde{f}'^{-1} \circ g \circ \tilde{f}) \circ (d_{\tilde{x}} \tilde{f})^{-1} = d_X g.$$

Rem. Any chart $\mu = f^{-1} : M \cap U \to D$ is differentiable with derivative

$$d_X \mu = (d_{\mu(X)} f)^{-1}$$

by definition: clearly $\mathrm{id}^{-1} \circ \mu \circ f = \mathrm{id}$ is differentiable at $x = \mu(X)$ and

$$d_X \mu = d_x \mathrm{id} \circ d_x \mathrm{id} \circ (d_x f)^{-1} = (d_x f)^{-1}.$$

With this notion of differentiability many of the usual theorems can be proved for maps between manifolds — by simply "transporting" the theorems onto manifolds by means of parametrizations resp charts.

Lemma (Chain rule). *If $g : M \to M'$ and $h : M' \to M''$ are differentiable then $h \circ g : M \to M''$ is with*

$$d_X(h \circ g) = d_{g(X)} h \circ d_X g.$$

Problem 8. Prove the chain rule for maps between manifolds.

Rem. The Leibniz rule for products of maps defined on manifolds follows from the chain rule and the usual Leibniz rule.

Similarly, linearity of the derivative for vector space valued maps follows.

Inverse Mapping Theorem. *Suppose that $g : M \to M'$ is smooth and $d_X g : T_X M \to T_{g(X)} M'$ is an isomorphism for some $X \in M$. Then g restricts to a* diffeomorphism *around X, i.e., there is a neighbourhood U of X in M so that $g|_U : U \to g(U)$ has a smooth inverse.*

Problem 9. Prove the Inverse Mapping Theorem on (sub-)manifolds.

Rem. If $H : \mathcal{E} \to \mathbb{R}$ is differentiable and $M \subset \mathcal{E}$ is a submanifold then $h := H|_M : M \to \mathbb{R}$ is differentiable with

$$d_X h = d_X H|_{T_X M} : T_X M \to \mathbb{R}.$$

Namely: h is clearly differentiable, as $H \circ f$ is for any parametrization f; moreover, if $X = f(x)$ and $v = d_x f(w) \in T_X M$ then

$$d_X h(v) = d_x(h \circ f)(w) = d_x(H \circ f)(w) = d_X H(v).$$

Problem 10. Prove the *Lagrange multiplier theorem*: if $M = F^{-1}(\{0\})$ is a submanifold in $\mathcal{E}$, where $F : \mathcal{E} \to \mathbb{R}$, and $H : \mathcal{E} \to \mathbb{R}$ is differentiable then $X \in M$ is a critical point (hence candidate for an extremum) of $h := H|_M : M \to \mathbb{R}$ iff there is $\lambda \in \mathbb{R}$ so that (λ, X) is a critical point of

$$\mathbb{R} \times \mathcal{E} \ni (\lambda, X) \mapsto H(X) - \lambda F(X) \in \mathbb{R}.$$

Rem. Let $M \subset \mathcal{E}$ be a manifold and $X \in M$. A *derivation* at X is a map

$$v : C^\infty(M) \to \mathbb{R}, \;\; h \mapsto vh$$

that is linear and satisfies Leibniz rule, $v(hg) = (vh)g(X) + h(X)(vg)$.

Every tangent vector $v \in T_X M$ gives rise to a derivation

$$v : C^\infty \to \mathbb{R}, \;\; h \mapsto vh := d_X h(v).$$

It can be shown that every derivation at X arises in this way, hence the tangent space $T_X M$ of M at X can be identified with the space of derivations at X, equipped with the usual linear structure on spaces of real-valued maps.

5.3 Lie groups

One way to investigate the geometry of a curve or surface was to attach a (suitably adapted) *moving frame* to the object: its change then provided *structure equations* that encoded the geometry (e.g., curvatures, torsion) of the object under investigation. More conceptually, such a frame was obtained as the image of the standard basis of $\mathbb{R}^3$ under a (smooth) map into a subgroup $G \leq \mathsf{Gl}(\mathbb{R}^3)$.

Thus a *Lie group* is required: a group G that is, at the same time, a manifold so that the group operations are differentiable.

Assumption: to simplify formulations we now assume *finite dimensions*.

Example. Let V be a Euclidean vector space. $\mathsf{Gl}(V) \overset{\circ}{\subset} \mathsf{End}(V)$ is a (sub-)manifold and a single parametrization is given by the inclusion map

$$\mathsf{Gl}(V) \hookrightarrow \mathsf{End}(V), \ \ g \mapsto g,$$

that we may use to investigate differentiability of the group operations,

$$\mu : \mathsf{Gl}(V) \times \mathsf{Gl}(V) \to \mathsf{Gl}(V), \ \ (g, g') \mapsto \mu(g, g') := g \circ g'$$

and

$$\iota : \mathsf{Gl}(V) \to \mathsf{Gl}(V), \ \ g \mapsto \iota(g) := g^{-1}.$$

Clearly, μ is differentiable as (restriction of) a bilinear map, with derivative

$$\mathsf{End}(V) \times \mathsf{End}(V) \ni (y, y') \mapsto d_{(g,g')}\mu(y, y') = g \circ y' + y \circ g' \in \mathsf{End}(V);$$

in particular, with $g' = g^{-1}$,

$$\mathsf{End}(V) \ni y' \mapsto d_{(g,g^{-1})}\mu(0, y') = g \circ y' \in \mathsf{End}(V)$$

is an isomorphism, so that the Implicit Mapping Theorem implies that the (unique) local solution ι of $\mu(h, \iota(h)) = \mu(g, g^{-1}) = \mathrm{id}$ is differentiable with

$$d_h\iota(y) = y' = -h^{-1} \circ y \circ h^{-1}.$$

Rem. $\mathsf{End}(V)$ becomes a Euclidean vector space with the inner product

$$(x, y) \mapsto \mathsf{tr}(x^* \circ y),$$

where $x^* \in \mathsf{End}(V)$ denotes the adjoint endomorphism. However, since $\mathsf{End}(V)$ is finite-dimensional, the notions of topology or differentiability in $\mathsf{End}(V)$ do not depend on this choice of inner product.

<u>*Rem*</u>. For a subgroup $G \leq \mathsf{Gl}(V)$ it suffices to be a (sub-)manifold in $\mathsf{End}(V)$ in order to be a Lie group: by the chain rule the group operations

$$\mu|_{G\times G} : G \times G \to G \text{ and } \iota|_G : G \to G$$

of G are differentiable, as restrictions of the group operations on $\mathsf{Gl}(V)$.

As we only consider submanifolds, we only introduce the notion of a Lie group for subgroups $G \leq \mathsf{Gl}(V)$ of a general linear group $\mathsf{Gl}(V)$:

Def. *$G \leq \mathsf{Gl}(V)$ is a* Lie group, *if $G \subset \mathsf{End}(V)$ is a submanifold.*

<u>*Example*</u>. The *orthogonal group* of a Euclidean vector space $(V, \langle .,. \rangle)$,

$$\mathsf{O}(V) := \{g \in \mathsf{Gl}(V) \,|\, \forall v, w \in V : \langle g(v), g(w)\rangle = \langle v, w\rangle\}$$

is a Lie group as it is a submanifold of $\mathsf{End}(V)$: consider

$$\mathsf{O}(V) = \beta^{-1}(\{0\}) \text{ with } \beta : \mathsf{Gl}(V) \to \mathsf{sym}(V), \; g \mapsto \beta(g),$$

where $\beta(g) \in \mathsf{sym}(V)$ denotes the symmetric bilinear form

$$\beta(g) : V \times V \to \mathbb{R}, \;\; \beta(g)(v, w) := \langle g(v), g(w)\rangle - \langle v, w\rangle;$$

we aim to show that $\beta : \mathsf{Gl}(V) \to \mathsf{sym}(V)$ is a submersion, i.e., that

$$\mathsf{End}(V) \ni y \mapsto (d_g\beta(y))(v, w) = \langle g(v), y(w)\rangle + \langle y(v), g(w)\rangle \in \mathsf{sym}(V)$$

surjects, which follows by the Riesz representation lemma — namely, using that

$$(v, w) \mapsto \sigma_0(v, w) := \langle g(v), g(w)\rangle$$

is positive definite, we deduce that

$$\forall \sigma \in \mathsf{sym}(V) \exists s \in \mathsf{Sym}_{\sigma_0}(V) \forall v, w \in V : \sigma(v, w) = 2\sigma_0(v, s(w)),$$

that is,

$$\sigma = d_g\beta(y) \text{ with } y = g \circ s.$$

Note that $\sigma_0 = \langle .,. \rangle$ for $g \in \mathsf{O}(V)$, so that

$$T_g\mathsf{O}(V) = \ker d_g\beta = \{g \circ s \,|\, \forall v, w \in V : \langle v, s(w)\rangle + \langle s(v), w\rangle = 0\}.$$

Def. *Let $G \leq \mathsf{Gl}(V)$ be a Lie group. Its* Lie algebra *is*

$$\mathfrak{g} := T_1 G;$$

its adjoint action *is the group action*

$$Ad : G \times \mathfrak{g} \to \mathfrak{g}, \ \ (g, y) \mapsto Ad_g(y) := g \circ y \circ g^{-1}.$$

<u>*Recall*</u>. A *group action* of a group G on a vector space W is a map

$$G \times W \to W, \ \ (g, w) \mapsto gw,$$

satisfying $g(g'w) = (g \circ g')w$ and $1\,w = w$ for $g, g' \in G$ and $w \in W$.

<u>*Example*</u>. $\mathfrak{gl}(V) = \mathsf{End}(V)$ is the Lie algebra of $\mathsf{Gl}(V)$.

<u>*Rem*</u>. The adjoint action is the derivative of conjugation: for fixed g, the map

$$G \ni h \mapsto \rho_g(h) := g \circ h \circ g^{-1}$$

satisfies $\rho_g(1) = 1$, so that its derivative $d_1 \rho_g = Ad_g : \mathfrak{g} \to \mathfrak{g}$.

Lemma. *$Ad : G \to \mathsf{End}(\mathfrak{g})$ is differentiable with derivative*

$$ad := d_1 Ad : \mathfrak{g} \to \mathsf{End}(\mathfrak{g}), \ \ ad_x(y) = [x, y] := x \circ y - y \circ x.$$

Proof. For fixed $y \in \mathfrak{g}$ consider $G \ni g \mapsto Ad_g(y) \in \mathfrak{g}$ to compute the derivative:

$$(d_1 Ad(x))(y) = x \circ y \circ 1^{-1} + 1 \circ y \circ (-1^{-1} \circ x \circ 1^{-1}) = [x, y],$$

where we used $d_g \iota(x) = -g^{-1} \circ x \circ g^{-1}$ for $\iota(g) = g^{-1}$. ∎

<u>*Rem*</u>. Thus the adjoint action yields a canonical (group) representation of a Lie group G on its Lie algebra $\mathfrak{g}$. If this representation is faithful, then it yields a natural ambient space for the Lie group as a submanifold.

Cor & Def. *The Lie algebra $\mathfrak{g}$ of a Lie group $G \leq \mathsf{Gl}(V)$ is a Lie algebra in the algebraic sense, i.e., comes with a* Lie bracket $[.,.]$ *satisfying*

(i) *skew symmetry: $\forall x, y \in \mathfrak{g} : [x, y] + [y, x] = 0$;*

(ii) *Jacobi identity: $\forall x, y, z \in \mathfrak{g} : [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.*

Proof. As $ad : \mathfrak{g} \to \mathsf{End}(\mathfrak{g})$, it yields a multiplication on $\mathfrak{g}$,

$$\mathfrak{g} \times \mathfrak{g} \ni (x, y) \mapsto ad_x(y) = [x, y] \in \mathfrak{g}.$$

It is trivial to verify skew symmetry and the Jacobi identity for the commutator of endomorphisms. ∎

Example. The *orthogonal algebra* of a Euclidean vector space $(V, \langle .,. \rangle)$ is the space

$$\mathfrak{o}(V) = T_1\mathsf{O}(V) = \{y \in \mathfrak{gl}(V) \,|\, \forall v, w \in V : \langle v, yw \rangle + \langle yv, w \rangle = 0\}.$$

Any other tangent space is obtained by *left translation*

$$l_g : T_1\mathsf{O}(V) \to T_g\mathsf{O}(V), \;\; y \mapsto l_g(y) := g \circ y.$$

Problem 11. Let $\Delta : V^n \to \mathbb{R}$ denote a volume form on V. Prove that the *special linear group* $\mathsf{Sl}(V) := \{g \in \mathsf{Gl}(V) \,|\, \det g = 1\}$ of volume preserving endomorphisms is a Lie group and determine its Lie algebra.

5.4 Grassmannians

Grassmannians not only provide an interesting example for manifolds but will also be paramount in our further analysis of vector bundles and the geometry of submanifolds. A key issue in identifying a Grassmannian as a (sub-)manifold in our context is to embed it into a suitable ambient Euclidean geometry: much of the analysis reduces to linear algebra and analysis, which we will discuss in detail here.

Def. *For vector subspaces $S, S' \leq V$ we introduce an equivalence relation*

$$S' \sim S :\Leftrightarrow \exists \psi \in \mathsf{Gl}(V) : S' = \psi(S).$$

An equivalence class $G_S(V) = \{S' \leq V \,|\, S' \sim S\}$ is a Grassmannian.

Rem. Assuming $\dim V < \infty$ we also write $G_k(V)$ for the Grassmannian of k-dimensional subspaces of V, as

$$S' \sim S \;\; \Leftrightarrow \;\; \dim S' = \dim S \;\text{ for }\; S, S' \leq V.$$

Example. $G_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$ is the real *projective n-space.*

Rem. In a (finite-dimensional) Euclidean vector space $(V, \langle .,. \rangle)$ the above equivalence relation may equivalently be formulated as

$$S' \sim S :\Leftrightarrow \exists \psi \in \mathsf{O}(V) : S' = \psi(S).$$

If the inner product is indefinite, then this alternative formulation leads to the *Grassmannians* of k-dimensional subspaces with fixed signature.

Lemma. *Suppose $V = S \oplus T$ and $S' \leq V$. Then $V = S' \oplus T$ iff*
$$\exists! g \in \mathsf{Hom}(S,T) : S' = \{x + g(x) \,|\, x \in S\}.$$

Proof. First suppose that $S' = \{x+g(x) \,|\, x \in S\}$ is the graph of a linear map $g : S \to T$. Then

- $V = S' + T$: take $v = s + t \in S + T = V$, then
$$v = (s + g(s)) + (t - g(s)) \in S' + T;$$
- $\{0\} = S' \cap T$: let $v = s + g(s) = t \in S' \cap T$, then
$$s = t - g(s) \in S \cap T = \{0\} \;\;\Rightarrow\;\; s = 0 = t = s + g(s).$$

Conversely, suppose that $V = S' \oplus T$ and denote the corresponding projection onto T by $\pi' : V \to T,\; v = s' + t \mapsto t$. Now set
$$g : S \to T,\;\; s \mapsto g(s) := -\pi'(s);$$
then, with the graph $G := \{x + g(x) \,|\, x \in S\}$ of g,

- $G \subset S'$: $s + g(s) = s - \pi'(s) \in S'$ for $s \in S$;
- $G \supset S'$: as $V = S \oplus T$ we can (uniquely) decompose $s' \in S'$,
$$s + t = s' \in S' \;\;\Rightarrow\;\; 0 = \pi'(s') = \pi'(s) + \pi'(t) = -g(s) + t.$$

A map g is (uniquely determined by) its graph, hence the uniqueness claim holds trivially. ∎

<u>*Rem*</u>. This lemma already provides “local” parametrizations
$$\mathsf{Hom}(S,T) \to G_S(V);$$
in the case of *projective space*, these are the usual affine coordinates.

Unfortunately, we did not identify $G_S(V)$ as a subset of some Euclidean space $\mathcal{E}$, hence do not have a notion of differentiability (or even topology) on $G_S(V)$ yet.

To remidy this defect we introduce an inner product $\langle .,. \rangle$ on V, which induces the aforementioned inner product on $\mathsf{End}(V) = \mathfrak{gl}(V)$,
$$\langle .,. \rangle : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to I\!R,\;\; (\lambda, \lambda') \mapsto \mathsf{tr}\, \lambda^* \lambda',$$
and then embed $G_S(V) \hookrightarrow \mathfrak{gl}(V)$ by identifying a subspace $S \leq V$ with the reflection ρ in S:

Lemma. *Given a scalar product* $\langle .,. \rangle$ *on* V, *where* $\dim V =: n \in \mathbb{N}$,

$$G_k(V) \stackrel{1:1}{\longleftrightarrow} \{\rho \in \mathfrak{gl}(V) \,|\, \rho^*\rho = \rho^2 = 1, \operatorname{tr}\rho = 2k - n\}.$$

Proof. Given $S \in G_k(V)$ we have $V = S \oplus S^\perp$; let π and $\pi^\perp$ denote the corresponding (orthogonal) projections and set

$$\rho := \pi - \pi^\perp = \begin{cases} 1 & \text{on } S, \\ -1 & \text{on } S^\perp; \end{cases}$$

hence $\rho^2 = 1$, $\rho^* = \rho$ and $\operatorname{tr}\rho = \dim S - \dim S^\perp = k - (n-k)$.
Conversely, if $\rho^* = \rho$ and $\rho^2 = 1$ then ρ is diagonalizable with eigenvalues ± 1 and orthogonal eigenspaces $S = \ker(\rho - 1)$ resp $S^\perp = \ker(\rho + 1)$; further, $2k - n = \operatorname{tr}\rho = 2\dim S - n$ yields $S \in G_k(V)$. ∎

Rem. Alternatively, the Grassmannian $G_k(V)$ can be identified with a set of orthogonal projections,

$$G_k(X) \stackrel{1:1}{\longleftrightarrow} \{\rho \in \operatorname{End}(X) \,|\, \rho^* = \rho = \rho^2, \operatorname{tr}\rho = k\}.$$

Problem 12. Let X be a Hilbert space, $\dim X = n$. Show that

$$G_k(X) \stackrel{1:1}{\longleftrightarrow} \{\rho \in \operatorname{End}(X) \,|\, \rho^* = \rho, \rho^2 = \rho, \operatorname{tr}\rho = k\}.$$

Recall. The *adjoint* $\lambda^* : W \to V$ of $\lambda \in \operatorname{Hom}(V, W)$, defined by

$$\forall v \in V \forall w \in W : \langle w, \lambda(v)\rangle_W = \langle \lambda^*(w), v\rangle_V,$$

satisfies

$$\ker\lambda^* = (\lambda(V))^\perp \text{ and } \lambda^*(W) = (\ker\lambda)^\perp.$$

We now reformulate the above local parametrizations in terms of reflections:

Lemma. *Let* $S \in G_k(V)$ *and* $S' = \{s + g(s) \,|\, s \in S\}$ *the graph of a linear map* $g \in \operatorname{Hom}(S, S^\perp)$; *let* ρ *resp* ρ' *denote the corresponding reflections, and* $\pi = \frac{1}{2}(1 + \rho) : V \to S$ *the orthogonal projection to* S; *then*

$$\rho' = (1 + \psi)\rho(1 + \psi)^{-1} \text{ with } \psi := (g\pi) - (g\pi)^*.$$

Conversely, let $S' \in G_k(V)$ *and suppose that* $|\rho' - \rho| < 2$; *then there is a unique* $g \in \operatorname{Hom}(S, S^\perp)$ *so that* $S' = \{s + g(s) \,|\, s \in S\}$ *is the graph of* g, *where*

$$g = (\rho' - \rho)(\rho' + \rho)^{-1}|_S.$$

Proof. First note that, clearly, $\psi^* + \psi = 0$, and that $\psi\rho + \rho\psi = 0$ since

$$S \xrightarrow{g\pi} S^\perp \xrightarrow{g\pi} \{0\} \text{ and } S^\perp \xrightarrow{(g\pi)^*} S \xrightarrow{(g\pi)^*} \{0\}$$

Further we learn that S and $S^\perp$ are invariant subspaces of

$$1 - \psi^2 = (1+\psi)^*(1+\psi) = 1 + (g\pi)^*(g\pi) + (g\pi)(g\pi)^*.$$

As a consequence, $(v, w) \mapsto \langle v, (1-\psi^2)\, w\rangle$ defines a *positive definite* inner product on V, as

$$|(1+\psi)w|^2 = \begin{cases} |w|^2 + |g(w)|^2 & \text{for } w \in S, \\ |w|^2 + |g^*(w)|^2 & \text{for } w \in S^\perp. \end{cases}$$

Hence $1 + \psi \in \mathsf{Gl}(V)$ is an automorphism, and satisfies

$$(1+\psi)(S^\perp) = ((1+\psi)(S))^\perp = \{s + g(s) \,|\, s \in S\}^\perp.$$

This proves the first claim:

$$\rho' = (1+\psi)\rho\,(1+\psi)^{-1} = (1+\psi)(1-\psi)^{-1}\rho.$$

Using this formula it is straightforward to recover g from ρ': we obtain

$$\rho' \pm \rho = \begin{cases} 2(1-\psi)^{-1}\rho, \\ 2\psi(1-\psi)^{-1}\rho; \end{cases}$$

hence

$$\psi = (\rho' - \rho)(\rho' + \rho)^{-1} \text{ and } g = \psi|_S.$$

To prove the second claim we show that $S^\perp$ is a complementary subspace of $S' \in G_k(V)$ as long as $|\rho' - \rho| < 2$: observe that,

$$(\rho' - \rho)(v) = 2v \text{ for } v \in S' \cap S^\perp;$$

thus $|\rho' - \rho|^2 = \mathrm{tr}(\rho' - \rho)^2 \geq 4$ as soon as $S' \cap S^\perp \neq \{0\}$, contrary to our assumption. ∎

Cor. *The Grassmannian $G_k(V)$ is a $k(n-k)$-dimensional manifold.*

Proof. We prove that $\{\rho \in \mathfrak{gl}(V) \,|\, \rho^*\rho = \rho^2 = 1, \operatorname{tr}\rho = 2k - n\}$ is a submanifold. Thus for $S \in G_k(V)$ we consider as a local parametrization:

$$f : \mathrm{Hom}(S, S^\perp) \to G_k(V), \;\; g \mapsto f(g) := \rho',$$

where ρ' is the reflection in the graph $S' = \{s + g(s) \,|\, s \in S\} \in G_k(V)$ of g. By the previous lemma f injects and its inverse

$$\rho' \mapsto g = (\rho' - \rho)(\rho' + \rho)^{-1}|_S$$

is the restriction of a differentiable, hence continuous, map — defined on a suitable neighbourhood $U \subset \mathfrak{gl}(V)$ of ρ.

To see that f qualifies as a local parametrization it therefore suffices to show that it immerses: we use

$$f(g) = (1+\psi)(1-\psi)^{-1}\rho \;\text{ and }\; d_g\psi(h) = (h\pi) - (h\pi)^* =: \eta$$

to compute

$$d_g f(h) = 2(1-\psi)^{-1}\eta\,(1-\psi)^{-1}\rho;$$

consequently $d_g f(h) = 0$ implies $\eta = 0$, hence $h = \eta|_S = 0$. ∎

Rem. Another approach is to describe a Grassmannian as a symmetric space:

$$G_S(V) = \mathrm{O}(V)/(\mathrm{O}(S) \times \mathrm{O}(S^\perp)),$$

using Lie groups as discussed in the previous section.

5.5 Vector bundles

A vector bundle over a manifold M is

- a manifold that locally "looks like" the product $M \times V$ of M and a vector space V;
- a family of (isomorphic) vector spaces $X \mapsto V_X$ that is smoothly parametrized by M.

The usual definition, to be found in many textbooks, takes the first approach. We take the second point of view:

Def. *Let $M \subset \mathcal{E}$ be a manifold. A* vector bundle over M *is a smooth map*
$$S : M \to G_k(V)$$
for some $k \in \mathbb{N}$ (the rank *of S) and some Hilbert space V; the* fibre *of S at X is the image S_X of X, the* total space *of S is its "graph"*
$$\{(X, Y) \in M \times V \,|\, Y \in S_X\}.$$

Rem. We will not distinguish between a k-dimensional subspace $S \leq V$ and the reflection $\rho \in \mathfrak{gl}(V)$ in S (from now on also denoted by S).

Rem. In a similar way, a *principal bundle over M* can be defined as a smooth map $M \ni X \mapsto G_X \leq \mathsf{Gl}(V)$ assigning isomorphic Lie subgroups G_X of some $\mathsf{Gl}(V)$ to every point of a manifold M.

Expl & Def. If $\dim V = n$ then $G_n(V) = \{V\}$ and the constant map
$$S : M \to G_n(V), \;\; X \mapsto S_X = V$$
is called a *trivial (vector) bundle* over $M \subset \mathcal{E}$.

Rem. Thus a "vector bundle" is, in the sense of the above definition, a "vector (sub-)bundle (of the trivial bundle $M \times V$)" — similar to our definition of a manifold as a submanifold of $\mathcal{E}$.

Def. *A* section *of a vector bundle $S : M \to G_k(V)$ is a smooth map*
$$\sigma : M \to V, \;\; \textit{where} \;\; \forall X \in M : \sigma(X) \in S_X.$$
A local section *is one that is defined locally, on some open $U \subset M$.*

Notations.

- $\Gamma V := \{\sigma : M \to V \,|\, \forall X \in M : \sigma(X) \in S_X\}$ the space of sections;
- $\Gamma_X V := \{\sigma : U \to V \,|\, \forall Y \in U : \sigma(Y) \in S_Y\}$ the space of local sections, defined on some open neighbourhood $U \subset M$ of $X \in M$.

Rem. In a similar way, a *(local) frame* $F : M \to \mathsf{Gl}(V)$ is a (local) section of a principal bundle $M \ni X \mapsto G_X \leq \mathsf{Gl}(V)$.

Rem. Given vector bundles $S : M \to G_k(V)$ and $S' : M \to G_{k'}(V')$
$$X \mapsto \mathsf{Hom}(S_X, S'_X) \subset \mathsf{End}(V, V') \;\; \text{(trivial extensions to } S_X^\perp)$$
defines another, rank kk', vector bundle (a *tensor bundle*) $\mathsf{Hom}(S, S')$.

Def. *A section* $\Phi \in \Gamma\mathsf{Hom}(S, S')$ *is called* vector bundle homomorphism*; it is a* vector bundle isomorphism *if every* Φ_X *bijects.*

Lemma. *A vector bundle* $S : M \to G_k(V)$ *admits* local basis sections*:*
$$\forall X \in M \exists b_i \in \Gamma_X S \forall Y \in U : S_Y = [(b_1(Y), \ldots, b_k(Y))].$$
Hence, any S *is locally (isomorphic to a) trivial (bundle).*

Proof. Fix $X \in M$ and $U \subset M$ so that $\forall Y \in U : |V_Y - V_X| < 2$. Hence, for any $Y \in U$, the orthogonal projection $\pi : V \to S_X$ yields an isomorphism
$$\pi|_{S_Y} : S_Y \to S_X.$$
Thus choose a basis $(c_1, \ldots, c_k)$ of S_X; then, for $i = 1, \ldots, k$,
$$U \ni Y \mapsto b_i(Y) := (\pi|_{S_Y})^{-1}(c_i) \in S_Y$$
define local sections of S, $b_i \in \Gamma_X S$, so that $(b_1, \ldots, b_k)(Y)$ is a basis of S_Y for each $Y \in U$. Hence a (local) vector bundle isomorphism
$$M \supset U \ni Y \mapsto \Phi_Y \in \mathsf{Hom}(S_Y, S_X), \;\; \Phi_Y : b_j(Y) \mapsto c_j,$$
is obtained from S to the trivial bundle $M \supset U \to G_k(S_X) = \{S_X\}$. ∎

Rem. If $b_i : M \to V$ $(i = 1, \ldots, k)$ are smooth and (pointwise) linearly independent then they span a (smooth) rank k vector bundle
$$S : M \to G_k(V), \;\; X \mapsto S_X := [(b_1(X), \ldots, b_k(X))].$$

Problem 13. Prove this claim.

Cor. *The total space of a vector bundle* $S : M^m \to G_k(V)$ *over an* m*-dimensional manifold is an* $(m+k)$*-dimensional manifold in* $\mathcal{E} \times V$*.*

Remark. Clearly, the graph of a vector bundle S is a (sub-)manifold
$$\{(X, S_X) \in M \times G_k(V) \,|\, X \in M\} \subset \mathcal{E} \times \mathsf{End}(V).$$
By the corollary, it is a manifold if the fibres S_X are thought of as subsets of V rather than as points in $G_k(V) \subset \mathsf{End}(V)$.

Proof. Fix $X \in M$ and a local parametrization
$$f : O \to M \subset \mathcal{E} \;\text{ around }\; X = f(x) \in f(O) =: U \overset{\circ}{\subset} M.$$

By the previous lemma $S : M \to G_k(V)$ admits local basis sections, wlog defined on U:
$$\exists b_i \in \Gamma_X S : S|_U = [(b_1, \dots, b_k)].$$
Hence, for any $v \in S_X$, that is, point (X, v) of the total space of S,
$$F : O \times \mathbb{R}^k \to \mathcal{E} \times V,\ \ (y, w) \mapsto (f(y), \textstyle\sum_{i=1}^k (b_i \circ f)(y)\, w_i)$$
yields a local parametrization of the total space of S around (X, v). ∎

Def & Cor. *Let $M \subset \mathcal{E}$ be an m-dimensional manifold; its* tangent bundle
$$TM : M \to G_m(V),\ \ X \mapsto T_X M,$$
is a rank m vector bundle. A vector field *on M is a section $\xi \in \Gamma(TM)$.*

Proof. We need to show that $TM : M \to G_m(V)$ is differentiable: this follows since
$$TM|_U = [(\tfrac{\partial}{\partial x_1}, \dots, \tfrac{\partial}{\partial x_m})],$$
where
$$U \ni X \mapsto \tfrac{\partial}{\partial x_i}|_X := d_{f^{-1}(X)} f(e_i) \in T_X M,$$
with the standard basis $(e_1, \dots, e_m)$ of $\mathbb{R}^m$, denote the *Gaussian basis fields* of a local parametrization $f : O \to f(O) =: U \subset M$. ∎

Remark. The tangent bundle $TM : M \to G_m(V)$ of an m-dimensional manifold $M \subset \mathcal{E}$ is rank m vector bundle over M, thus a $2m$-dimensional manifold in $\mathcal{E} \times V$.

Lemma & Def. *Given vector fields $\xi, \eta \in \Gamma(TM)$ there is a unique vector field $\zeta \in \Gamma(TM)$ so that*
$$\forall h \in C^\infty(M) : \zeta h = \xi(\eta h) - \eta(\xi h).$$
$[\xi, \eta] := \zeta$ is called the Lie bracket *of ξ and η.*

Proof. First note that, by Schwarz lemma,
$$\forall h \in C^\infty(M) : \tfrac{\partial}{\partial x_i} \tfrac{\partial}{\partial x_j}\, h - \tfrac{\partial}{\partial x_j} \tfrac{\partial}{\partial x_i}\, h = 0 \ \Rightarrow\ [\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}] = 0$$
for the Gaussian basis fields $\frac{\partial}{\partial x_i}$ of a local parametrization $f : O \to U$. Now write $\xi|_U = \sum_{i=1}^m v_i \frac{\partial}{\partial x_i}$ and $\eta|_U = \sum_{i=1}^m w_j \frac{\partial}{\partial x_j}$ to compute
$$[\xi, \eta]\, h \ = \ \textstyle\sum_{i,j=1}^m v_i \frac{\partial w_j}{\partial x_i} \frac{\partial h}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \frac{\partial h}{\partial x_i} \ \text{ for } h \in C^\infty(U),$$

hence

$$[\xi,\eta]|_U \;=\; \sum_{i,j=1}^m v_i \frac{\partial w_j}{\partial x_i}\frac{\partial}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j}\frac{\partial}{\partial x_i} \in \Gamma(TM|_U)$$

yields indeed a vector field, i.e., second order derivatives vanish. ∎

Rem. With the Lie bracket $\Gamma(TM)$ is an (infinite dimensional) Lie algebra: clearly $[.,.]$ is bilinear and skew-symmetric; the Jacobi identity is readily checked.

5.6 Connections on vector bundles

Connections already made an appearance earlier in this text: as the *Levi-Civita connection* of a surface or as the *normal connection* of a curve. These connections provided a method to take derivatives of tangential resp normal vector fields. A *linear connection* on a vector bundle is a straightforward generalization of these two notions:

Def. *A* (linear) connection *on a vector bundle* $S : M \to G_k(V)$ *is a map*

$$\nabla : \Gamma(TM \times S) \to \Gamma(S), \;\; (\xi,\sigma) \mapsto \nabla_\xi\sigma$$

so that

(i) ∇ *is* $C^\infty(M)$*-linear in the first argument:*

$$\nabla_{\xi+\eta}\sigma = \nabla_\xi\sigma + \nabla_\eta\sigma \;\text{ and }\; \nabla_{h\xi}\sigma = h\nabla_\xi\sigma;$$

(ii) ∇ *is a* derivation *in the second argument:*

$$\nabla_\xi(\sigma+\tau) = \nabla_\xi\sigma + \nabla_\xi\tau \;\text{ and }\; \nabla_\xi(h\sigma) = h\nabla_\xi\sigma + (\xi h)\,\sigma.$$

Rem. The *Lie derivative* $\mathcal{L}_\xi\eta := [\xi,\eta]$ yields a way to differentiate vector fields of the tangent bundle TM of a (sub-)manifold; however, this is *not* a linear connection on TM since $\xi \mapsto \mathcal{L}_\xi\eta$ is not $C^\infty(M)$-linear.

Expl & Def. Ordinary differentiation on a trivial vector bundle,

$$S : M \to G_n(V) \;\text{ with }\; \dim V = n,$$

yields a *trivial connection*

$$\nabla : \Gamma(TM \times S) \to \Gamma(S), \;\; \nabla_\xi\sigma|_X := d_X\sigma(\xi_X) \;\text{ for }\; X \in M.$$

Rem. Any two connections ∇ and ∇' on a vector bundle S differ by a tensor field: $\beta := \nabla' - \nabla : \Gamma(TM) \to \Gamma\mathrm{End}(S)$ satisfies

$$\forall h \in C^\infty(M) : \beta(h\sigma) = h\,\beta\sigma.$$

Def. *A section $\sigma \in \Gamma(S)$ of a vector bundle $S : M \to G_k(V)$ is called* parallel *if*

$$\nabla\sigma \equiv 0.$$

Two connections ∇, ∇' on vector bundles S and S' over M will be called gauge equivalent *if*

$$\exists \Phi \in \Gamma\mathrm{Hom}(S, S') : \nabla' \circ \Phi = \Phi \circ \nabla \text{ and } \forall X \in M : \Phi_X^{-1} \text{ exists.}$$

The map $\nabla \mapsto \nabla' = \Phi \circ \nabla \circ \Phi^{-1}$ is called a gauge transformation *of ∇.*

Lemma. *A connection is gauge equivalent to a trivial connection iff it admits a parallel basis field.*

Proof. Suppose $(\sigma_1, \dots, \sigma_k)$ is a parallel basis field of S, that is, $\nabla\sigma_i = 0$ for $i = 1, \dots, k$; let $S' = M \times \mathbb{R}^k$ denote a(ny) trivial rank k vector bundle with trivial connection d and (constant) basis $(b_1, \dots, b_k)$. Then define

$$\Phi \in \Gamma\mathrm{Hom}(S, S') \text{ by } \Phi(\sigma_i) := b_i \text{ for } i = 1, \dots, k$$

to obtain the sought gauge transformation:

$$(d \circ \Phi)(\sigma_i) = db_i = 0 = \Phi(0) = (\Phi \circ \nabla)(\sigma_i).$$

The converse follows by the very same computation, where Φ is used to obtain $(\sigma_1, \dots, \sigma_k)$. ∎

Rem. Any vector bundle S admits local basis fields, i.e., is *locally trivial*. However, a vector bundle S with connection ∇ may not even be locally gauge equivalent to a vector bundle S' with a trivial connection d.

Local triviality of a (vector bundle with) connection is detected by the corresponding curvature tensor:

Def. *The* curvature tensor *of a connection ∇ on a vector bundle S is the tensor field $\mathrm{R} : \Gamma(TM \times TM \times S) \to \Gamma(S)$,*

$$(\xi, \eta, \sigma) \mapsto \mathrm{R}_{\xi,\eta}\sigma := \nabla_\xi\nabla_\eta\sigma - \nabla_\eta\nabla_\xi\sigma - \nabla_{[\xi,\eta]}\sigma.$$

Rem. R is a *tensor field*, that is, $C^\infty(M)$-linear in all entries.

Problem 14. Prove that the curvature tensor R of a connection ∇ on a vector bundle S is a tensor field.

Thm & Def. ∇ *is locally trivial iff it is* flat, *that is, iff* $\mathrm{R} \equiv 0$.

Proof. If ∇ is locally trivial we use a local parallel basis field $(\sigma_1, \ldots, \sigma_k)$ to compute

$$\forall i = 1, \ldots, k : \mathrm{R}\sigma_i \equiv 0.$$

To prove the converse, we show that any vector $\sigma_o \in S_O$, $O \in M$, can be extended to a local parallel section $\sigma \in \Gamma_O S$; then a basis at any point can be extended to a local parallel basis field, hence ∇ is locally trivial.

We use induction over $m = \dim M$: let $\dim M = m + 1$ and fix a local parametrization $\mathbb{R}^m \times \mathbb{R} \ni (x,t) \mapsto f(x,t) \in M$ around $O = f(0,0)$; fix $\sigma_o \in S_O$ and let $M_0 := \{f(x,0)\}$ denote the $t = 0$ "sheet" in M. Since the restriction of ∇ to M_0 is flat there is a local parallel section

$$\tilde{\sigma} : M_0 \to V, \;\; \tilde{\sigma}(f(x,0)) \in S_{f(x,0)};$$

now use parallel transport along the t-curves to extend $\tilde{\sigma}$ to a local section over M, cf Sect 1.3: by the Picard-Lindelöf theorem there are unique solutions $t \mapsto \sigma(f(x,t))$ of the initial value problems

$$\nabla_{\frac{\partial}{\partial t}} \sigma = 0, \;\; \sigma(f(x,0)) = \tilde{\sigma}(f(x,0)).$$

Taking for granted (smooth dependence on the initial value) that σ is differentiable we now verify that σ is indeed parallel: since

$$\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial x_i}} \sigma = \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial t}} \sigma - \mathrm{R}_{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial t}} \sigma = 0 \text{ and } \nabla_{\frac{\partial}{\partial x_i}} \sigma|_{M_0} = 0$$

we conclude that $\nabla_{\frac{\partial}{\partial x_i}} \sigma = \nabla_{\frac{\partial}{\partial t}} \sigma = 0$, that is, $\nabla \sigma = 0$. ∎

Rem. If $d = \nabla + \beta$ with a tensor field $\beta : \Gamma(TM) \to \Gamma\mathrm{End}(S)$ then

$$0 = \mathrm{R}^d_{\xi,\eta}\sigma = (\mathrm{R}^\nabla_{\xi,\eta} + [\beta_\xi \circ \beta_\eta])\,\sigma + ((\nabla_\xi \beta_\eta) - (\nabla_\eta \beta_\xi) - \beta_{[\xi,\eta]})\,\sigma, \;\; (*)$$

with the *covariant derivative of a tensor field*,

$$(\nabla_\xi \beta_\eta)\sigma := \nabla_\xi(\beta_\eta \sigma) - \beta_\eta(\nabla_\xi \sigma) \text{ for } \beta_\eta \in \Gamma\mathrm{End}(S).$$

If $M \subset \mathcal{E}^3$ is a surface in Euclidean 3-space (over $V = \mathbb{R}^3$) and $S \equiv V$, the *Gauss-Weingarten equations* yield a decomposition $d = \nabla + \beta$ of the trivial connection (differentiation) on $S = M \times \mathbb{R}^3$, and $(*)$ become the corresponding *Gauss-Codazzi equations*.

Problem 15. Revise the Gauss-Codazzi equations.

5.7 Geometry of submanifolds

We take up the remark formulated at the end of the previous section to generalize the geometric notions of the first two chapters of this text to submanifolds. In order to analyze the structure of the geometric invariants and their relations in more depth we first consider the structure equations in an affine setting, before specializing to a Euclidean setting: thus we initially use a "normal bundle" that complements the tangent bundle, but is not necessarily orthogonal:

Def. *Two vector bundles $S : M \to G_k(V)$ and $S' : M \to G_{k'}(V)$ are* complementary *if*
$$\forall X \in M : V = S_X \oplus S'_X.$$

Notation. Throughout this section $M \subset \mathcal{E}$ will be an m-dimensional submanifold in a Euclidean space $\mathcal{E}$ over V. We will identify V with the trivial vector bundle $V : M \to G_n(V)$, where $\dim V = n$, and let d denote a trivial connection on V.

Lemma (Gauss-Weingarten equations). *Suppose that TM and N' are complementary vector bundles over $M \subset \mathcal{E}$ and denote the respective projections by $\pi \in \Gamma\mathrm{Hom}(V, TM)$ and $\pi' \in \Gamma\mathrm{Hom}(V, N')$; let*
$$\begin{array}{rclcrcl} \nabla & := & \pi \circ d \circ \pi, & \quad & \mathrm{S} & := & -\pi \circ d \circ \pi', \\ \mathrm{I\!I} & := & \pi' \circ d \circ \pi, & \quad & \nabla' & := & \pi' \circ d \circ \pi'. \end{array}$$
Then ∇ and ∇' are connections on TM and N', respectively, while
$$\Gamma(TM \times TM) \ni (\xi, \eta) \mapsto \mathrm{I\!I}(\xi, \eta) \in \Gamma(N')$$
and
$$\Gamma(N') \ni \nu \mapsto \mathrm{S}_\nu \in \Gamma\mathrm{End}(TM)$$
are tensors, i.e., $C^\infty(M)$-linear; further, $\mathrm{I\!I}$ is symmetric, and ∇ is torsion free, *that is, its* torsion tensor T *vanishes,*
$$\forall \xi, \eta \in \Gamma(TM) : \mathsf{T}(\xi, \eta) := \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta] = 0.$$

Rem. Thus, for $\xi, \eta \in \Gamma(TM)$ and $\nu \in \Gamma(N')$,
$$\begin{array}{rcccccc} \xi\eta & = & d\eta(\xi) & = & \nabla_\xi \eta & + & \mathrm{I\!I}(\xi, \eta), \\ \xi\nu & = & d\nu(\xi) & = & -\mathrm{S}_\nu \xi & + & \nabla'_\xi \nu. \end{array}$$

Further note that the torsion only makes sense for a connection on TM.

Proof. For $\xi, \eta \in \Gamma(TM)$ and $h \in C^\infty(M)$

$$\xi(h\eta) = (\pi + \pi')((\xi h)\eta + h\,(\xi\eta)) = ((\xi h)\eta + h\nabla_\xi \eta) + h\,\mathrm{II}(\xi, \eta),$$

showing that ∇ and II are a derivtation resp $C^\infty(M)$-linear in the second argument; clearly both are $C^\infty(M)$-linear in the first argument. Hence, ∇ is a connection and II a tensor.

A similar computation/argument applies to S and ∇'.

Finally, as the trivial connection d is torsion free (Schwarz' lemma),

$$[\xi, \eta] = \xi\eta - \eta\xi = (\nabla_\xi \eta - \nabla_\eta \xi) + (\mathrm{II}(\xi, \eta) - \mathrm{II}(\eta, \xi))$$

is tangential, showing that II is symmetric and ∇ torsion free. ∎

Gauss-Codazzi-Ricci equations. *Suppose that TM and N' are complementary vector bundles over a submanifold $M \subset \mathcal{E}$; then*

(G) $\mathrm{R}_{\xi,\eta}\zeta = \mathrm{S}_{\mathrm{II}(\zeta,\eta)}\xi - \mathrm{S}_{\mathrm{II}(\zeta,\xi)}\eta$ — Gauss equation,

(C) $(\nabla'_\xi \mathrm{II})(\eta, \zeta) = (\nabla'_\eta \mathrm{II})(\xi, \zeta)$ — Codazzi equation *(for II)*,

$(\nabla_\xi \mathrm{S})_\nu\, \eta = (\nabla_\eta \mathrm{S})_\nu\, \xi$ — Codazzi equation *(for S)*,

(R) $\mathrm{R}'_{\xi,\eta}\nu = \mathrm{II}(\xi, \mathrm{S}_\nu \eta) - \mathrm{II}(\mathrm{S}_\nu \xi, \eta)$ — Ricci equation,

where R and R' denote the curvature tensors of ∇ resp ∇', and with the covariant derivatives of II resp $\mathrm{S} : \Gamma(N' \times TM) \to \Gamma(TM)$,

$$\begin{array}{lcl} (\nabla'_\xi \mathrm{II})(\eta, \zeta) & := & \nabla'_\xi \mathrm{II}(\eta, \zeta) - \mathrm{II}(\nabla_\xi \eta, \zeta) - \mathrm{II}(\eta, \nabla_\xi \zeta), \\ (\nabla_\xi \mathrm{S})_\nu \eta & := & \nabla_\xi(\mathrm{S}_\nu \eta) - \mathrm{S}_\nu \nabla_\xi \eta - \mathrm{S}_{\nabla'_\xi \nu}\eta. \end{array}$$

Proof. The equations are obtained in a straightforward way, by expanding

$$0 = \xi(\eta\zeta) - \eta(\xi\zeta) - [\xi, \eta]\,\zeta \ \text{ and } \ 0 = \xi(\eta\nu) - \eta(\xi\nu) - [\xi, \eta]\,\nu$$

and decomposing the results into $\Gamma(TM)$- and $\Gamma(N')$-components. ∎

Rem. These are just the equation $(*)$ from the previous section, with $\nabla + \nabla'$ as a connection on $TM \oplus N'$ and with $\beta = \mathrm{II} - \mathrm{S}$ as its deviation from the trivial connection.

Example. Suppose that ∇ is torsion free and $\nu \in \Gamma(N')$ is ∇'-parallel. Then

$$\mathrm{S}_\nu = \kappa_\nu \mathrm{id}_{TM} \;\;\Rightarrow\;\; (\nabla_\xi \mathrm{S}_\nu) = (\xi\kappa_\nu)\,\mathrm{id}_{TM}.$$

Thus, if $\dim M \geq 2$ and $\mathrm{S}_\nu = \kappa_\nu \mathrm{id}_{TM}$ then $\kappa_\nu \equiv const$ by the Codazzi equation (as in Sect 2.4), hence, as $\pi' \circ d\nu = \nabla'\nu = 0$,

$$0 = \kappa_\nu \mathrm{id}_{TM} - \mathrm{S}_\nu = \kappa_\nu + d\nu \;\;\Rightarrow\;\; \begin{cases} X \mapsto \nu(X) \equiv const & \text{or} \\ X \mapsto X + \frac{\nu(X)}{\kappa_\nu(X)} \equiv const, \end{cases}$$

depending on whether $\kappa_\nu \equiv 0$ or $\neq 0$. In the former case, $M \subset \mathcal{H}$ for some hyperplane $\mathcal{H} \subset \mathcal{E}$; in the latter case, M is a submanifold in some hypersphere $S^{n-1}(\frac{1}{\kappa_\nu}) \subset \mathcal{E}$ as soon as ν is a unit normal field.

We now turn to a Euclidean setting — thus require the presence of an inner product — where, for example, the orthogonal complement of the tangent bundle yields a canonical complementary vector bundle. The main consequences will be the existence of a distinguished tangential connection, the *Levi-Civita connection* of a submanifold, and a relation between the second fundamental form II and the shape operator S of a submanifold, procured by the *induced metric*. To start we set the scene:

Def. *A vector bundle $S : M \to G_k(V)$ is* Euclidean *if it carries a* metric

$$\langle .,. \rangle : \Gamma(S \times S) \to C^\infty(M);$$

that is, each $\langle .,. \rangle_X : S_X \times S_X \to \mathbb{R}$ is a positive definite inner product. A connection ∇ on S is then metric *if it satisfies Leibniz' rule*

$$\xi\,\langle \sigma, \tau \rangle = \langle \nabla_\xi \sigma, \tau \rangle + \langle \sigma, \nabla_\xi \tau \rangle.$$

Expl & Def. If V is equipped with a scalar product $\langle .,. \rangle$ then the tangent bundle $TM : M \to G_m(V)$ of $M \subset \mathcal{E}$ inherits a metric

$$\mathrm{I} : \Gamma(TM \times TM) \to C^\infty(M), \;\; (\xi, \eta) \mapsto \mathrm{I}(\xi, \eta) := \langle \xi, \eta \rangle,$$

the *induced metric* or *first fundamental form* on M. In this situation

$$NM : M \to G_{n-m}(V), \;\; X \mapsto N_X M := T_X M^\perp$$

yields a natural complementary vector bundle, the *normal bundle* of M.

Decomposing the trivial (metric) connection d on $M \to G_n(V) = \{V\}$ according to the Gauss-Weingarten equations,

$$d\sigma = \begin{cases} \nabla\sigma & + & \mathrm{I\!I}(\sigma, .) & \text{for } \sigma \in \Gamma(TM), \\ -S_\sigma & + & \nabla^\perp\sigma & \text{for } \sigma \in \Gamma(NM), \end{cases}$$

yields

- the *covariant derivative* or *Levi-Civita connection* ∇ of $M \subset \mathcal{E}$;
- its *second fundamental form* $\mathrm{I\!I}$ and *shape operator* S;
- and its *normal connection* $\nabla^\perp$.

Notation. From now on we assume that $M \subset \mathcal{E}$ be a sub(!)manifold of a Euclidean space $\mathcal{E}$ over a vector space V with inner product $\langle ., . \rangle$.

Lemma. *The shape operator* S *of a submanifold* $M \subset \mathcal{E}$ *is symmetric (with respect to the first fundamental form* I*) and is related to its second fundamental form* $\mathrm{I\!I}$ *by*

$$\forall \xi, \eta \in \Gamma(TM) \forall \nu \in \Gamma(NM) : \langle \nu, \mathrm{I\!I}(\xi, \eta) \rangle = \mathrm{I}(\mathrm{S}_\nu \xi, \eta).$$

Proof. For $\xi, \eta \in \Gamma(TM)$ and $\nu \in \Gamma(NM)$ we compute

$$0 = \xi \langle \eta, \nu \rangle = \langle \xi\eta, \nu \rangle + \langle \eta, \xi\nu \rangle = \langle \mathrm{I\!I}(\xi, \eta), \nu \rangle - \langle \eta, \mathrm{S}_\nu \xi \rangle,$$

proving the claimed relation. Symmetry of S_ν then follows from the symmetry of $\mathrm{I\!I}$. ∎

Lemma. *The normal connection* $\nabla^\perp$ *of a submanifold* M *is metric.*

Proof. The trivial connection d on $M \to G_n(V)$, $X \mapsto V$, is metric, hence

$$d\langle \nu, \tilde\nu \rangle = \langle d\nu, \tilde\nu \rangle + \langle \nu, d\tilde\nu \rangle = \langle \nabla^\perp \nu, \tilde\nu \rangle = \langle \nu, \nabla^\perp \tilde\nu \rangle$$

for $\nu, \tilde\nu \in \Gamma(NM)$. ∎

Expl & Def. If $M \subset \mathcal{E}$ is a *hypersurface*, i.e., $\dim \mathcal{E} = \dim M + 1$, then the normal connection $\nabla^\perp$ of M is locally trivial, hence flat: namely, normalizing a local basis field we obtain a (local) *Gauss map* $\nu \in \Gamma(NM)$ (which is unique up to sign); then (ν) is a parallel basis field of NM as

$$0 = d|\nu|^2 = 2\langle \nu, d\nu\rangle \;\;\Rightarrow\;\; d\nu = -\mathrm{S}_\nu \in \Gamma\mathrm{End}(TM).$$

As $\mathrm{S}_\nu \in \Gamma\mathrm{End}(TM)$ is symmetric it diagonalizes and, where the *principal curvatures* of M (i.e., eigenvalues κ_i of S_ν) do not change multiplicities, there is a local orthonormal basis field of *curvature directions*

$$(\xi_1, \dots, \xi_m) \text{ with } \mathrm{S}_\nu \xi_i = \kappa_i \xi_i \text{ for } i = 1, \dots, m.$$

If M is *totally umbilic*, $\mathrm{S}_\nu = \kappa\, \mathrm{id}_{TM}$, and $\dim M \geq 2$ we already know that

$$\kappa \equiv const.$$

Thus:

- if $\kappa = 0$ then $\nu \equiv const$, hence, for any fixed $X_o \in M$,
$$M \ni X \mapsto \langle X - X_o, \nu\rangle \equiv const \in \mathbb{R},$$
showing that M lies in a hyperplane with unit normal ν;
- if $\kappa \neq 0$ then $X \mapsto C := X + \nu\,\frac{1}{\kappa} \equiv const$, hence
$$M \ni X \mapsto |X - C|^2 = \tfrac{1}{|\kappa|^2} \equiv const \in \mathbb{R},$$
showing that M lies in a hypersphere with centre C and radius $\frac{1}{|\kappa|}$.

Lemma. *The covariant derivative ∇ of a manifold $M \subset \mathcal{E}$ is torsion free and metric (with respect to the induced metric).*

Proof. We already know that ∇ is torsion free (from Schwarz' lemma). For vector fields $\xi, \eta, \zeta \in \Gamma(TM)$ one computes

$$\xi\mathrm{I}(\eta, \zeta) = \xi\langle \eta, \zeta\rangle = \langle \xi\eta, \zeta\rangle + \langle \eta, \xi\zeta\rangle = \mathrm{I}(\nabla_\xi \eta, \zeta) + \mathrm{I}(\eta, \nabla_\xi \zeta),$$

showing that ∇ is a metric connection. ∎

Lemma (Koszul's formulas). *Let M be a manifold with a metric* I*. Any torsion free and metric connection ∇ on TM satisfies*

$$\begin{array}{rcccccc} 2\mathrm{I}(\nabla_\xi\eta,\zeta) & = & \xi\,\mathrm{I}(\eta,\zeta) & + & \eta\,\mathrm{I}(\xi,\zeta) & - & \zeta\,\mathrm{I}(\xi,\eta) \\ & - & \mathrm{I}(\xi,[\eta,\zeta]) & - & \mathrm{I}(\eta,[\xi,\zeta]) & + & \mathrm{I}(\zeta,[\xi,\eta]). \end{array}$$

Proof. This is a straightforward computation: if ∇ is metric then

$$\begin{array}{rcccccc} \xi\,\mathrm{I}(\eta,\zeta) & = & & & \mathrm{I}(\nabla_\xi\zeta,\eta) & + & \mathrm{I}(\nabla_\xi\eta,\zeta) \\ \eta\,\mathrm{I}(\zeta,\xi) & = & \mathrm{I}(\nabla_\eta\zeta,\xi) & & & + & \mathrm{I}(\nabla_\eta\xi,\zeta) \\ \zeta\,\mathrm{I}(\xi,\eta) & = & \mathrm{I}(\nabla_\zeta\eta,\xi) & + & \mathrm{I}(\nabla_\zeta\xi,\eta) & & \end{array}$$

and then the claim follows by using that ∇ is torsion free. ∎

Cor (Levi-Civita connection). *Given a metric* I *on a manifold M, there is a unique torsion free and metric connection ∇ on TM.*

Proof. Any metric and torsion free connection ∇ must satisfy Koszul's formula, hence is uniquely determined by the metric via Riesz' representation lemma.

On the other hand, Koszul's formula defines indeed a metric and torsion free connection on TM: the right hand side of Koszul's formula

- is $C^\infty(M)$-linear in the 1^{st} argument,
- behaves like a derivation in the 2^{nd} argument,
- symmetrizing the 2^{nd} and 3^{rd} arguments yields $\xi\,\mathrm{I}(\eta,\zeta)$,
- skew-symmetrizing 1^{st} and 2^{nd} arguments yields $\mathrm{I}([\xi,\eta],\zeta)$. ∎

Cor. *The curvature (tensor)* R *of the Levi-Civita connection ∇ of a manifold $M \subset \mathcal{E}$ depends on the metric* I *on M alone.*

Proof. Clear. ∎

Theorema egregium. *The* Gauss curvature $K := \det \mathrm{S}_\nu$ *of a surface* $M^2 \subset \mathcal{E}^3$ *in Euclidean* 3*-space depends on the induced metric alone.*

Proof. By the Gauss equation

$$\begin{aligned} \mathrm{I}(\mathrm{R}_{\xi,\eta}\eta,\xi) &= \mathrm{I}(\mathrm{S}_{\mathrm{II}(\eta,\eta)}\xi,\xi) - \mathrm{I}(\mathrm{S}_{\mathrm{II}(\eta,\xi)}\eta,\xi) \\ &= \langle \mathrm{II}(\xi,\xi), \mathrm{II}(\eta,\eta)\rangle - \langle \mathrm{II}(\xi,\eta), \mathrm{II}(\xi,\eta)\rangle \\ &= \mathrm{I}(\mathrm{S}_\nu\xi,\xi)\,\mathrm{I}(\mathrm{S}_\nu\eta,\eta) - \mathrm{I}(\mathrm{S}_\nu\xi,\eta)^2 \\ &= \det \mathrm{S}_\nu \end{aligned}$$

for any (local) orthonormal basis field (ξ, η) of TM. ∎

<u>*Rem*</u>. The Gauss curvature K is independent of the choice (sign) of the Gauss map $\nu \in \Gamma(NM)$.

For higher dimensional hypersurfaces $M \subset \mathcal{E}$, a similar argument shows that products $K_{ij} := \kappa_i\kappa_j$ of principal curvatures κ_i and κ_j for $i \neq j$, the *sectional curvatures* of a hypersurface M, are *intrinsic quantities* (that is, depend on I only).

Epilogue

In this rather concise introduction to the wide field of differential geometry we were only able to touch upon few topics — however, I hope to have given a clear outline of the basic methods of (local) differential geometry in a way that will enable the interested student to study further topics independently.

In particular, this lecturer hopes that the underlying principle of how connections on vector bundles lead to geometric results has been elucidated clearly, by recovering some of the key results for (parametrized) surfaces from Chap 2 in the setting of submanifolds of Chap 5.

Using these methods, it should now for example be rather straightforward to generalize those results on special surfaces from Chap 4, that did not depend on a specific dimension or codimension, to the general setting of a submanifold in a Euclidean space.

I hope that the presented material has sparked your interest in the beautiful field of differential geometry, and that it has provided a solid base to read and study further, for example, using the books referenced in the introduction. Anyhow, I wish you *all the best for your future (studies)*!

Appendix: Tools from algebra and analysis

The following notations, definitions and formulas of linear algebra and analysis are used throughout the text, without further comment or explanation; they may be found in many standard textbooks. It will make a good exercise to verify any unfamiliar identities, by example and/or proof.

A.1 Euclidean geometry

We study the differential geometry of objects in a Euclidean ambient geometry — here we recall the notions of a Euclidean space resp motion, and collect some useful facts and formulas in the context.

Euclidean space. A triplet $(\mathcal{E}, V, \tau)$, consisting of a set $\mathcal{E}$ of points, a Euclidean vector space $(V, \langle .,. \rangle)$, where $\langle .,. \rangle : V \times V \to \mathbb{R}$ is a positive definite inner product, and an action $\tau : V \times \mathcal{E} \to \mathcal{E}$ of V on $\mathcal{E}$ by *translations*:

(i) $\tau_0 = \mathrm{id}_{\mathcal{E}}$ and $\forall v, w \in V : \tau_v \circ \tau_w = \tau_{v+w}$ (*group action*);

(ii) $\forall X, Y \in \mathcal{E} \exists! v \in V : \tau_v X = Y$ (*simple transitivity*).

For simplicity we write $\tau_v X =: X + v$.

Note that no assumption about the dimension is made.

Cartesian reference system. $(O; E)$, where $O \in \mathcal{E}$ is an *origin* and E is an orthonormal basis of $(V, \langle .,. \rangle)$. If $E = (e_1, \dots, e_n)$ then every point $X \in \mathcal{E}$ has (unique) *cartesian coordinates* $x_1, \dots, x_n \in \mathbb{R}$, where

$$X = O + \sum_{i=1}^{n} e_i x_i = O + (e_1, \dots, e_n)(x_1, \dots, x_n)^t.$$

Vector products. In general, a *product* is a bilinear map $\odot : V \times V \to W$ into some target vector space W. We use mainly two different products:

Euclidean inner product. $\langle .,. \rangle : V \times V \to \mathbb{R}$ is additionally *symmetric* and *positive definite*, that is,

$$\forall v, w \in V : \langle v, w \rangle = \langle w, v \rangle \text{ and } \forall v \in V \setminus \{0\} : \langle v, v \rangle > 0;$$

two vectors $v, w \in V$ are *orthogonal* or *perpendicular* if $\langle v, w \rangle = 0$ and, more generally, the *angle* $\alpha \in [0, \pi]$ of two vectors $v, w \in V$ can be defined by the equation

$$\langle v, w \rangle = |v|\,|w| \cos\alpha, \text{ where } |v| := \sqrt{\langle v, v \rangle}.$$

Cross product. $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a skew-symmetric vector product,

$$\forall v, w \in \mathbb{R}^3 : v \times w + w \times v = 0,$$

and a pair (v, w) of vectors is linear independent iff $v \times w \neq 0$. The cross product can serve to measure area: with the angle α of $v, w \in \mathbb{R}^3$

$$|v \times w|^2 = |v|^2|w|^2 - \langle v, w\rangle^2 = |v|^2|w|^2 \sin^2 \alpha;$$

another way to phrase this identity is the defining identity:

$$\forall u, v, w \in \mathbb{R}^3 : \langle u, v \times w\rangle = \det(u, v, w).$$

In particular, $v, w \perp v \times w$ and $\langle u \times v, w\rangle = \langle v, u \times w\rangle$. and a double cross product can be reduced by $u \times (v \times w) = v \langle w, u\rangle - w \langle v, u\rangle$.

The cross product is particular to $\mathbb{R}^3$; the *wedge product* is a product with similar properties and that generalizes to higher dimensions in a straightforward way:

$$\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \to \Lambda^2\mathbb{R}^3 \cong \mathfrak{o}(3), \;\; (v \wedge w)\, x := (v \times w) \times x.$$

Euclidean motion. This is an orientation and distance preserving transformation of a Euclidean space,

$$\mathcal{E} \ni X \mapsto \tilde{O} + A(X - O) \in \mathcal{E}, \;\text{ where }\; A \in \mathrm{SO}(V)$$

denotes the *linear part* of the (affine) transformation and $\tilde{O} \in \mathcal{E}$ is the image of the origin $O \in \mathcal{E}$. If $\dim \mathcal{E} = \dim V = 3$, wlog $V = \mathbb{R}^3$, then any *positively oriented* orthonormal basis (e_1, e_2, e_3) is mapped to a basis of the same type by $A \in \mathrm{SO}(3)$ as

$$\langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle \;\text{ and }\; \det(Ae_1, Ae_2, Ae_3) = \det(e_1, e_2, e_3).$$

Note that any $A \in \mathrm{SO}(3)$ is compatible with the cross product,

$$(Av) \times (Aw) = A(v \times w).$$

Linear transformations. More generally, we use the general linear and (special) orthogonal groups on a vector space V, with inner product $\langle ., .\rangle$ and volume distortion det:

$$\begin{aligned} \mathrm{Gl}(V) &:= \{A \in \mathrm{End}(V) \,|\, A^{-1} \text{ exists}\}; \\ \mathrm{O}(V) &:= \{A \in \mathrm{Gl}(V) \,|\, \forall v, w \in V : \langle Av, Aw\rangle = \langle v, w\rangle\}; \\ \mathrm{SO}(V) &:= \{A \in \mathrm{O}(V) \,|\, \det A = +1\}. \end{aligned}$$

If $E = (e_1, \ldots, e_n)$ is an orthonormal basis of V, then $A \in \mathsf{End}(V)$ can be identified with a matrix $X \in \mathbb{R}^{n\times n}$, where

$$(Ae_1, \ldots, Ae_n) = (e_1, \ldots, e_n)\, X;$$

the corresponding *matrix groups* are then

$$\begin{array}{rcl} \mathsf{Gl}(n) & = & \{X \in \mathbb{R}^{n\times n} \,|\, X^{-1} \text{ exists}\}; \\ \mathsf{O}(n) & = & \{X \in \mathsf{Gl}(n) \,|\, X^t X = E_n\}; \\ \mathsf{SO}(n) & = & \{X \in \mathsf{O}(n) \,|\, \det X = +1\}. \end{array}$$

When $A : I \to G$ is a group-valued curve with $A(0) = \mathrm{id}_V$, then its derivative $A'(0) \in \mathfrak{g}$ takes values in the corresponding *algebra*,

$$\begin{array}{rcl} \mathfrak{gl}(V) & = & \mathsf{End}(V); \\ \mathfrak{o}(V) & = & \{X \in \mathfrak{gl}(V) \,|\, \forall v, w \in V : \langle Xv, w\rangle + \langle v, Xw\rangle = 0\}; \\ \mathfrak{so}(V) & = & \mathfrak{o}(V); \end{array}$$

or, in matrix representations,

$$\mathfrak{gl}(n) = \mathbb{R}^{n\times n} \text{ and } \mathfrak{o}(n) = \{X \in \mathfrak{gl}(n) \,|\, X^t + X = 0\}.$$

A.2 Derivative and differentiation

Obviously, differentiation and derivative are key notions in differential geometry — here we clarify notations and review the most important differentiation rules.

Derivative. A map $X : \mathcal{D} \supset M \to \mathcal{E}$ between Euclidean spaces, over vector spaces U resp V, is *differentiable* at $p \in M$ if it can be approximated by a (continuous) affine map $A : \mathcal{D} \to \mathcal{E}$ to first order,

$$\tfrac{X(q)-A(q)}{|q-p|} = \tfrac{X(q)-X(p)-d_pX(q-p)}{|q-p|} \to 0 \text{ as } q \to p,$$

where $d_pX : U \to V$ denotes the linear part of the approximating affine map $q \mapsto A(q) = A(p) + d_pX(q-p)$, the *derivative of X at $p \in M$*.

Continuity is automatic if $\dim U < \infty$, else it needs to be assumed.

For parametrizations we use $\mathcal{D} = U = \mathbb{R}^n$, with $n = 1$ or $n = 2$ for curves resp surfaces. If $\mathcal{D} = \mathbb{R}^n$ then the derivative d_pX can be identified with the tupel of its partial derivatives, in particular, for $X : \mathbb{R} \supset I \to \mathcal{E}$, $t \mapsto X(t)$,

$$d_t X(x) = X'(t)\, x$$

and, for $X : \mathbb{R}^2 \supset M \to \mathcal{E}$, $(u, v) \mapsto X(u, v)$,

$$d_{(u,v)} X(\tbinom{x}{y}) = X_u(u, v)\, x + X_v(u, v)\, y = (X_u, X_v)|_{(u,v)} \tbinom{x}{y},$$

where subscripts denote partial derivatives, $X_u = \frac{\partial}{\partial u} X$, etc.

If $\dim \mathcal{E} < \infty$ and F is given in terms of cooordinate functions F_i with respect to a (cartesian) reference system $(O; e_1, \dots, e_m)$ then the *Jacobi matrix* of F is obtained, as a matrix representation of its derivative.

Problem 1. Compute the Jacobi matrix of

$$F : \mathbb{R}^2 \to \mathbb{R},\;\; (u, v) \mapsto F(u, v) := \begin{cases} \frac{uv^2\sqrt{u^2+v^2}}{u^2+v^4} & \text{for } (u, v) \neq (0, 0), \\ 0 & \text{for } (u, v) = (0, 0) \end{cases}$$

at $(u, v) = (0, 0)$ and prove that F is not differentiable at $(u, v) = (0, 0)$.

Chain rule. The derivative of a composition of maps is the composition of their derivatives: given differentiable maps $p : \mathcal{C} \to \mathcal{D}$ and $X : \mathcal{D} \to \mathcal{E}$, their derivative is differentiable and

$$d_t(X \circ p) = d_{p(t)} X \circ d_t p;$$

for example, for a composition $t \mapsto C(t) = X(u(t), v(t))$ we obtain

$$C'(t) = X_u(u(t), v(t))\, u'(t) + X_v(u(t), v(t))\, v'(t);$$

often this will be more clearly represented by (not entirely correctly) dropping arguments,

$$C' = X_u u' + X_v v'.$$

Linearity. If $Y, Z : \mathcal{D} \supset M \to V$ are *vector valued* differentiable maps, then any linear combination of Y and Z is differentiable and the activity of taking derivative is linear: for $p \in M$ and $y, z \in \mathbb{R}$,

$$d_p(Yy + Zz) = (d_p Y)\, y + (d_p Z)\, z.$$

Leibniz rule. Any (continuous) product $\odot : V \times V \to W$ on a vector space is differentiable and the chain rule yields the derivative of the product of two differentiable maps $Y, Z : \mathcal{D} \supset M \to V$: for $p \in M$,

$$d_p(Y \odot Z) = (d_p Y) \odot Z(p) + Y(p) \odot (d_p Z).$$

The most important application of the Leibniz rule in (Euclidean) differential geometry is the derivative of the inner product: for $Y, Z : \mathbb{R} \ni I \to V$, $t \mapsto Y(t), Z(t)$,

$$\langle Y, Z\rangle' = \langle Y', Z\rangle + \langle Y, Z'\rangle.$$

The Leibniz rule also holds for products of objects from different vector space, for example, for the scalar multiplication: for $Y : \mathbb{R} \ni I \to V$ and $y : I \to \mathbb{R}$ we obtain

$$(Yy)' = Y'y + Y\,y'.$$

Problem 2. Let $\beta : V^3 \to \mathbb{R}$ be tri-linear; assuming that β is differentiable show that

$$d_{(u,v,w)}\beta(x, y, z) = \beta(x, v, w) + \beta(u, y, w) + \beta(u, v, z).$$

Conclude that (the differentiable function) det : Gl(3) $\to \mathbb{R}$ has derivative

$$d_A \,\mathrm{det} : \mathfrak{gl}(3) \to \mathbb{R}, \;\; X \mapsto d_A \,\mathrm{det}(X) = \mathrm{det}(A)\;\mathrm{tr}(A^{-1}X).$$

A.3 Inverse & implicit mappings

If a smooth function $f : \mathbb{R} \supset I \to \mathbb{R}$ has derivative $f'(t) \neq 0$ then it is (locally) strictly increasing or decreasing around $t \in I$, hence it has a local inverse $(f|_{(t-\delta,t+\delta)})^{-1} : f(t-\delta, t+\delta) \to I$ around $t \in I$. A similar statement holds true for smooth maps between Euclidean spaces.

Inverse Mapping Theorem. Suppose that $X : \mathcal{D} \supset M \to \mathcal{E}$ is continuously differentiable and that $d_pX : U \to V$ is invertible at some $p \in M$. Then there is an open neighbourhood $B \subset M$ of p so that:

(i) $X|_B : B \to \mathcal{E}$ injects (so that $X|_B : B \to X(B)$ is invertible);

(ii) $X(B) \subset \mathcal{E}$ is open (so that $(X|_B)^{-1}$ may be differentiable);

(iii) $X^{-1} : X(B) \to B$ is continuously differentiable with

$$d_qX^{-1} = (d_pX)^{-1} \;\text{ for }\; q = X(p) \in X(B)$$

For short. A smooth map $X : \mathcal{D} \supset M \to \mathcal{E}$ has, locally, a smooth inverse where its derivative is invertible — and the derivative of the inverse is the inverse of the derivative, as obtained from the chain rule.

Remark. If $d_pX : U \to V$ is invertible then, necessarily, $\dim U = \dim V$. If, furthermore, $U = V$ and $\dim V < \infty$ then $d_pX \in \mathrm{End}(V)$ is invertible if and only if $\det d_pX \neq 0$.

Problem 3. Let $X : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$, $(u,v) \mapsto X(u,v) := (u^2 - v^2, 2uv)$. Show that X does not inject, but has a local inverse around every (u,v).

Implicit Mapping Theorem. Suppose that $F : \mathcal{E} \supset M \to \mathcal{F}$ is continuously differentiable, $V = U \oplus U'$ and that, for $o \in M$, the restriction

$$d_oF|_{U'} : U' \to W \;\text{ of }\; d_oF : V \to W$$

is invertible; then there are open neighbourhoods $B \subset U$ and $B' \subset U'$ of $0 \in V$ and a (unique) continuously differentiable map $g : B \to B'$ so that $g(0) = 0$ and, for $(u,v) \in B \times B'$,

$$F(o + u + v) = F(o) \Leftrightarrow v = g(u).$$

For short. The equation $F(u,v) = F(u_o, v_o)$ can be locally solved for v if and only if the equation $d_{(u_o,v_o)}F(\binom{x}{y}) = 0$ can be solved for y.

Problem 4. Use the Implicit mapping theorem to show that, for any point

$$X \in E = \{O + e_1u + e_2v \in \mathcal{E}^2 \,|\, (\tfrac{u}{a})^2 + (\tfrac{v}{b})^2 = 1\}$$

of the ellipse $E \subset \mathcal{E}^2$, the ellipse can locally be written as a graph over one of the coordinate axes of the cartesian reference system $(O; e_1, e_2)$.

Rem. The Implicit and Inverse mapping theorems are equivalent.

Problem 5. Prove the Implicit from the Inverse mapping theorem.

Useful notions. A smooth map $X : \mathcal{D} \supset M \to \mathcal{E}$ is called

- an *immersion* if d_pX injects for all $p \in M$;
- a *submersion* if d_pX surjects for all $p \in M$;
- a *diffeomorphism* if it has a smooth inverse.

A.4 ODEs: the Picard-Lindelöf theorem

Recall that an *ordinary differential equation* (of order n) is an equation

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)) \qquad (\dagger)$$

for an unknown function $x = x(t)$ which depends on a (real) variable t; for the time being, we think of x as a real valued function.

Any such ODE can be re-written as a (system of) ODE(s) of order $n = 1$ by introducing the derivatives as new functions: with $x_k := x^{(k-1)}$ the equation ($\dagger$) is equivalent to the system

$$\begin{array}{rcl} x_1'(t) & = & x_2(t) \\ & \vdots & \\ x_{n-1}'(t) & = & x_n(t) \\ x_n'(t) & = & f(t, x_1(t), \dots, x_n(t)). \end{array}$$

Hence we never need to think about higher order ODEs.

<u>*Picard-Lindelöf theorem*</u>. Let $\mathbb{R} \times \mathbb{R}^n \supset I \times U \ni (t, x) \mapsto f(t, x) \in \mathbb{R}^n$ be continuous and Lipschitz continuous in x and let $(t_o, x_o) \in I \times U$; then there is $\varepsilon > 0$ so that the *initial value problem*

$$x'(t) = f(t, x(t)), \quad x(t_o) = x_o \qquad (\star)$$

has a unique solution on $(t_o - \varepsilon, t_o + \varepsilon)$.

A proof and detailed explanations can be found in any analysis text book.

<u>*Special cases*</u>. Two special cases of the Picard-Lindelöf theorem are of particular interest in differential geometry:

(1) if $x \mapsto f(t, x) = f(x)$ is differentiable then ($\star$) has a unique local solution (prove it for $n = 1$!);

(2) if $x \mapsto f(t, x) = A(t)\,x$ is linear then ($\star$) has a unique global(!) solution $x : I \to \mathbb{R}^n$.

<u>*Problem 6*</u>. Let $t \mapsto \kappa(t)$ be some function. Find the solution of

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ with } \begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \qquad (*)$$

[Hint: write (x, y) in polar coordinates.]

A.5 PDEs: the Poincaré & Maurer-Cartan lemmas

Partial differential equations come in many different flavours. To us the following two (systems of) partial differential equations are of particular interest — generalizations to higher dimensional domains are straightforward.

Poincaré lemma. Given $\varphi = \varphi(u, v)$ and $\psi = \psi(u, v)$ the partial differential equation

$$dX = \varphi\, du + \psi\, dv \quad \Leftrightarrow \quad \begin{cases} X_u = \varphi \\ X_v = \psi \end{cases}$$

has a local (on simply connected domains) solution x iff $\varphi_v = \psi_u$. Moreover, the solution is unique up to an additive constant.

A proof of the Poincaré lemma can be found in any good analysis text book. The following theorem is less commonly found:

Maurer-Cartan lemma. Given $\Phi = \Phi(u, v), \Psi = \Psi(u, v) \in \mathfrak{gl}(n)$ the partial differential equation

$$dF = F \cdot (\Phi\, du + \Psi\, dv) \quad \Leftrightarrow \quad \begin{cases} F_u = F \cdot \Phi \\ F_v = F \cdot \Psi \end{cases} \tag{$\star$}$$

can locally (on small open sets) be solved to get $F = F(u, v) \in \mathrm{Gl}(n)$ iff

$$\Phi_v - \Psi_u = [\Phi, \Psi] := \Phi\Psi - \Psi\Phi. \tag{$\star\star$}$$

The solution is unique up to left multiplication by a constant matrix.

Proof. First prove that the *Maurer-Cartan equation* ($\star\star$) is necessary: if F is a solution of ($\star$) then

$$\begin{aligned} 0 &= (F_u)_v - (F_v)_u \\ &= F_v\Phi + F\Phi_v - F_u\Psi - F\Psi_u \\ &= F\,(\Psi\Phi + \Phi_v - \Phi\Psi - \Psi_u). \end{aligned}$$

To show that ($\star\star$) is also a sufficient condition suppose that Φ and Ψ are defined on $(-\varepsilon, \varepsilon)^2$ and satisfy ($\star\star$). We first use the Picard-Lindelöf theorem twice to obtain F:

(1) fix $v = 0$ and consider the initial value problem

$$F_u(u, 0) = F(u, 0)\Phi(u, 0), \;\; F(0, 0) = \mathrm{id}_{\mathbb{R}^n},$$

which is a linear system of ordinary differential equations, hence has a unique solution $u \mapsto F(u,0)$ by the Picard-Lindelöf theorem;

(2) now fix u and consider the initial value problem

$$F_v(u,v) = F(u,v)\Psi(u,v), \ \ F(u,0) \text{ as obtained in (1)},$$

which is again has a unique solution $v \mapsto F(u,v)$ by the Picard-Lindelöf theorem.

Now we got $F(u,v)$ at any $(u,v) \in (-\varepsilon,\varepsilon)^2$. Taking differentiability of F for granted, we now verify that F satisfies ($\star$); by construction (2), $F_v = F\Psi$ so that only $F_u = F\Phi$ needs to be verified. Thus compute

$$\begin{array}{rcl} (F_u - F\Phi)_v &=& F_{vu} - F_v\Phi - F\Phi_v \\ &=& (F\Psi)_u - F\Psi\Phi - F\Phi_v \\ &=& F_u\Psi + F(\Psi_u - \Phi_v - \Psi\Phi) \\ &=& (F_u - F\Phi)\Psi \end{array}$$

by ($\star\star$); which, as a linear system of ODEs (u is fixed), has the unique solution $F_u - F\Phi \equiv 0$ since $(F_u - F\Phi)(u,0) = 0$ by construction in (1).

Next we show that $F(u,v) \in \mathrm{Gl}(n)$ for all $(u,v) \in (-\varepsilon,\varepsilon)^2$. Suppose $F(u,v)$ was not invertible at some point (u,v); then $F(u,v)$ would not surject, hence there would exist $x^t \in (\mathbb{R}^n)^* \setminus \{0\}$ with $x^tF(u,v) = 0$. On the other hand, the function x^tF satisfies

$$(x^tF)_u = (x^tF)\Phi \ \text{ and } \ (x^tF)_v = (x^tF)\Psi,$$

which is a linear system of partial differential equations, thus has a unique solution by a similar argument as above. As $x^tF \equiv 0$ is a solution with the given initial value $x^tF(u,v) = 0$ we infer that $x^t = x^tF(0,0) = 0$, contradicting the initial assumption $x^t \neq 0$.

Finally we examine uniqueness: suppose that $\tilde{F}$ is another solution of ($\star$). Using

$$0 = (\mathrm{id}_{\mathbb{R}^n})_u = (FF^{-1})_u = F_uF^{-1} + F(F^{-1})_u,$$

hence

$$(F^{-1})_u = -F^{-1}F_uF^{-1},$$

we compute

$$(\tilde{F}F^{-1})_u = (\tilde{F}_u)F^{-1} - \tilde{F}(F^{-1}F_uF^{-1}) = \tilde{F}(\Phi - \Phi)F^{-1} = 0,$$

and similarly for $(\tilde{F}F^{-1})_v$, showing that $\tilde{F} = AF$ with constant A. ∎

Problem 7. Let $(u, v) \mapsto \Phi(u, v), \Psi(u, v) \in \mathfrak{gl}(2)$ be trace free. Prove that a solution $(u, v) \mapsto F(u, v) \in \mathsf{Gl}(2)$ of $F_u = F\Phi$ and $F_v = F\Psi$ has constant determinant. [Hint: verify that $(\det F)_u = \det F \ \mathrm{tr}(F^{-1}F_u)$.]

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