How packing triangles helps to study symplectic isoperimetry

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University of Luxembourg

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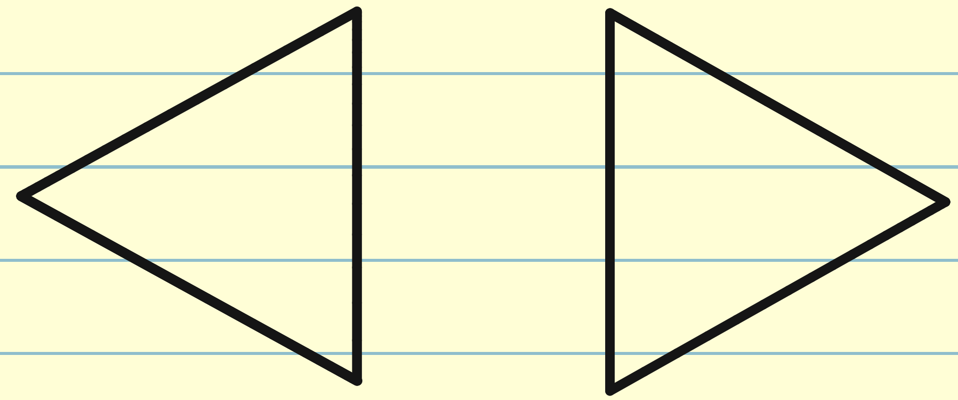
Prototype problem

- you are given a few triangles in $\mathbb{R}^2$

e.g. 

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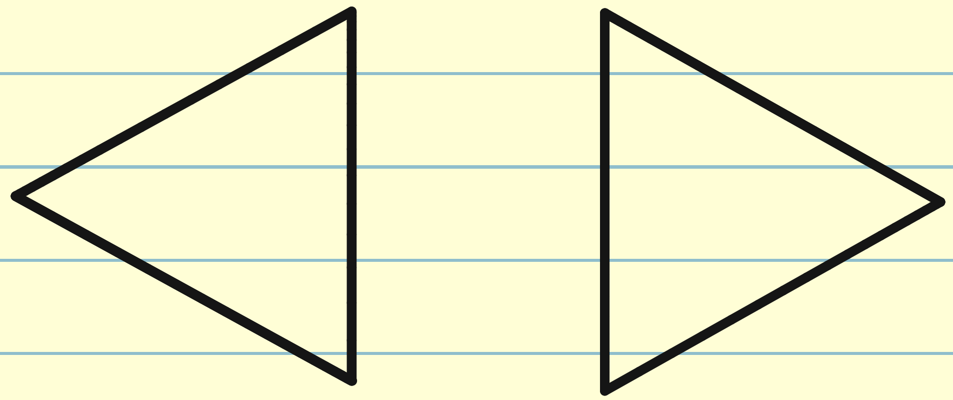
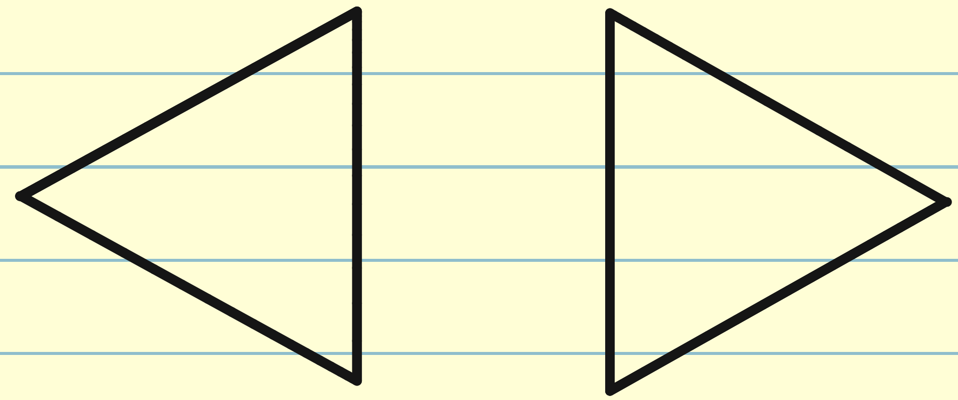
e.g. 

- you can translate them, but not rotate

Prototype problem

- you are given a few triangles in $\mathbb{R}^2$

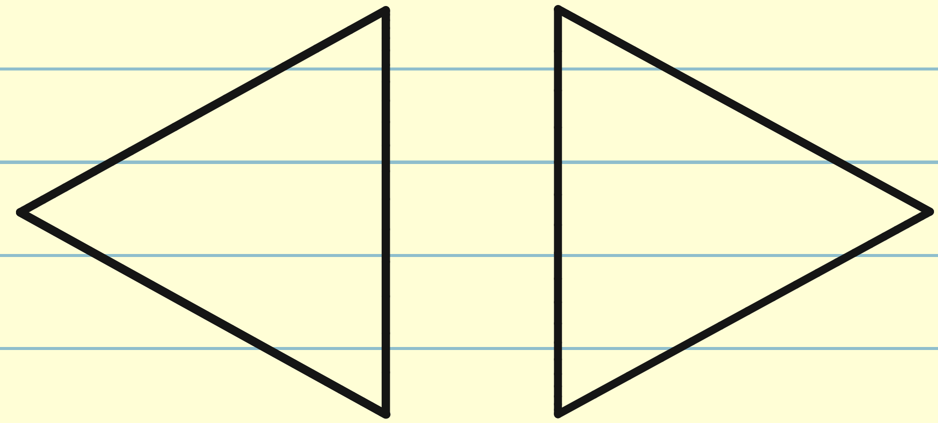
e.g. 

- you can translate them, but not rotate
- you want to place them (possibly with overlaps) inside a convex shape of minimal area

Example #1, silly

Problem: translate the triangles to minimize the area of their convex hull

triangles:



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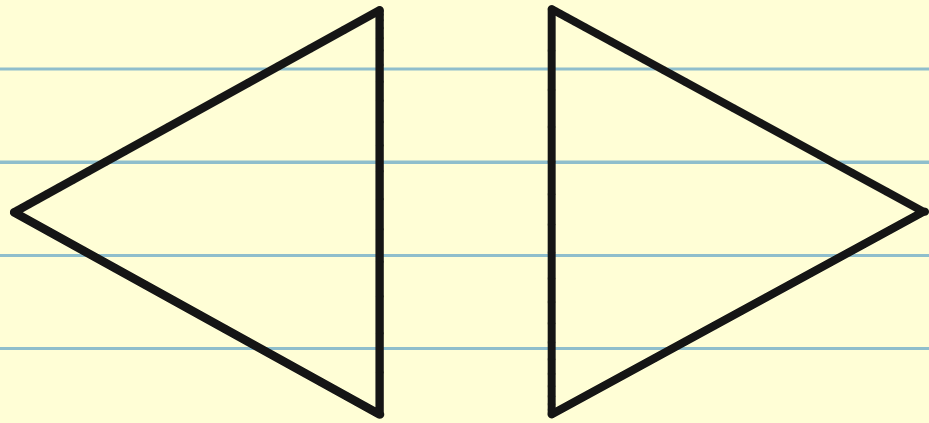

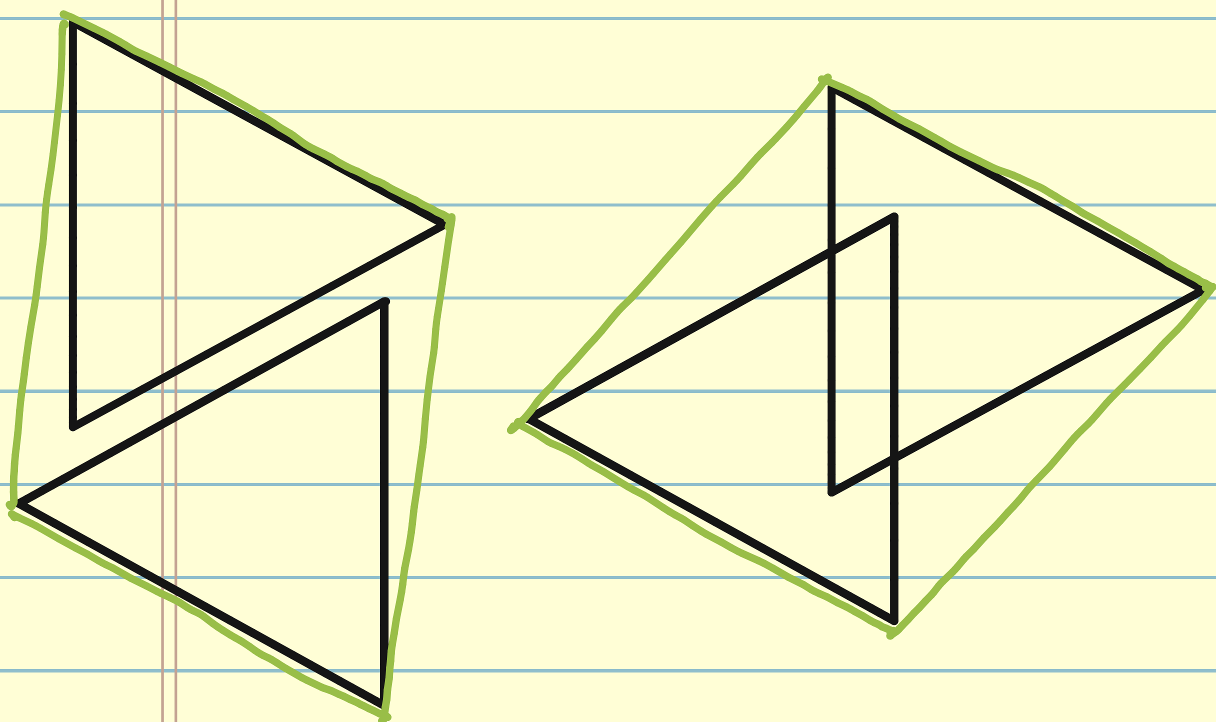

not minimal

Example #1, silly

Problem: translate the triangles to minimize the area of their convex hull

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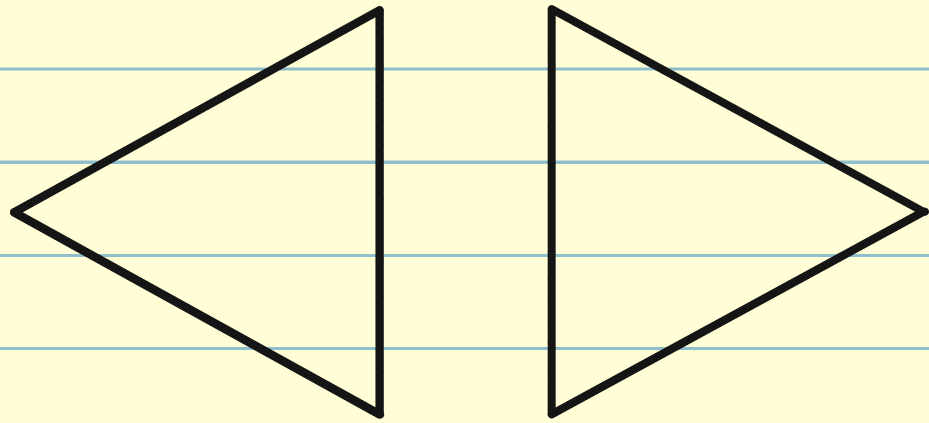
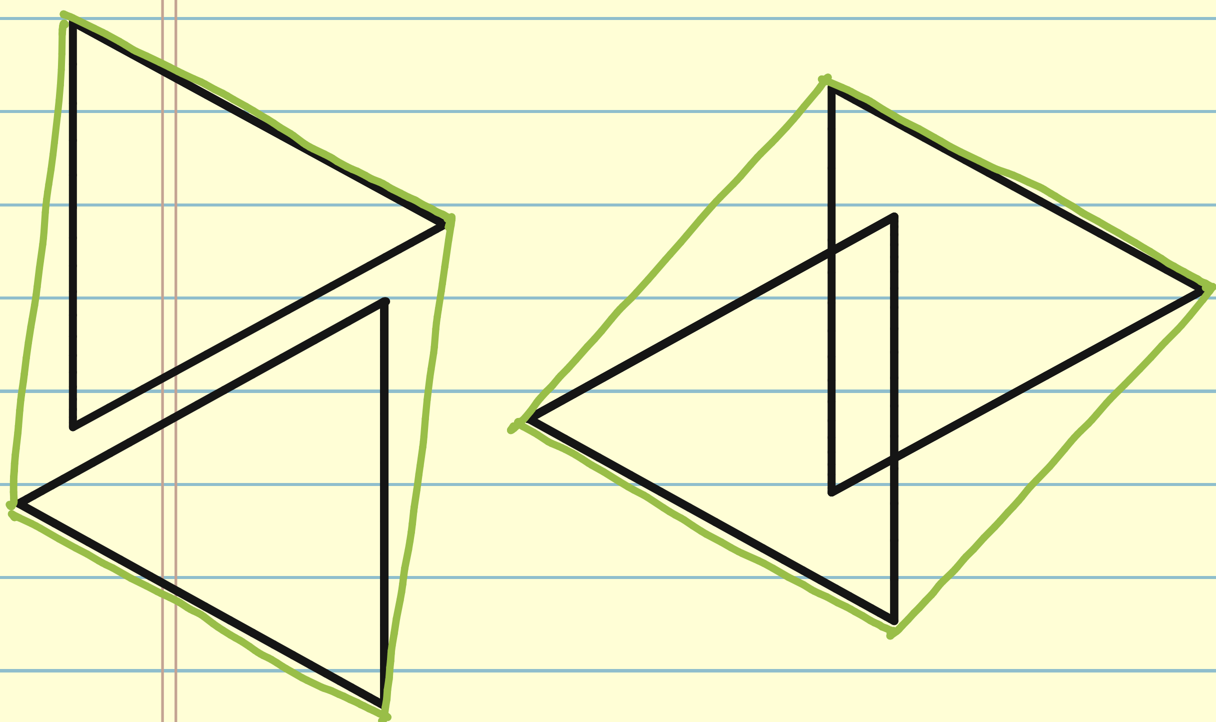
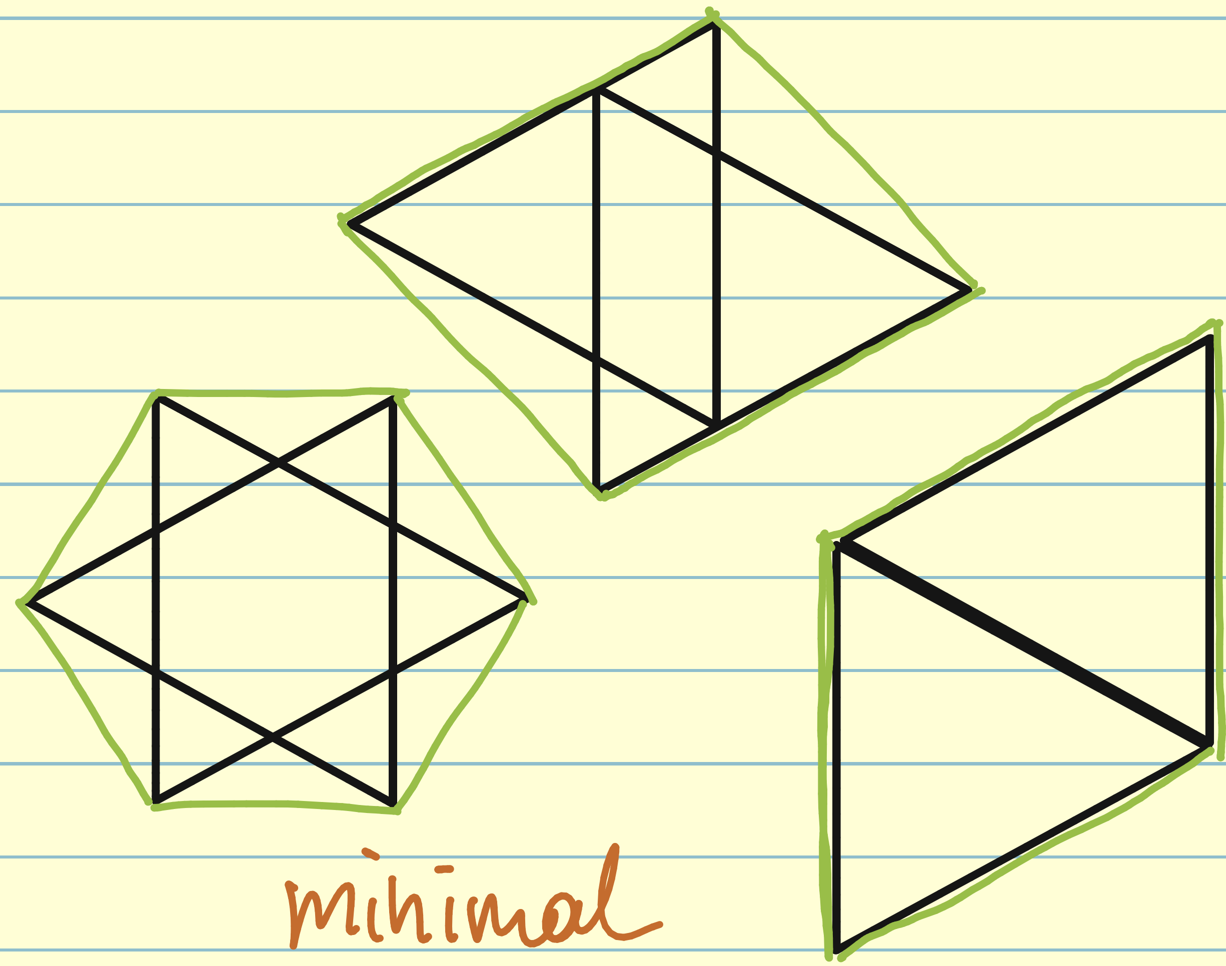




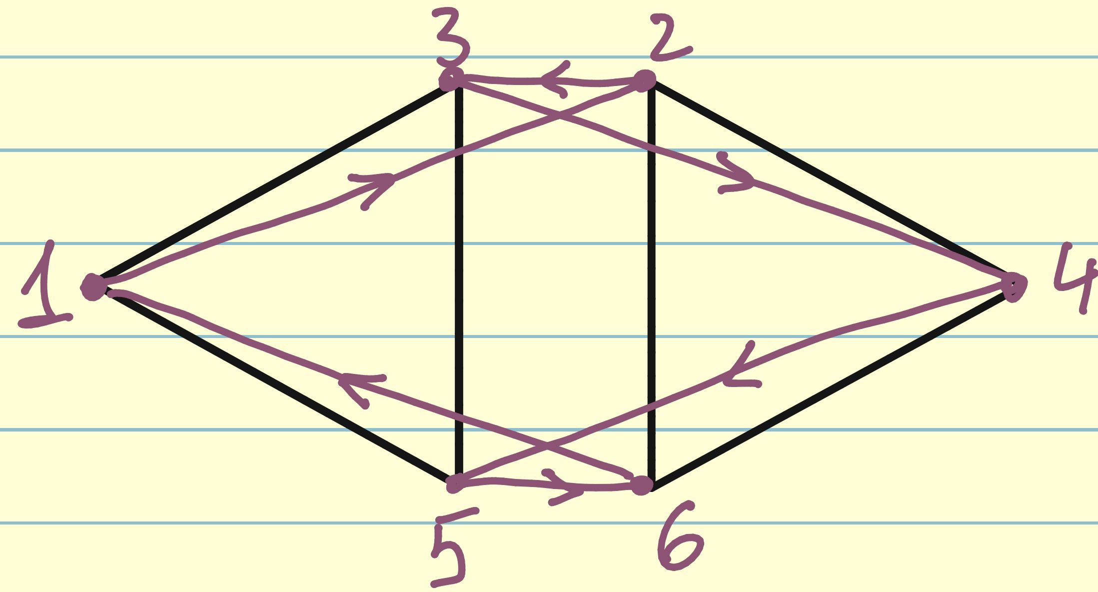
not minimal



minimal

An unnecessarily long argument for the silly example

Step 1 Consider this closed polygonal curve:



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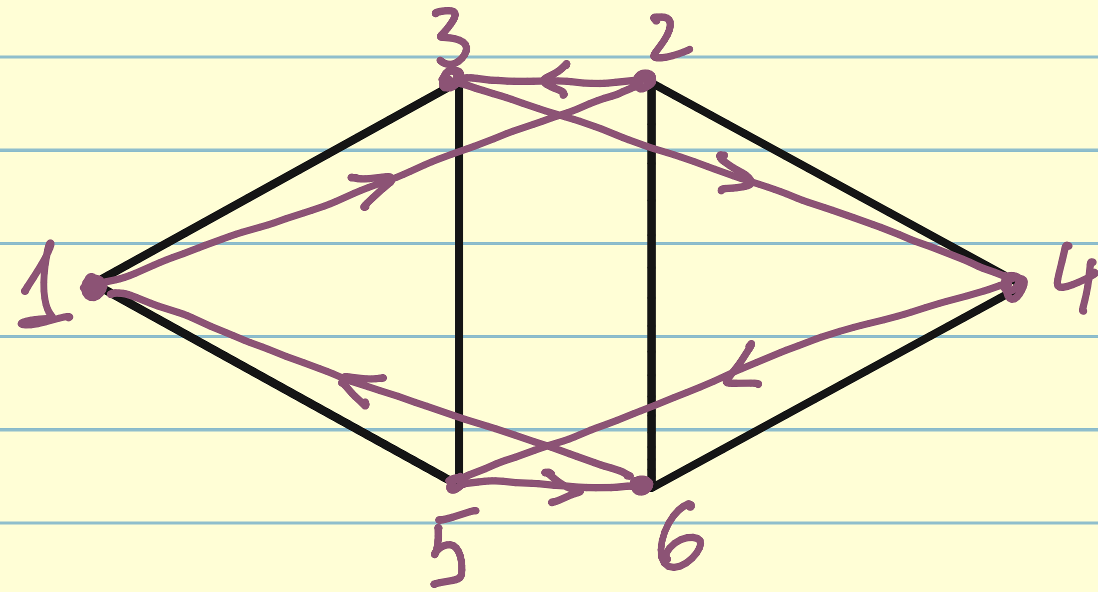
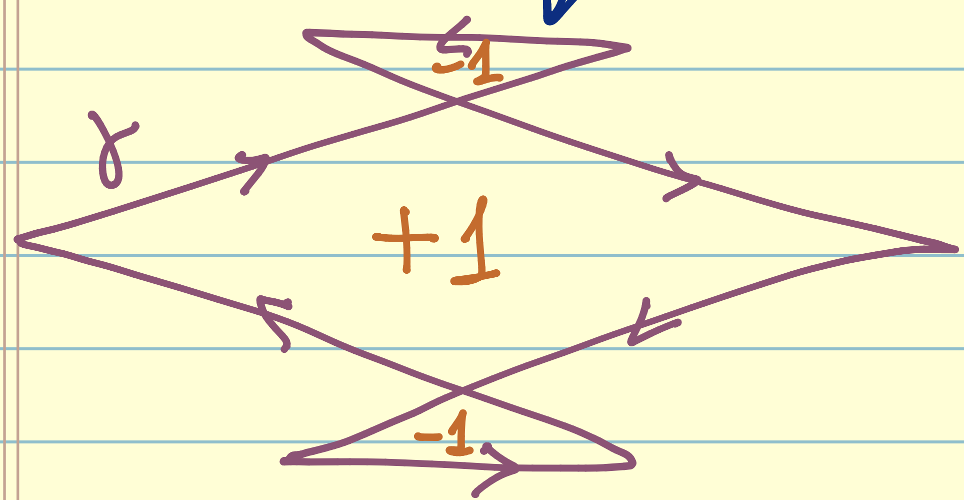


Its enclosed area is constant (as a function of the positions of two triangles)



$$\int_\gamma \frac{x\,dy - y\,dx}{2}$$

An unnecessarily long argument for the silly example

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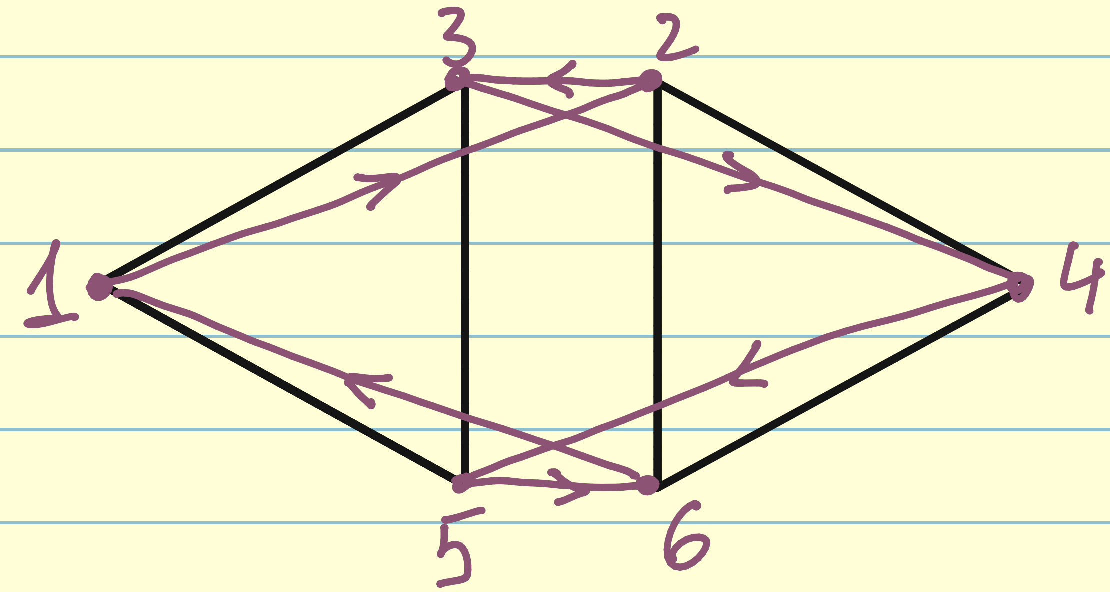

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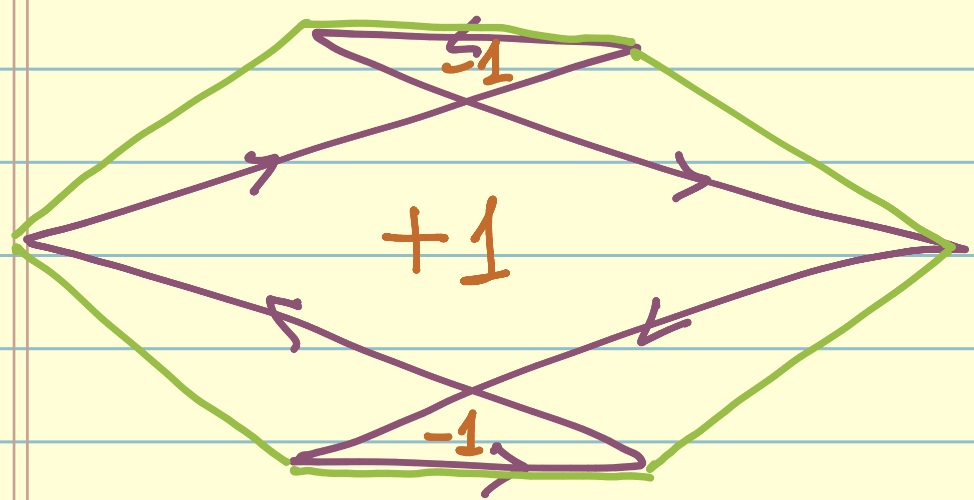


Step 2 The enclosed area $\leq$

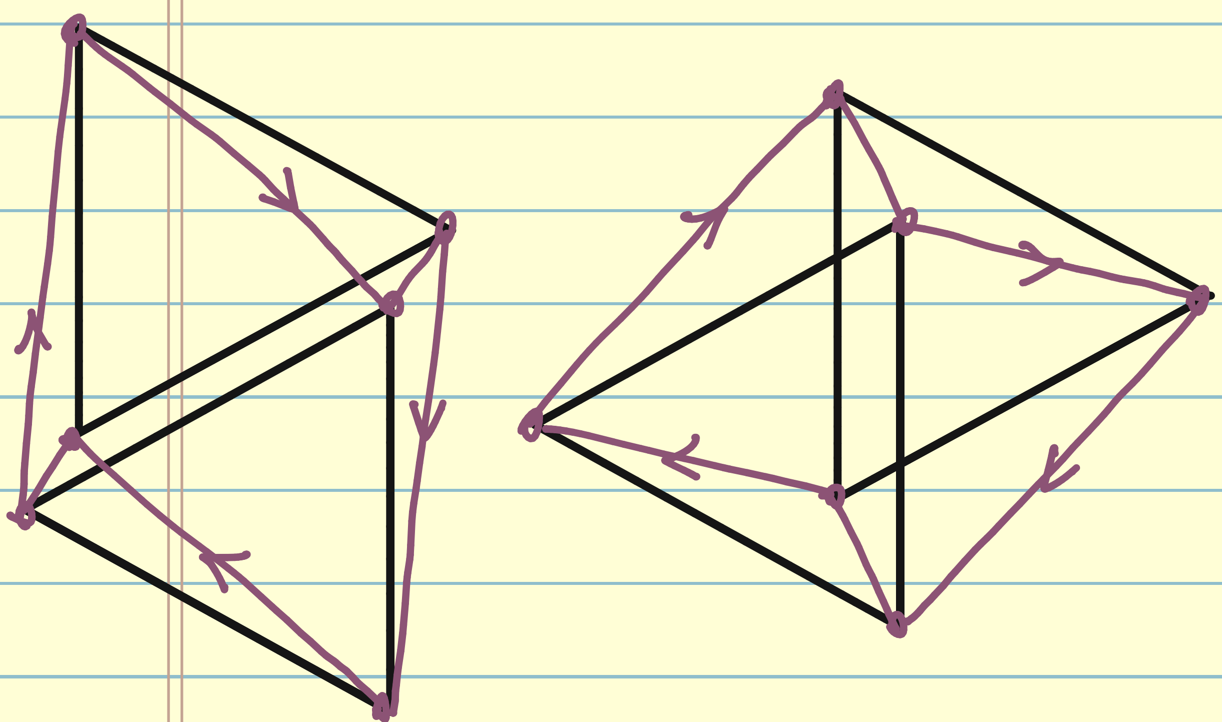
$\leq$ area of convex hull

(because winding numbers are ≤ 1)

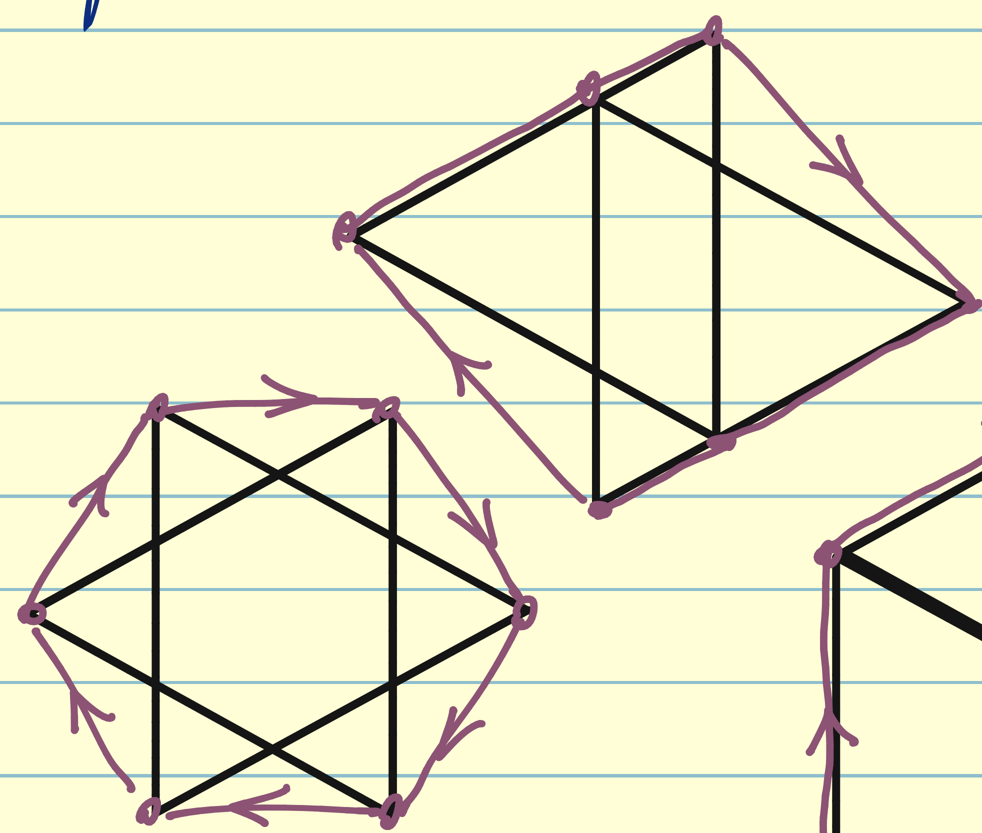
Example #1, silly

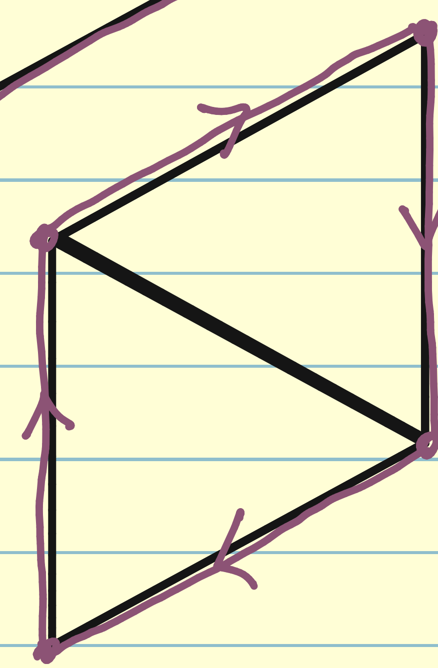
Problem: translate the triangles to minimize the area of their convex hull

Conclusion: area of convex hull is minimal whenever that polygonal curve is in convex position



not minimal

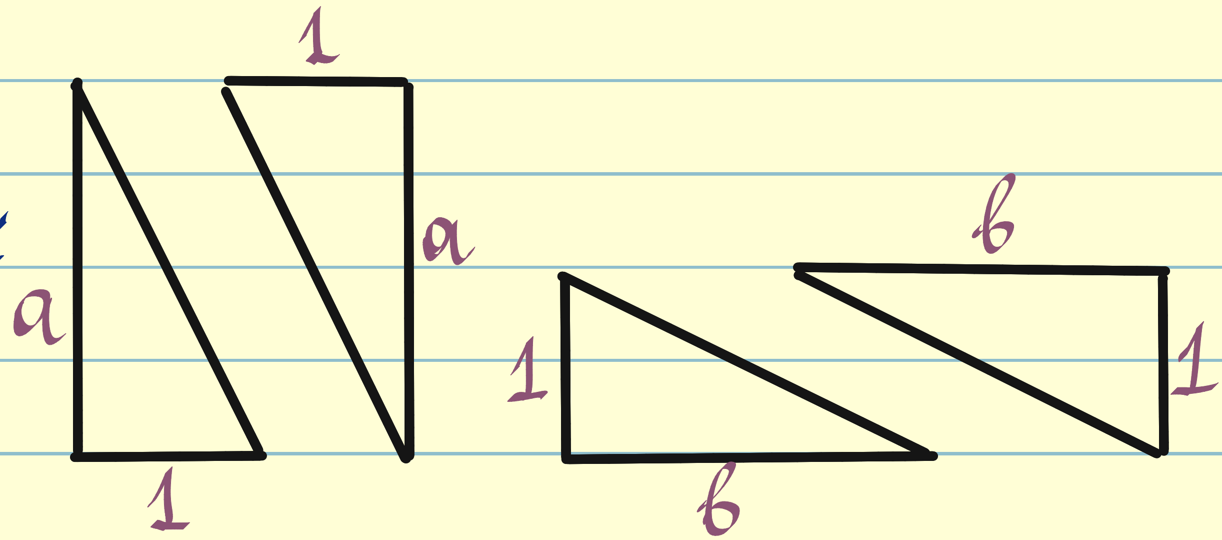




minimal

Example #2, not so silly

triangles:



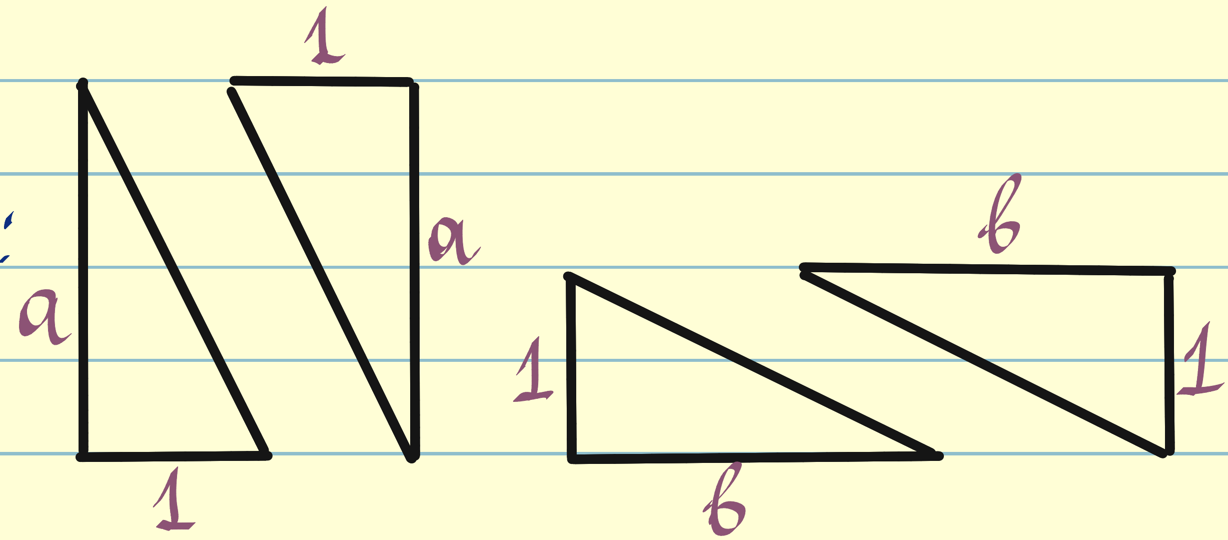
Problem: translate the triangles to minimize the area of their convex hull

$a > 1$, $b > 1$

Example #2, not so silly

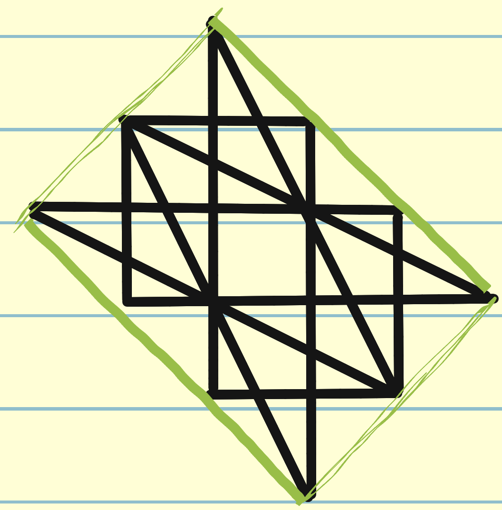
Problem: translate the triangles to minimize the area of their convex hull

triangles:



$a > 1$, $b > 1$

Answer (B.–Mitrofanov–Polyanskii):

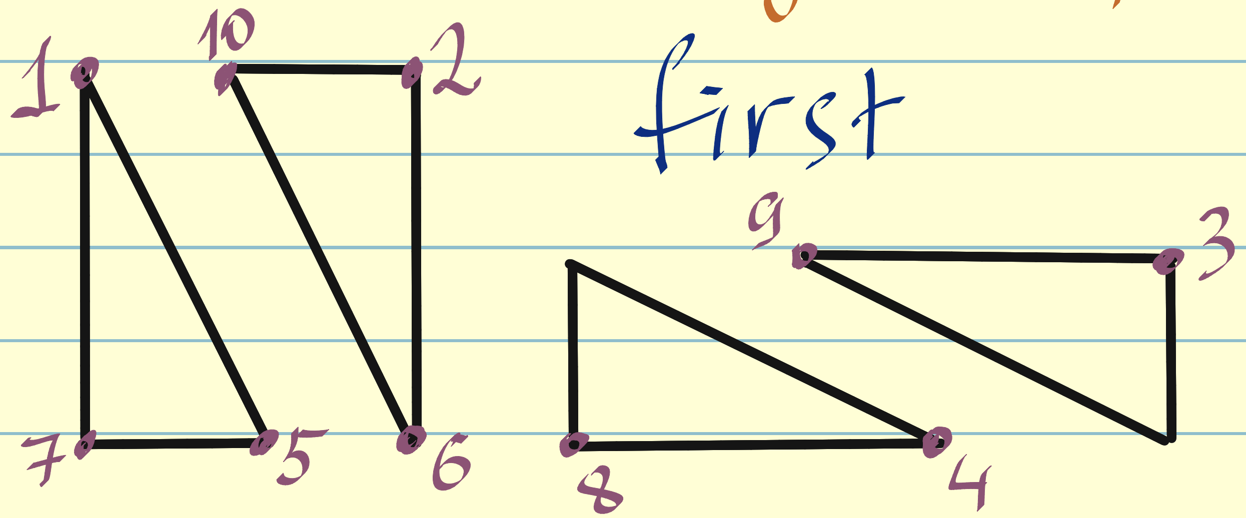


is minimal

& unique unless $(a-1)(b-1)=1$

An argument for the not so silly example

Step 1 Consider closed polygonal curves:



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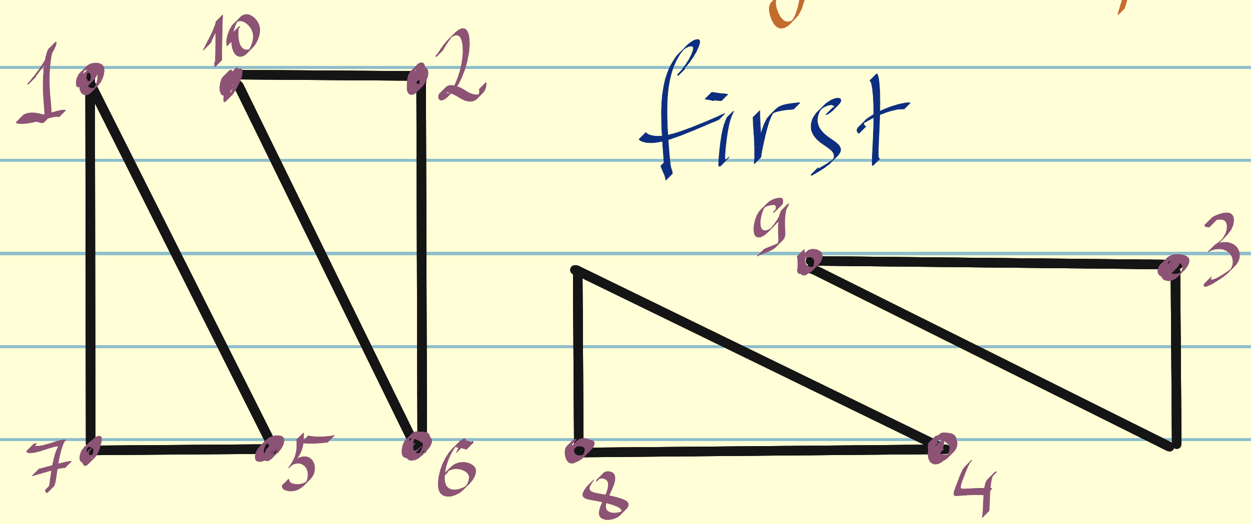

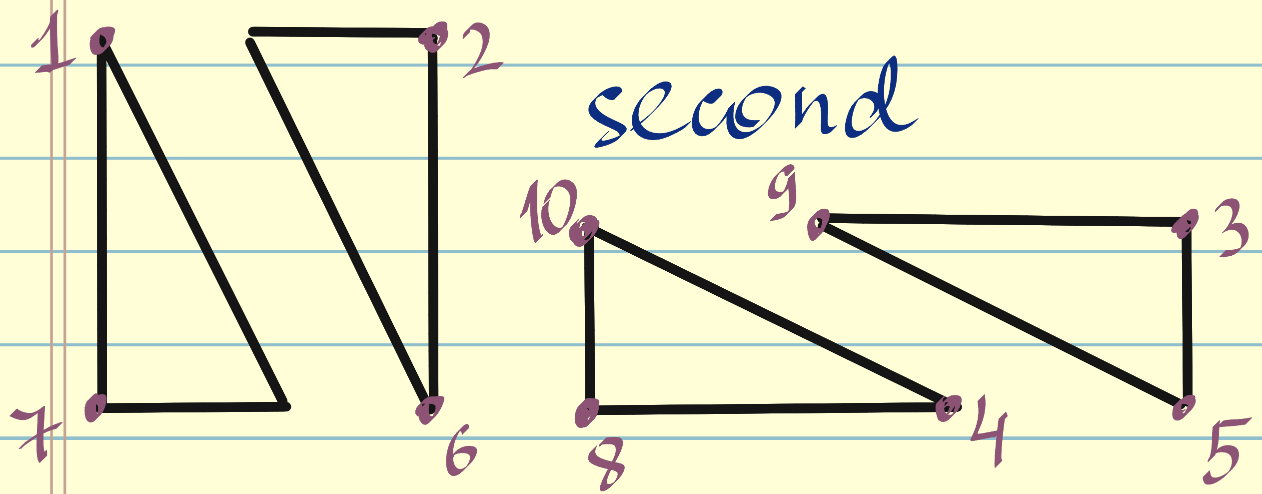

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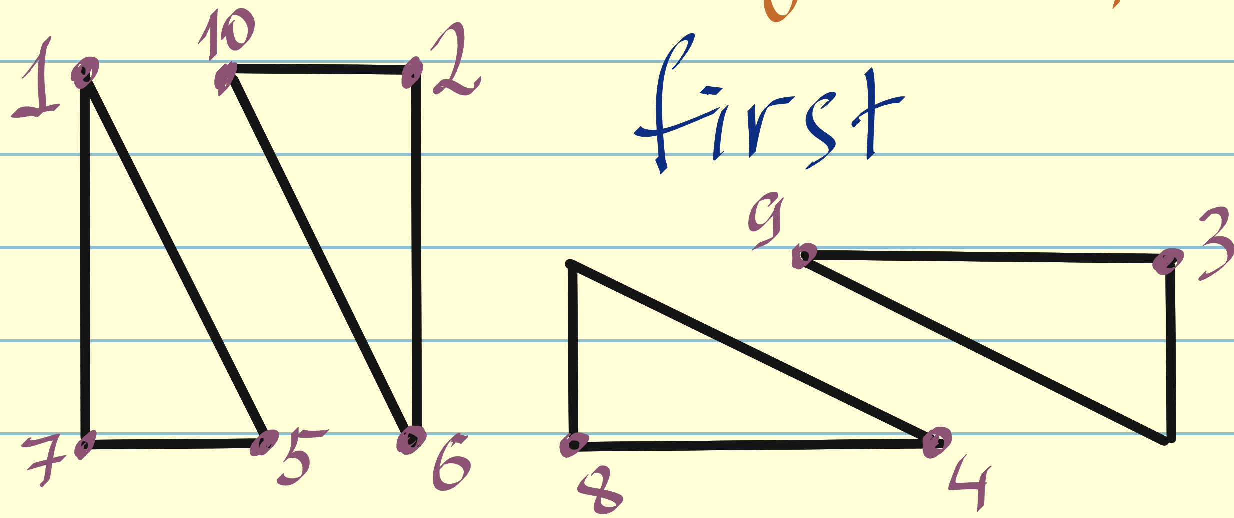
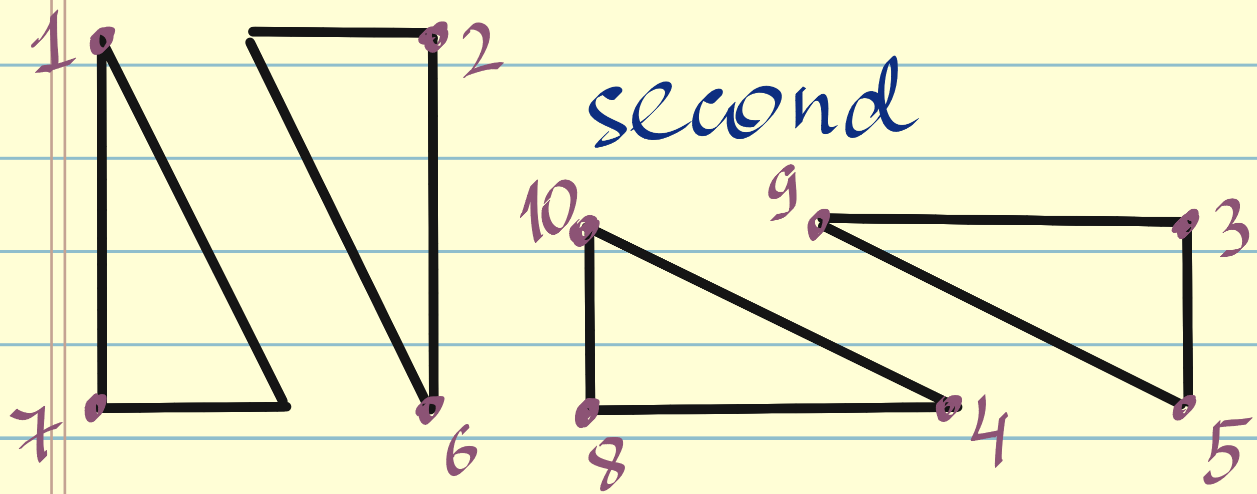
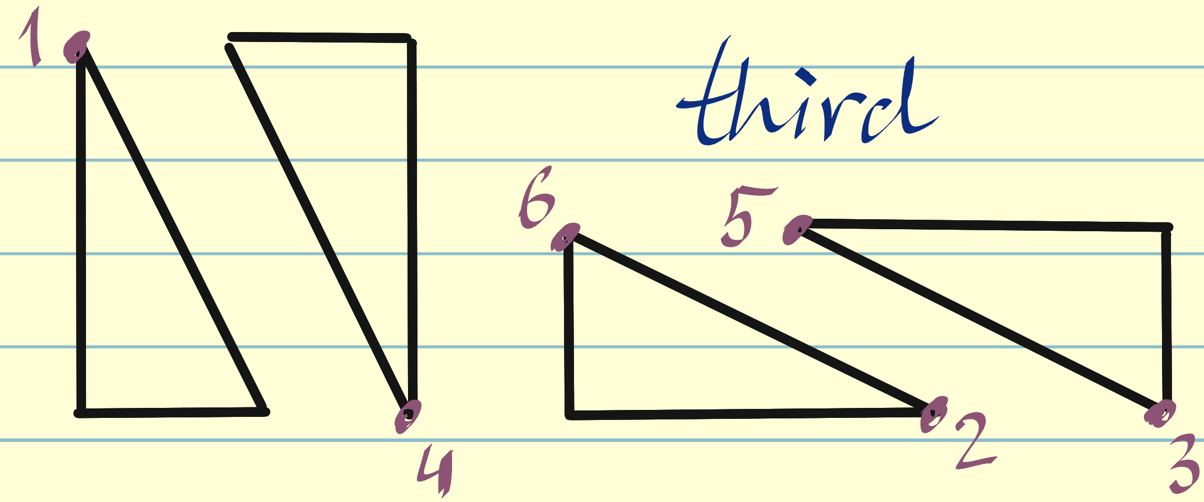


first



second

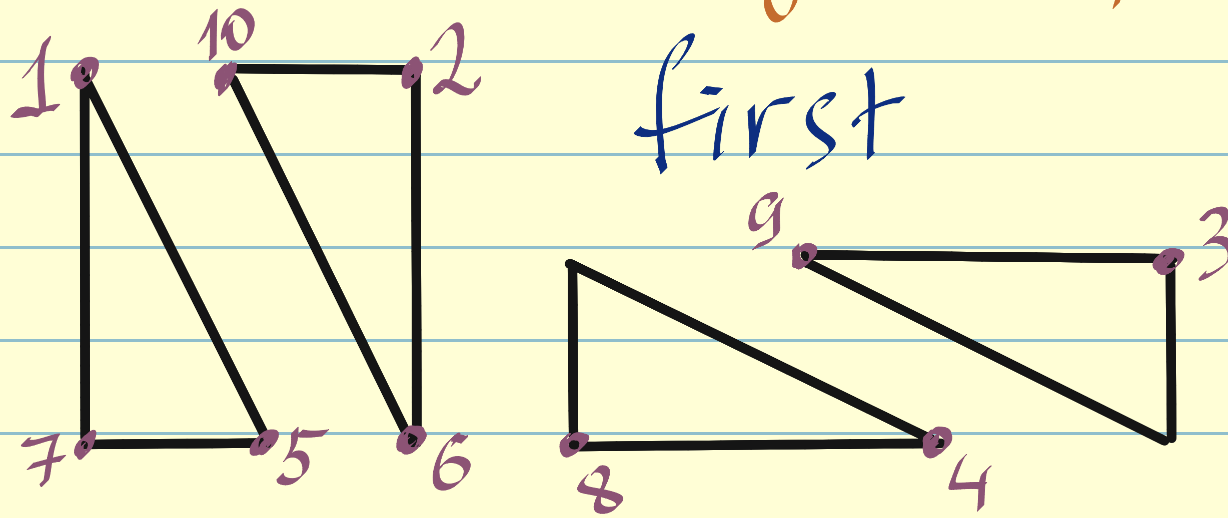


third

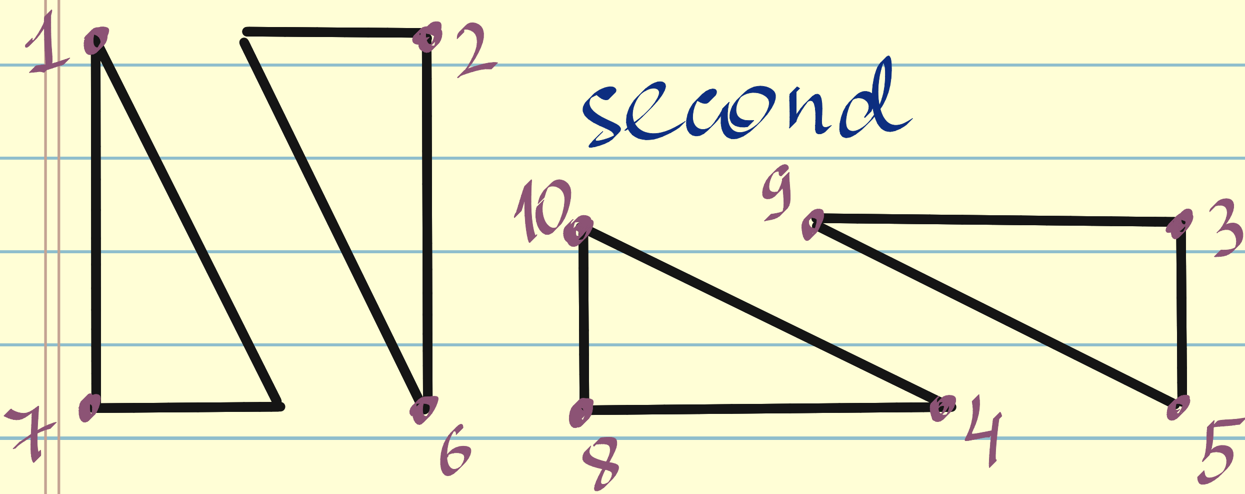
An argument for the not so silly example

Step 1 Consider closed polygonal curves:

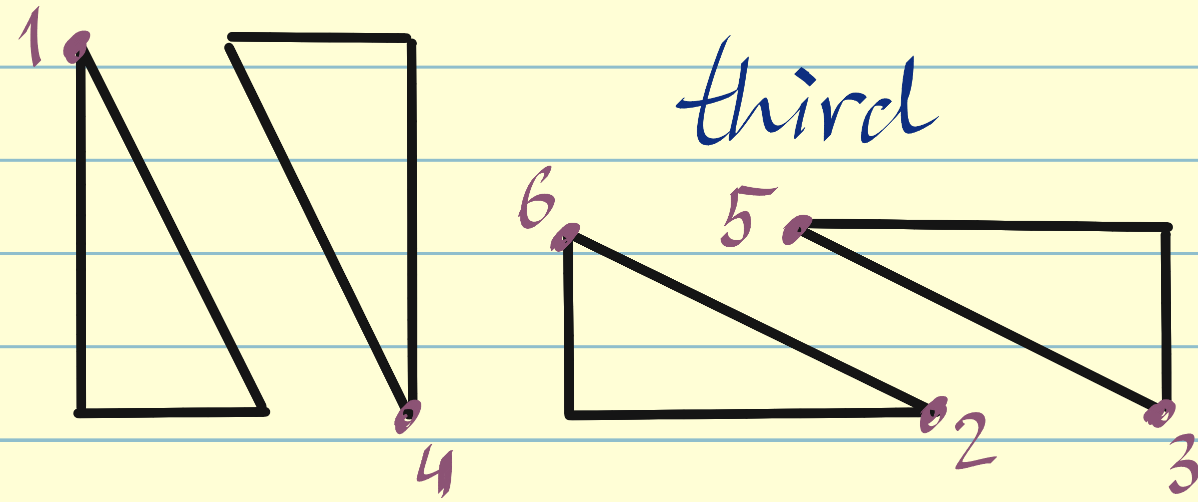
first

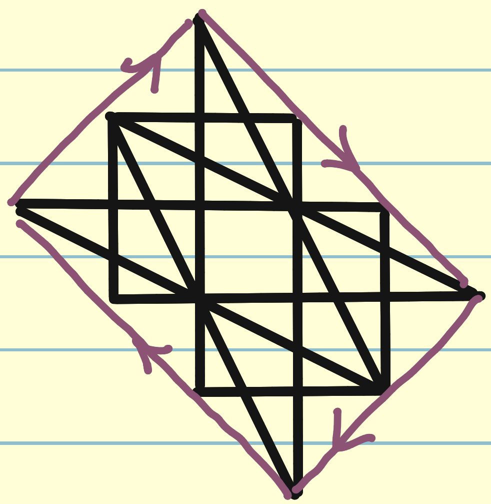


second



third

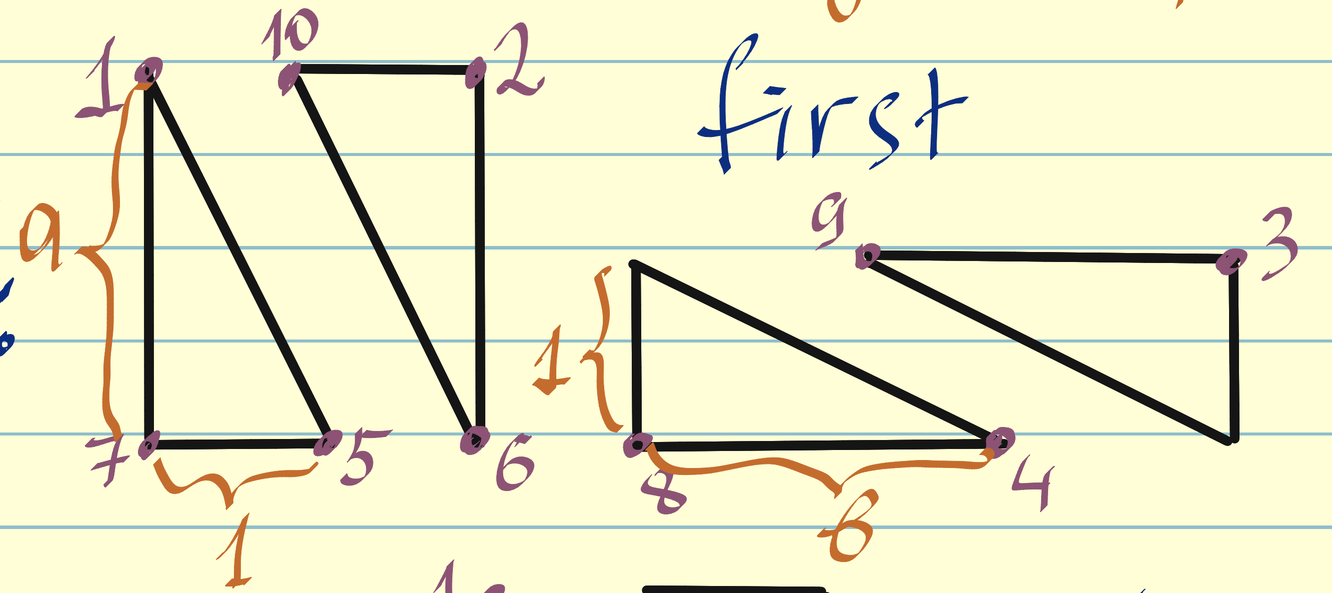


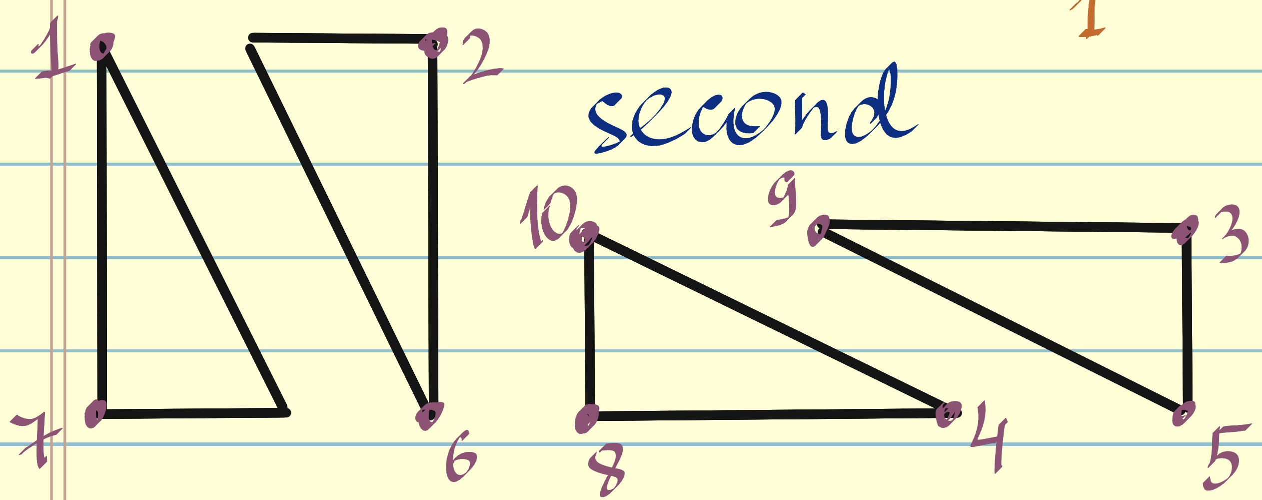


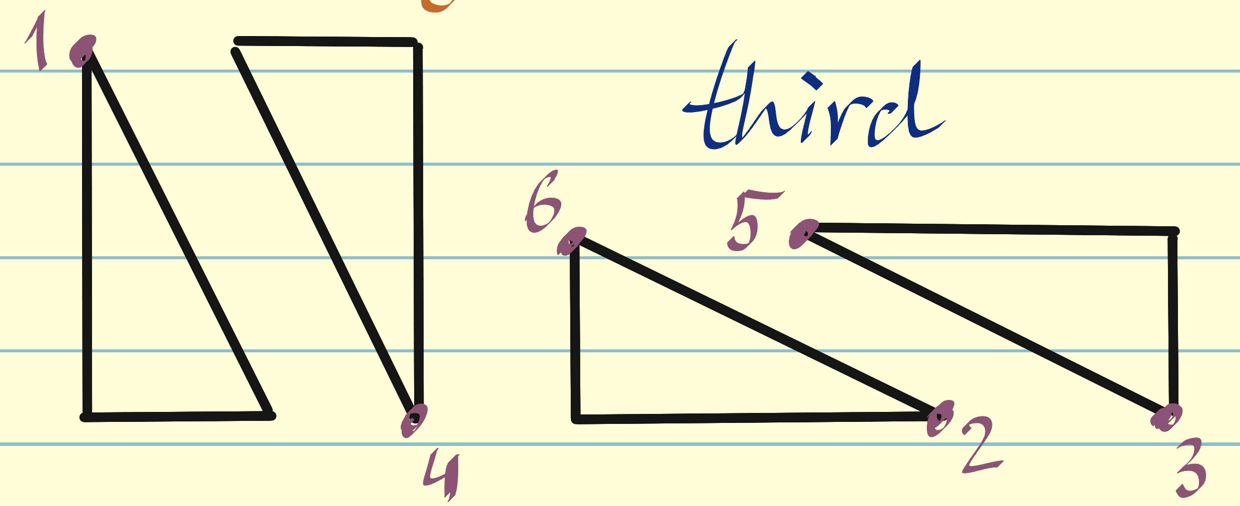
they all follow the boundary of the answer

An argument for the not so silly example

Step 1 Consider closed polygonal curves:







$$\frac{b}{b-1} \cdot \begin{matrix}\text{first}\\ \text{enclosed}\\ \text{area}\end{matrix} + \frac{1}{a-1} \cdot \begin{matrix}\text{second}\\ \text{enclosed}\\ \text{area}\end{matrix} + 1 \cdot \begin{matrix}\text{third}\\ \text{enclosed}\\ \text{area}\end{matrix} =$$

$$= \text{constant} = \left(\frac{b}{b-1} + \frac{1}{a-1} + 1\right) \cdot \text{area of the answer}$$

An argument for the not so silly example

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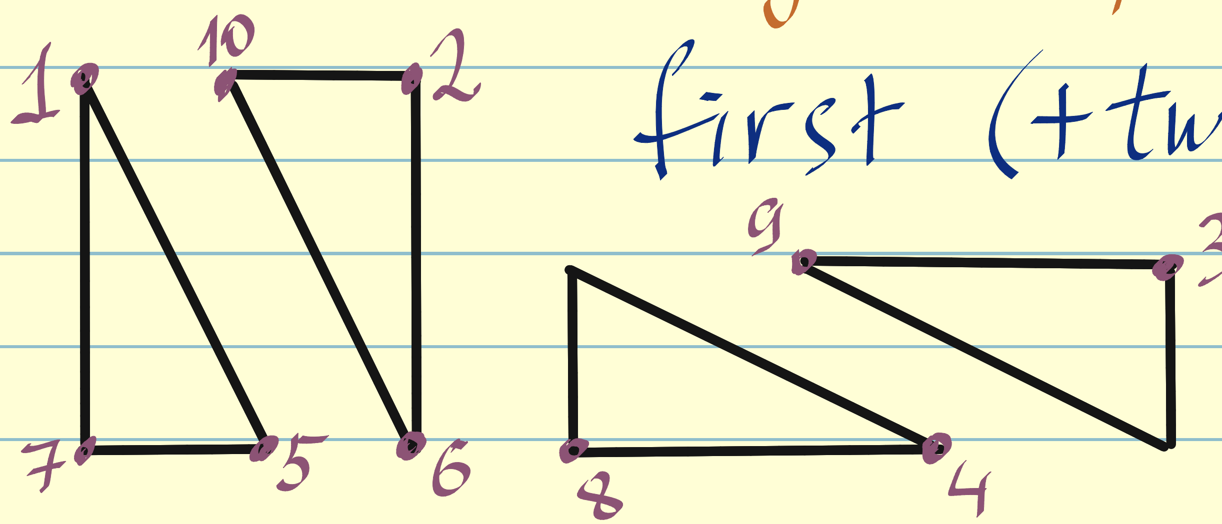

first (+ two more)

Step 2 Each enclosed area $\leq$ area of convex hull

(because winding numbers are ≤ 1 after a certain Steiner symmetrization, which preserves LHS and decreases RHS)

$$\frac{b}{b-1} \cdot \begin{matrix}\text{first}\\ \text{enclosed}\\ \text{area}\end{matrix} + \frac{1}{a-1} \cdot \begin{matrix}\text{second}\\ \text{enclosed}\\ \text{area}\end{matrix} + 1 \cdot \begin{matrix}\text{third}\\ \text{enclosed}\\ \text{area}\end{matrix} =$$

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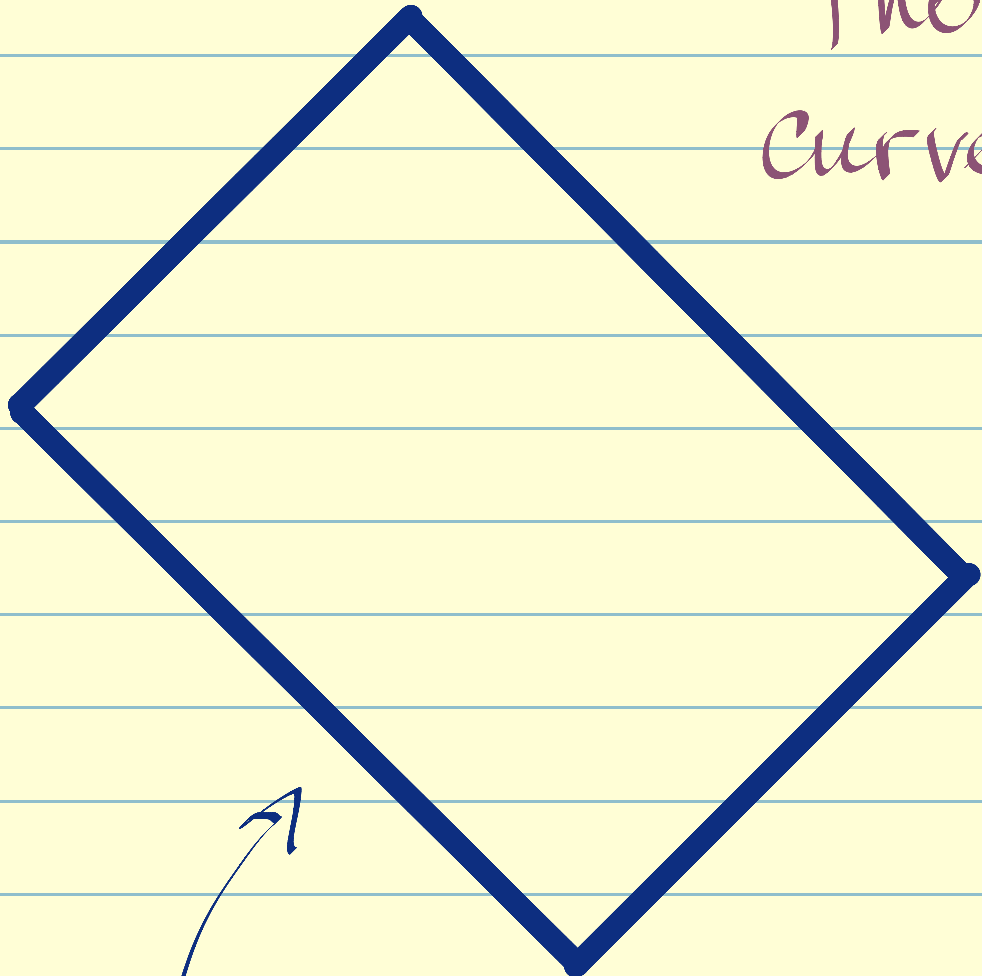
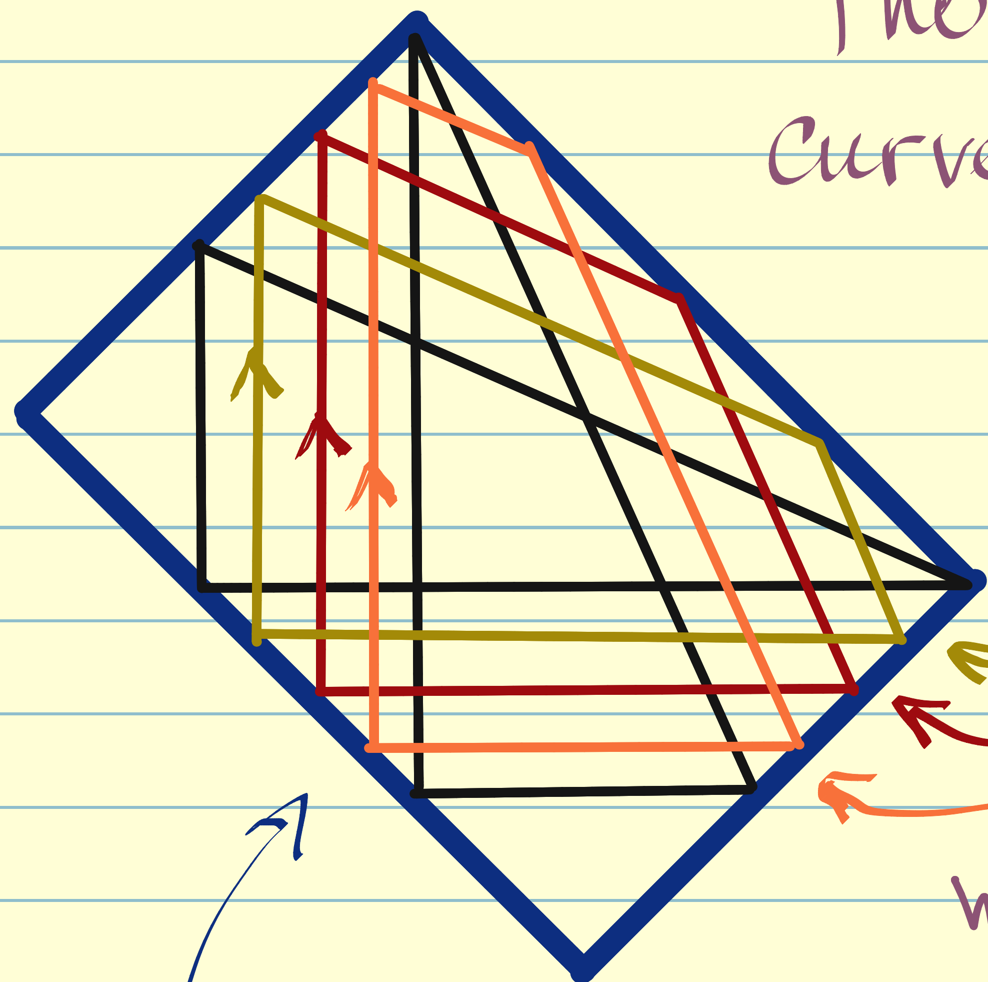
Those enigmatic closed polygonal curves come from certain billiard dynamics



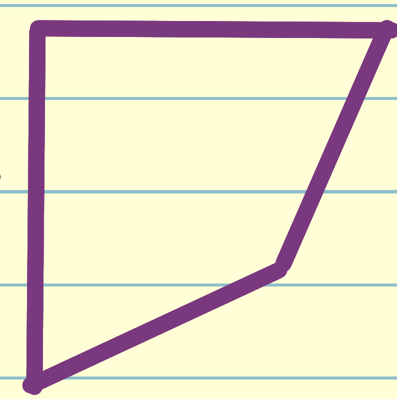
billiard table



billiard table

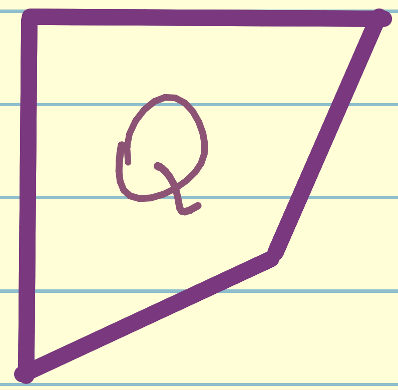
Those enigmatic closed polygonal curves come from certain billiard dynamics

closed billiard trajectories w.r.t. the reflection rule in the normed plane, whose dual unit ball is



General problem

- A convex polygon Q is given



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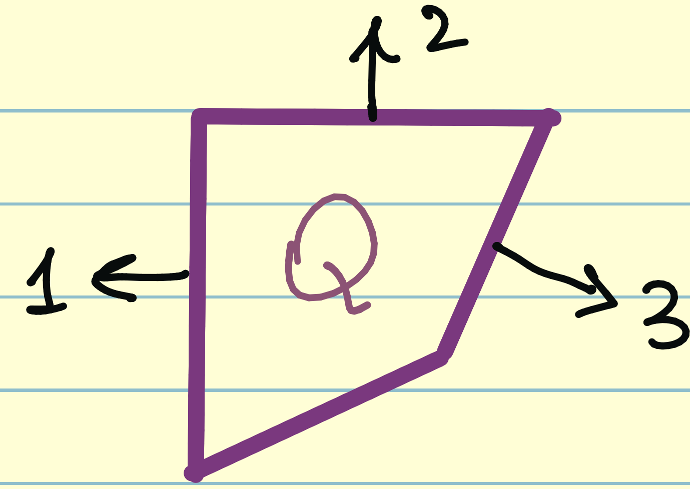
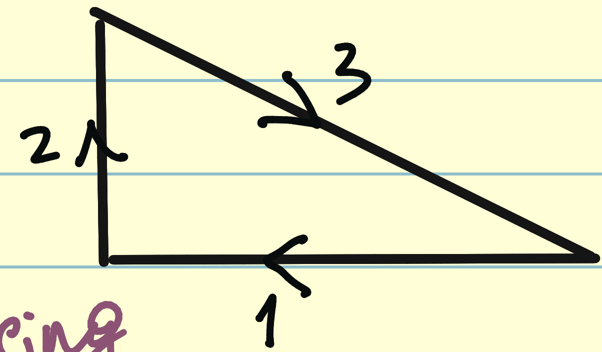


- A convex polygon Q is given
- Pick a triple of outer normals that admits a positive linear dependence, and draw the unique triangle following these vectors, with the perimeter length 1



measured using the norm whose dual unit ball is Q

General problem

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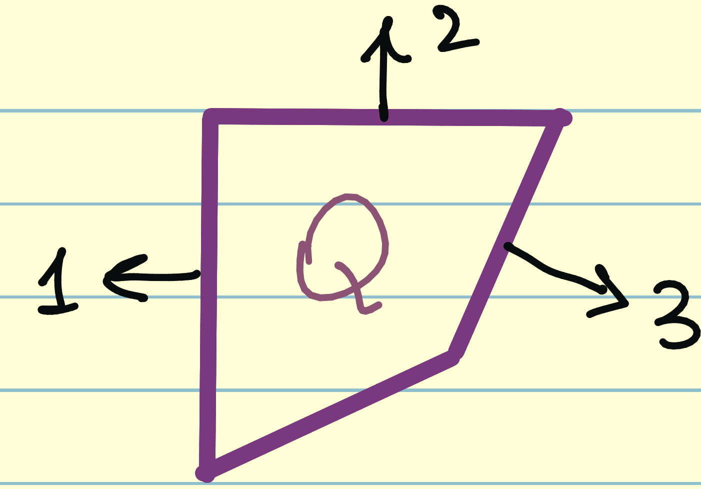

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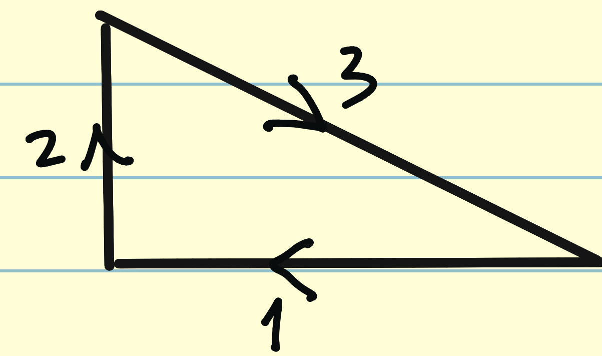
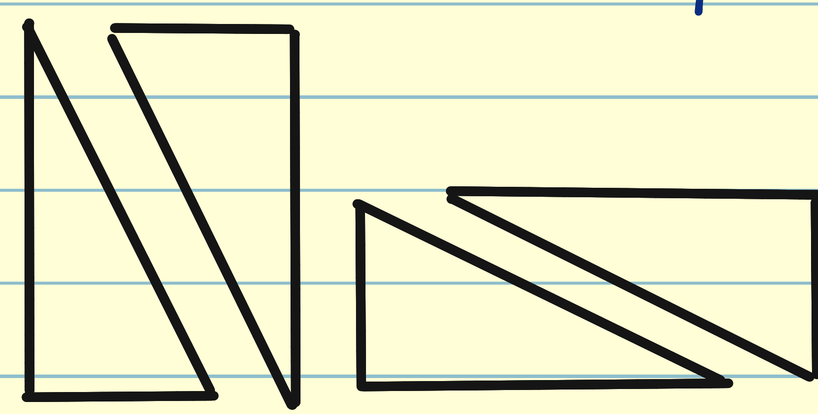


- Repeat for other triples of normals (and also pairs)



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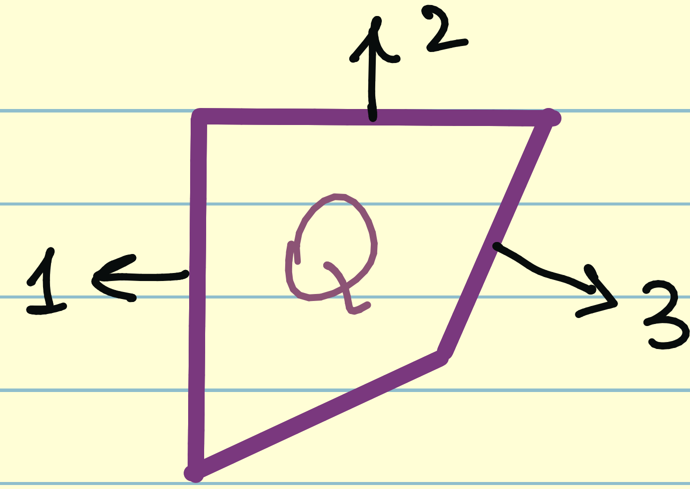

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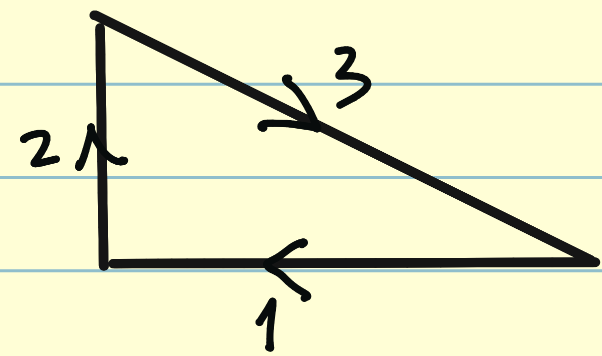
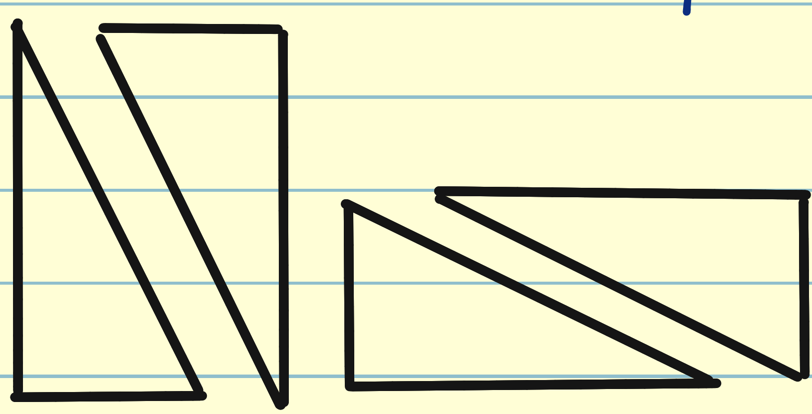
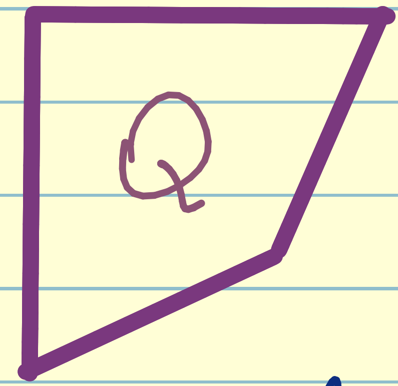


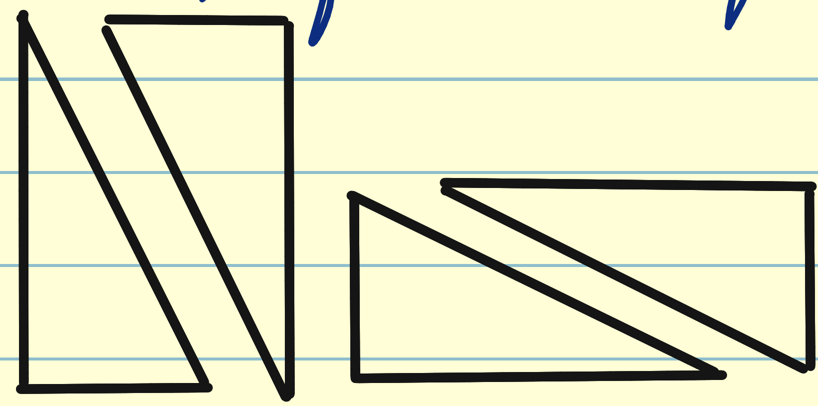
- Repeat for other triples of normals (and also pairs)
- Solve the prototype problem: pack these triangles in a convex shape of minimal area



General problem



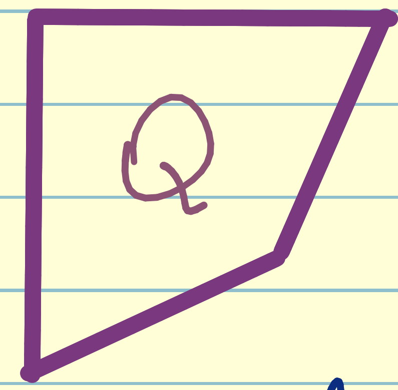
- A convex polygon Q is given
- Draw all distinguished triangles
  (over all pairs/triples of normal w/ positive dependence)
- Pack these triangles in
  a convex shape of minimal area



Conjecture: the area needed is $\geq \dfrac{1}{2 \cdot \text{area } Q}$

General problem

- A convex polygon Q is given



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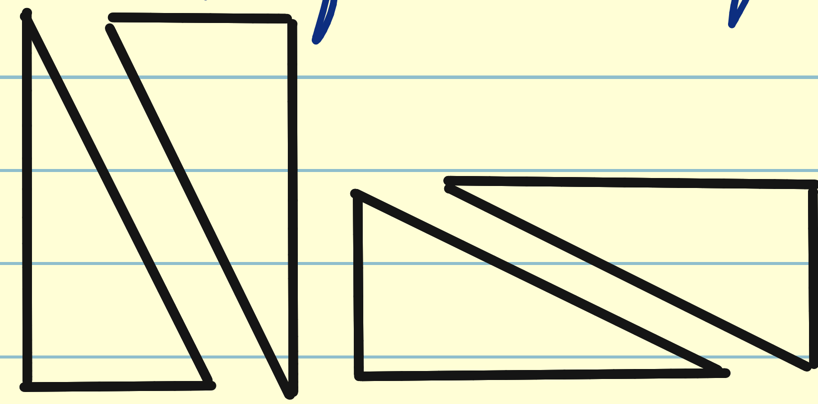
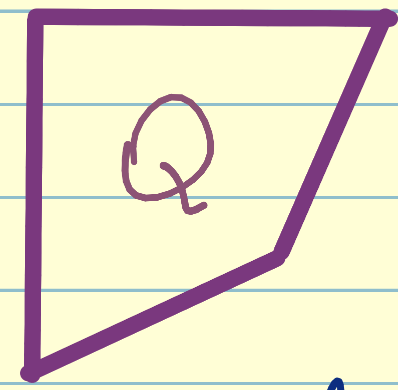


Conjecture: the area needed is $\geq \dfrac{1}{2 \cdot \text{area } Q}$

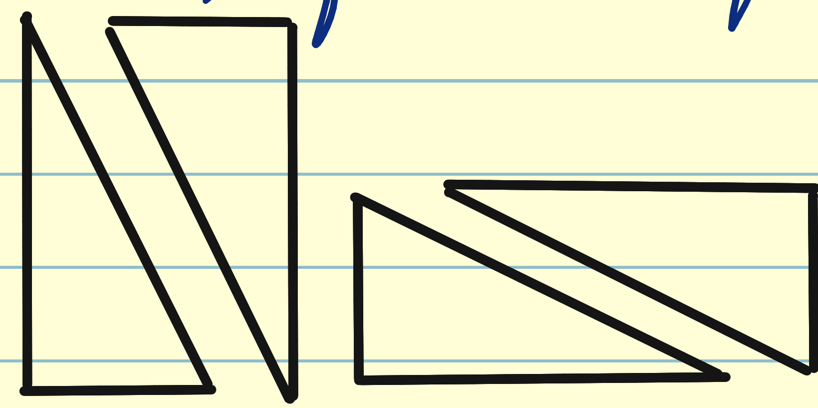
Silly example #1 $\longleftrightarrow$ Q is a triangle

Not so silly example #2 $\longleftrightarrow$ Q is a quadrilateral

General problem in $\mathbb{R}^n$

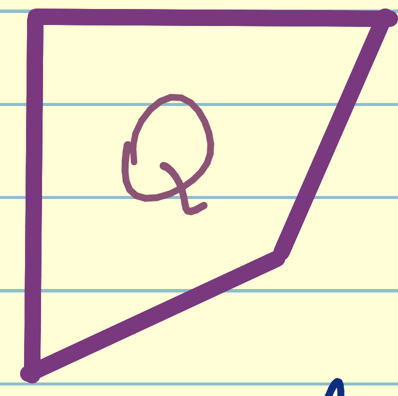


- A convex polytope Q is given
- Draw all distinguished simplices (over all pairs/triples/.../(n+1)-tuples of normal w/ positive dependence)
- Pack these simplices in a convex shape of minimal volume



Conjecture: the volume needed is $\geq \dfrac{1}{n!\,\mathrm{vol}\,Q}$

General problem in $\mathbb{R}^n$



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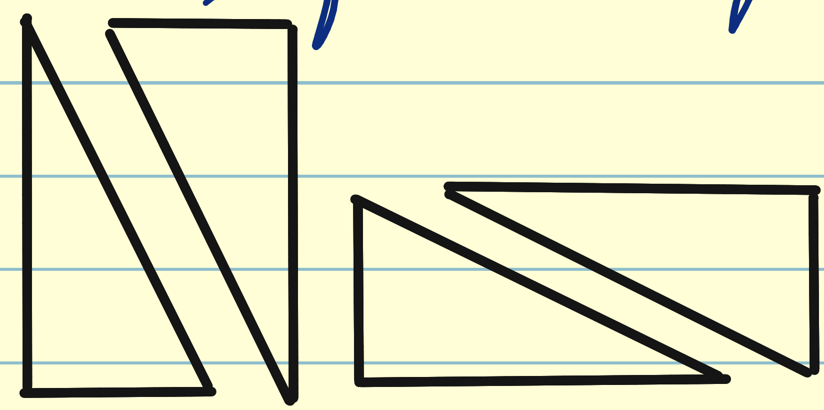
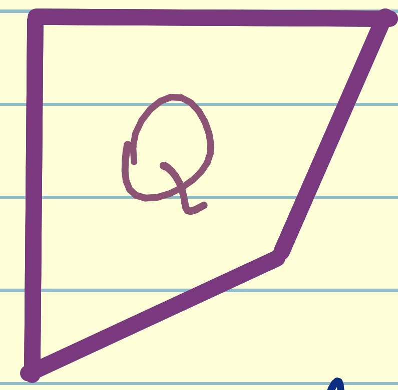


Conjecture: the volume needed is $\geq \dfrac{1}{n!\,\mathrm{vol}\,Q}$

- a special case of Viterbo's conjecture

$$\mathrm{vol}\left(\begin{array}{c}\text{convex body } X\\ \text{in sympletic } \mathbb{R}^{2n}\end{array}\right) \geq \frac{1}{n!}\left(\begin{array}{c}\text{sympletic}\\ \text{capacity of } X\end{array}\right)^n$$

General problem in $\mathbb{R}^n$



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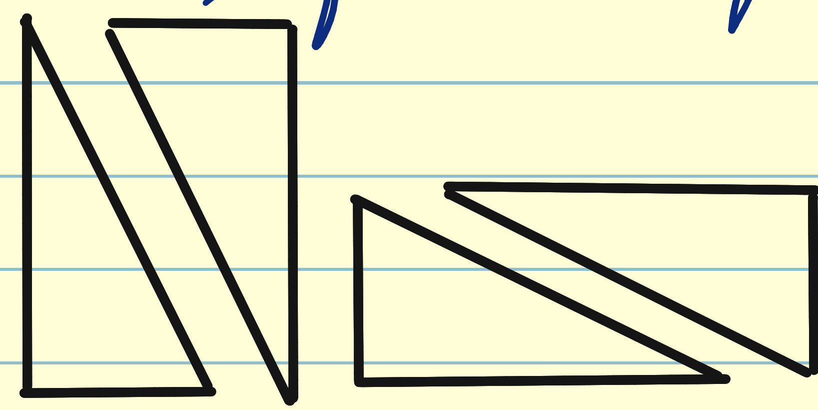


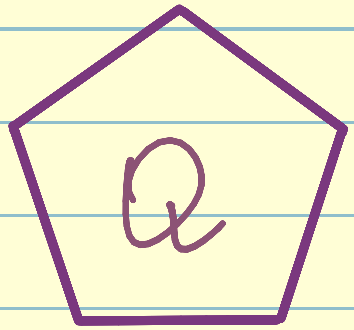
Conjecture: the volume needed is $\geq \dfrac{1}{n!\,\mathrm{vol}\,Q}$

- a special case of Viterbo's conjecture
- implies Mahler's conjecture from 1939
  $\mathrm{vol}(\text{convex } K \subset \mathbb{R}^n) \cdot \mathrm{vol}(K^\circ) \geq 4^n/n!$

Example #3, liberating

(Haim-Kislev, Ostrover, 2024)

Problem: translate distinguished triangles to minimize the area of their convex hull



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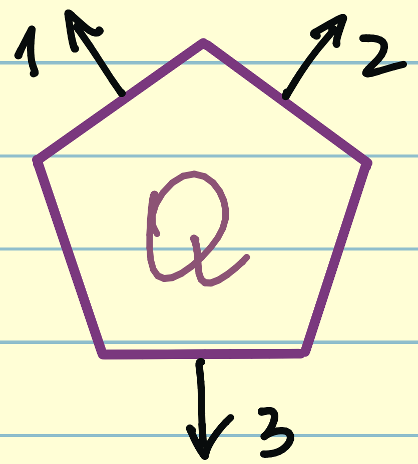
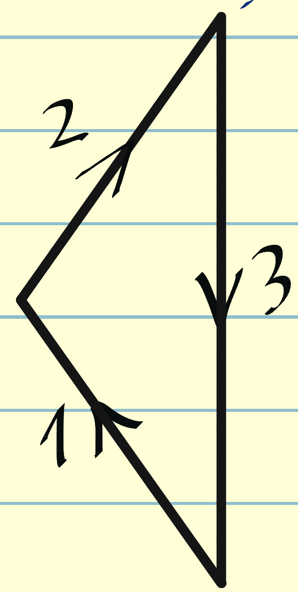


triangles:

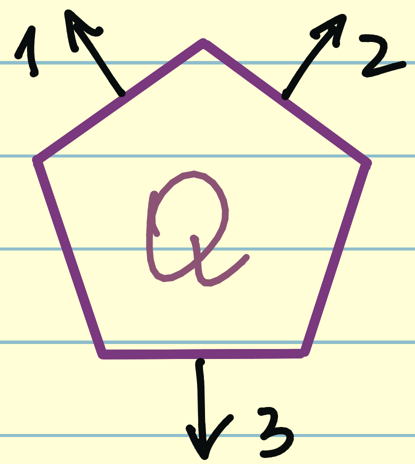


and 9 rotations of it

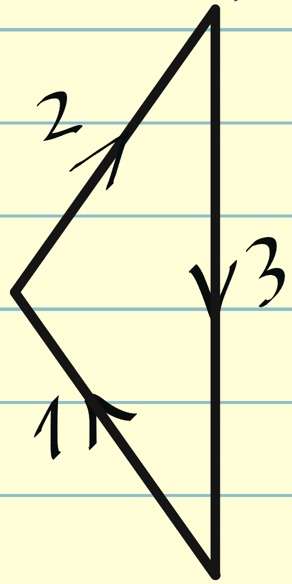
Example #3, liberating

(Haim-Kislev, Ostrover, 2024)

Problem: translate distinguished triangles to minimize the area of their convex hull

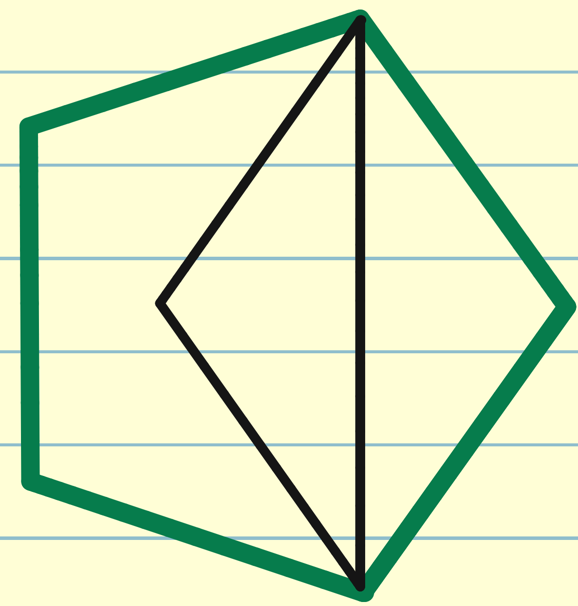


triangles:



and 9 rotations of it

optimal packing:



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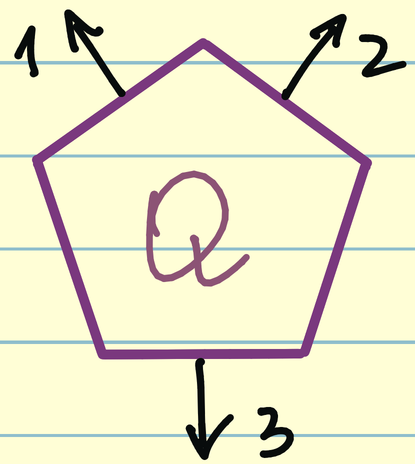
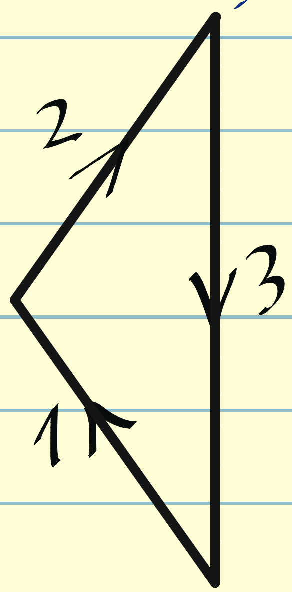


triangles: 

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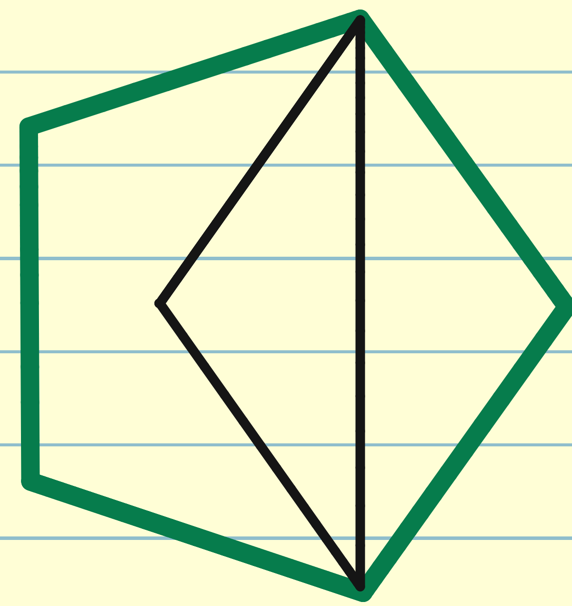
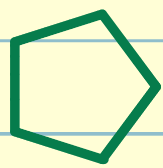
optimal packing: 

area  $< 0.96 \cdot \dfrac{1}{2 \text{ area } Q}$

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(Haim-Kislev, Ostrover, 2024)

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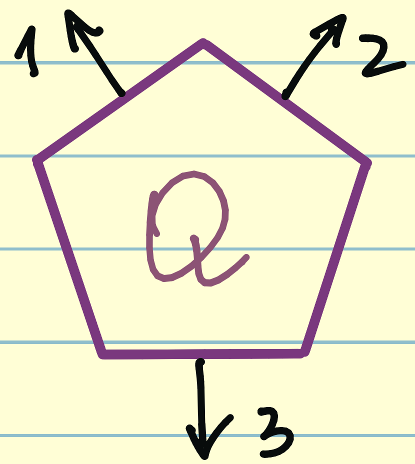
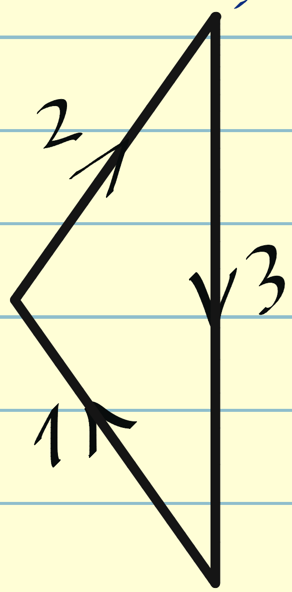


triangles: 

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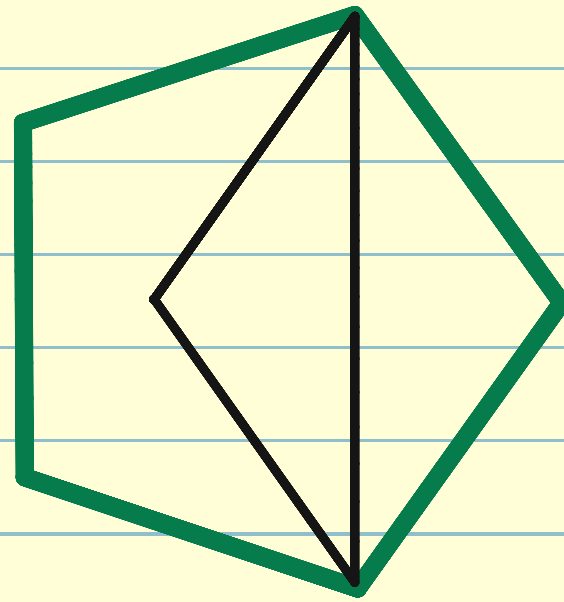
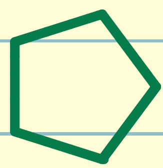
optimal packing: 

area  $< 0.96 \cdot \dfrac{1}{2 \text{ area } Q}$

Viterbo's conjecture is false!

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(Haim-Kislev, Ostrover, 2024)

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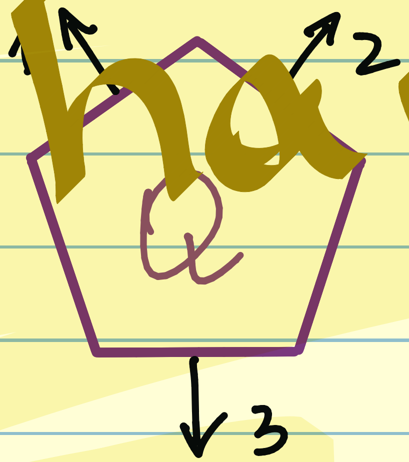
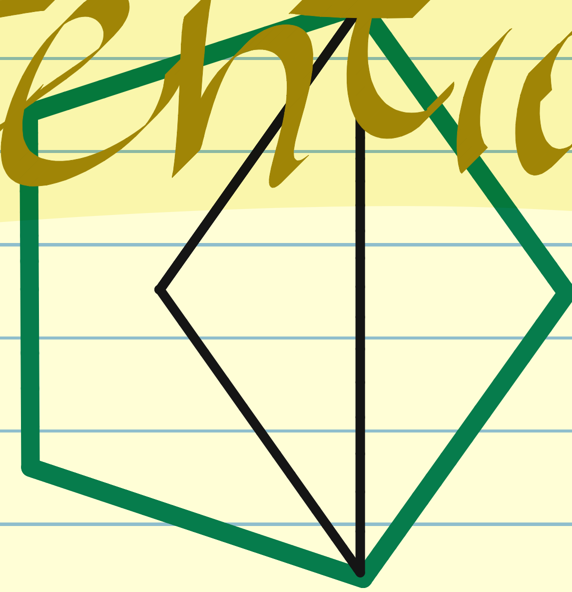


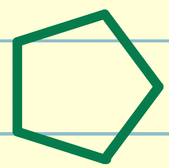
triangles:

and 9 rotations of it

Thank you for your attention!

Optimal packing:



area  $< 0.96 \cdot \frac{1}{2 \text{ area } Q}$

Viterbo's conjecture is false!