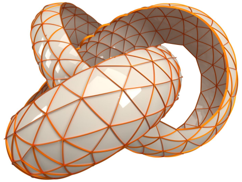
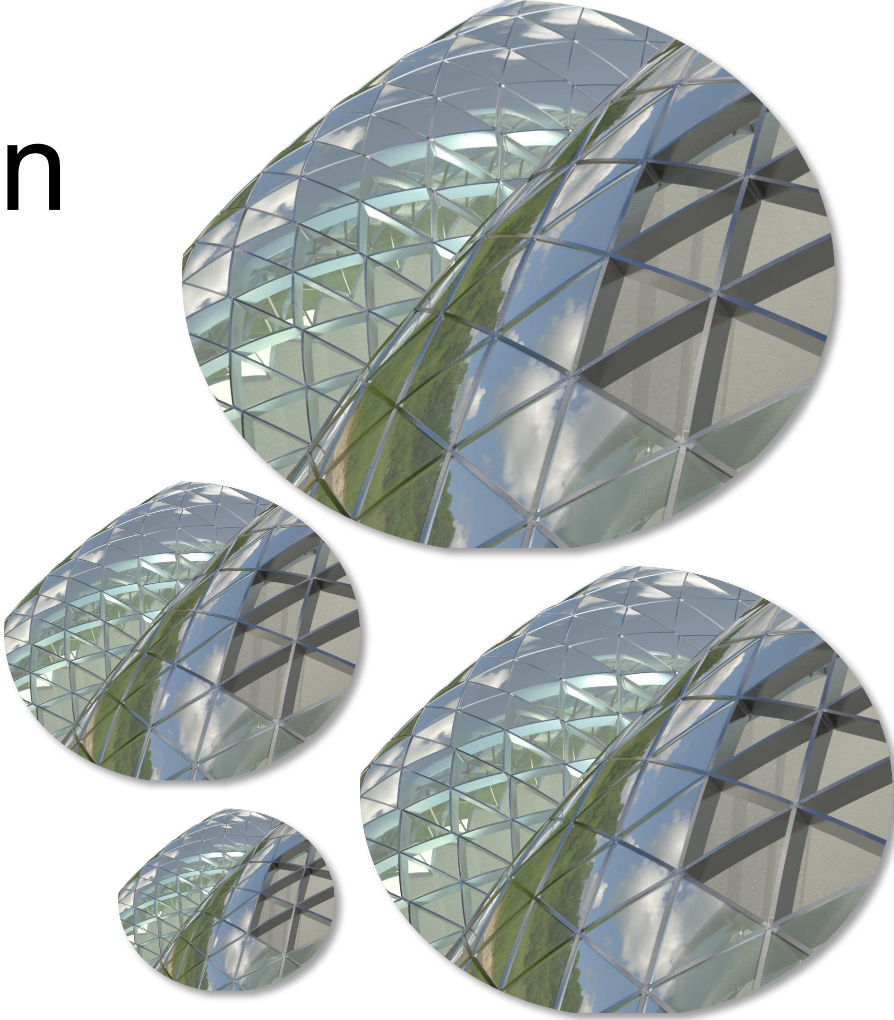


Approximation by Meshes with Spherical Faces



Anthony Ramos-Cisneros
Alisher Aikyn
Martin Kilian
Helmut Pottmann
Christian Mueller

KAUST
TU WIEN

Previous Work

Meshes with Spherical Faces

MARTIN KILIAN, TU Wien, Austria
ANTHONY S RAMOS CISNEROS, KAUST, Saudi Arabia
CHRISTIAN MÜLLER, TU Wien, Austria
HELMUT POTTMANN, KAUST, Saudi Arabia

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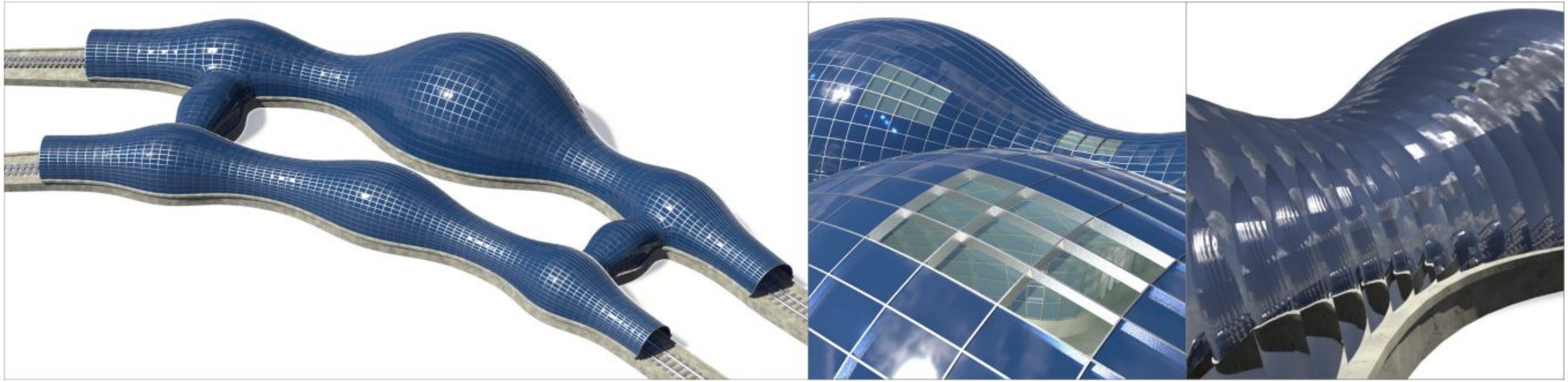
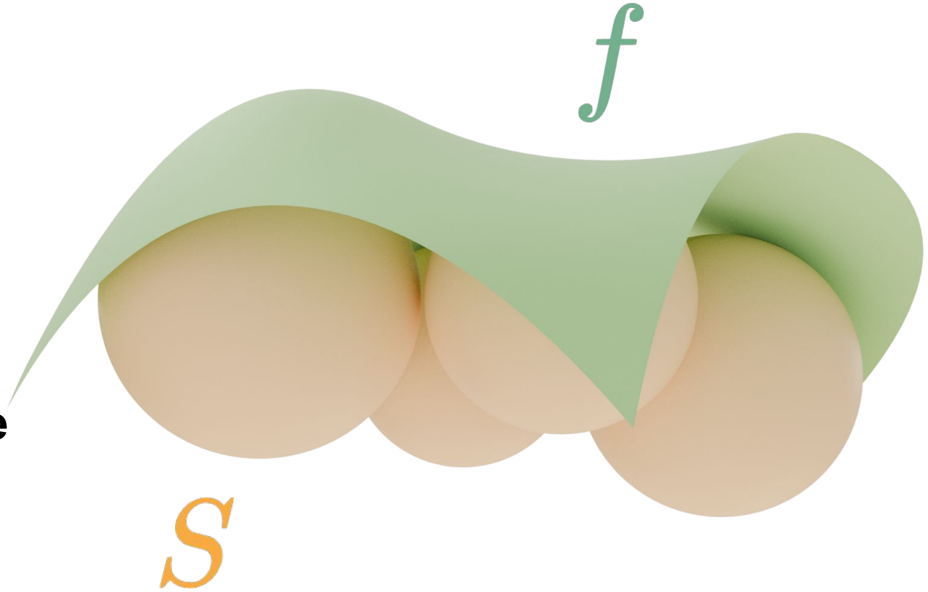
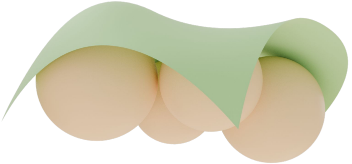


Overview

- Approximation by **discrete** family of spheres.
- **Smooth** appearance and **torsion free** support structure.
- Spherical Triangle meshes (ST Meshes) with **support structure** and **adaptable** face size.
- Spherical Quad meshes (SQ Meshes) **remeshing** and **support structure**.



Overview



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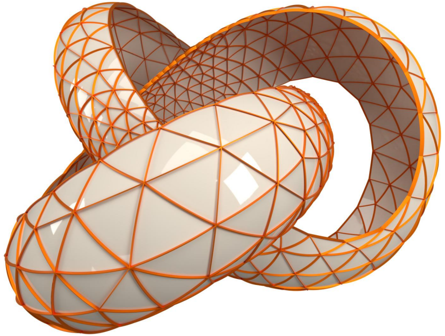
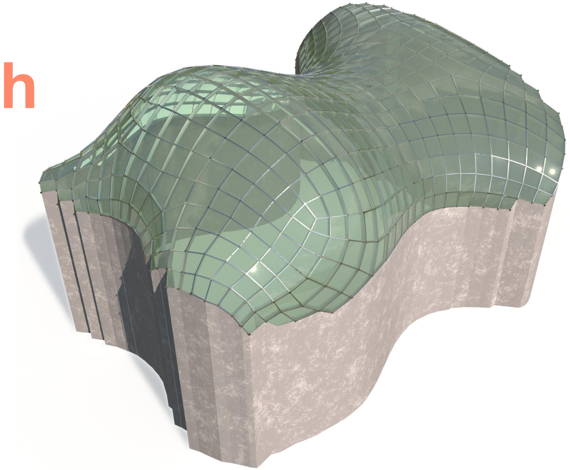
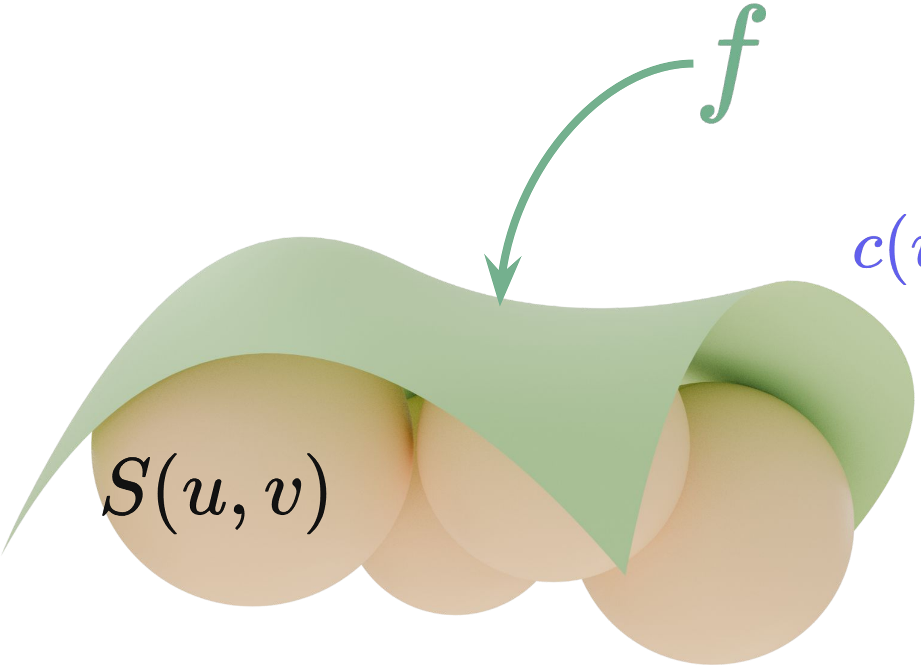


ST-Mesh

SQ-Mesh



Sphere Congruences and Line Congruences



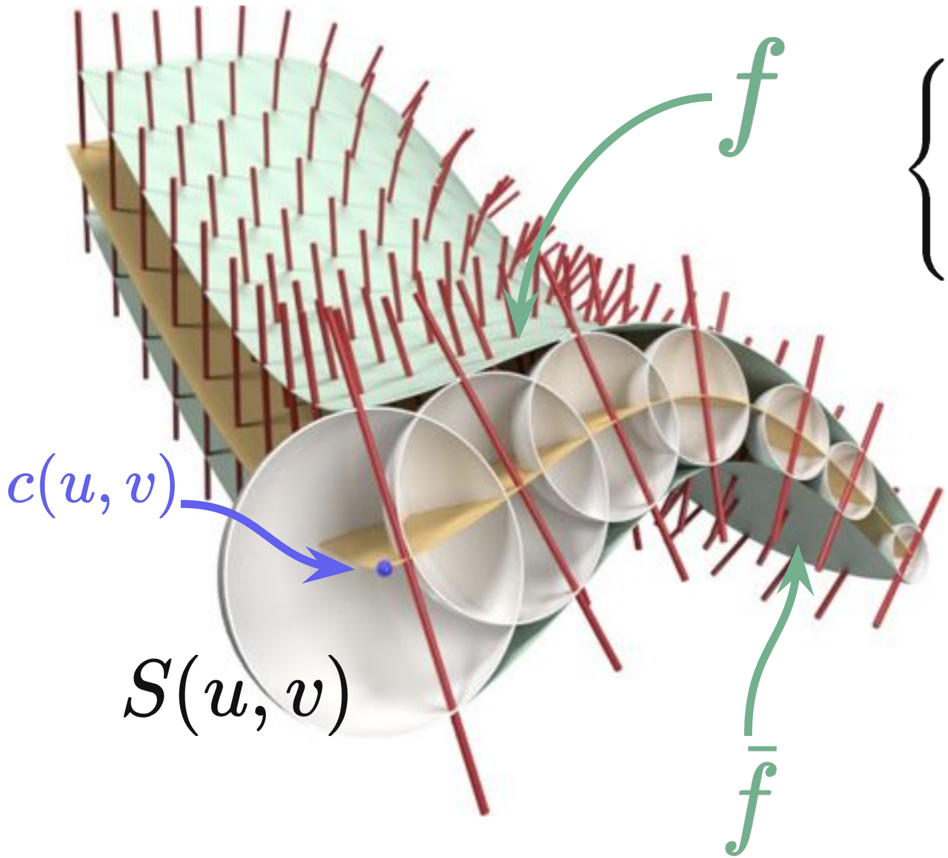
The centers $c(u,v)$ depend only on the radius function $r(u,v)$

$$c(u,v) = f(u,v) + r(u,v)\, n(u,v)$$

$$\downarrow$$

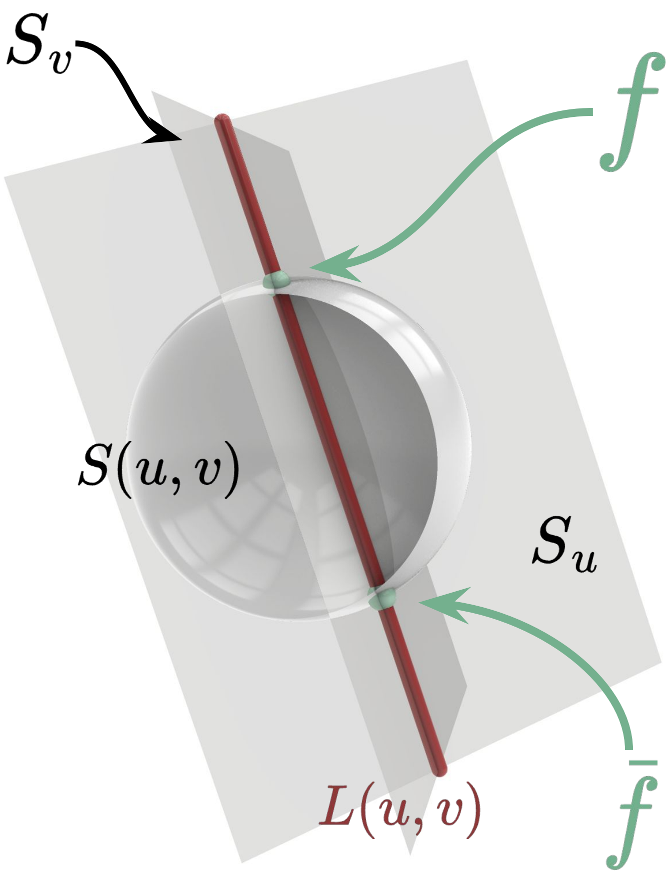
$$S(u,v)$$

Sphere Congruences and Line Congruences



$$\begin{cases} S: \quad ||x - c(u,v)||^2 - r(u,v)^2 = 0 \\ S_u: \langle x - c, c_u \rangle + r\, r_u = 0 \\ S_v: \langle x - c, c_v \rangle + r\, r_v = 0 \end{cases}$$

Sphere Congruences and Line Congruences



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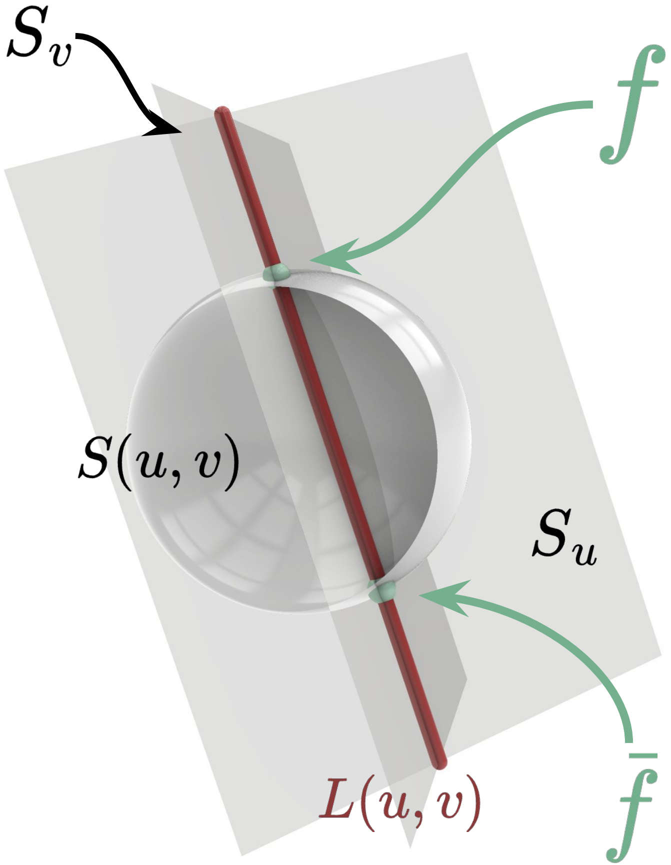
Associated line congruence

$$L(u,v) = S_u \cap S_v$$

Sphere Congruences and Line Congruences



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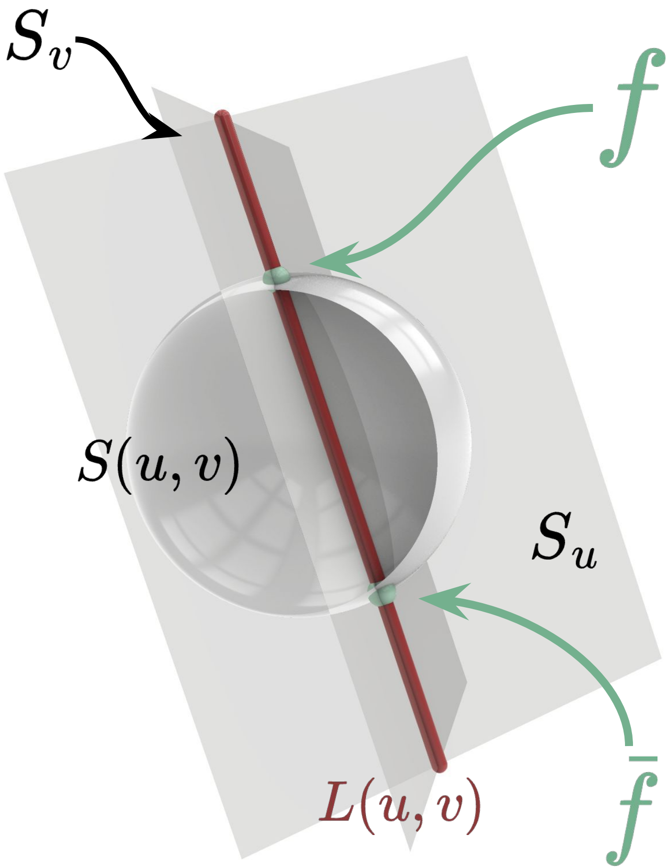
Associated line congruence

$$L(u,v) = S_u \cap S_v$$

The direction vectors of $L(u,v)$

$$l = c_u \times c_v$$

Sphere Congruences and Line Congruences



$$\begin{cases} S: \quad \|x - c(u,v)\|^2 - r(u,v)^2 = 0 \\ S_u: \langle x - c, c_u \rangle + r\, r_u = 0 \\ S_v: \langle x - c, c_v \rangle + r\, r_v = 0 \end{cases}$$

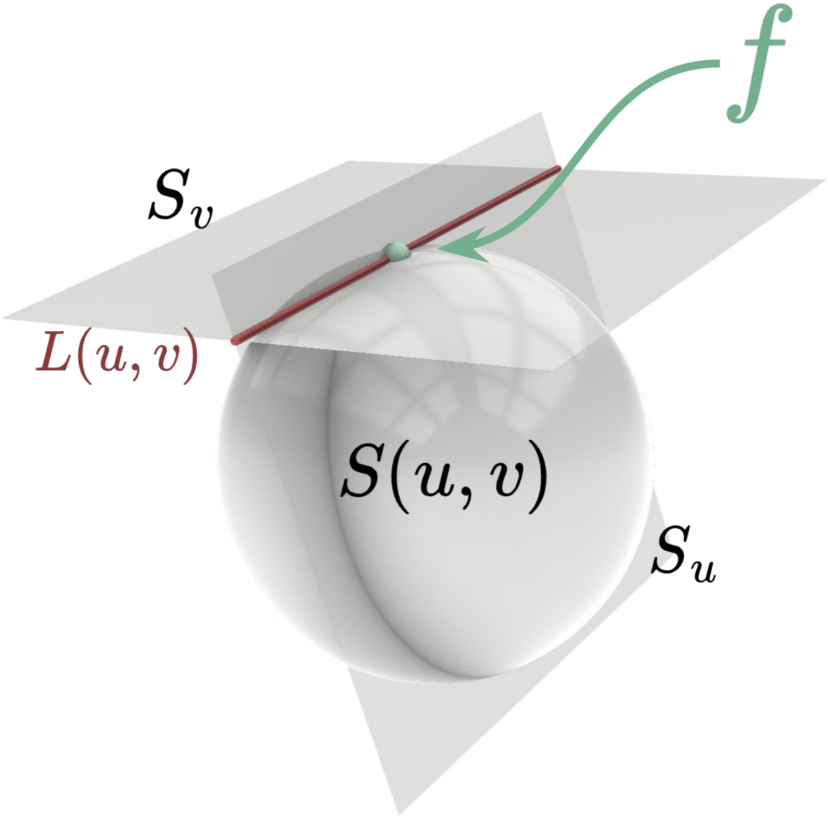
Associated line congruence

$$L(u,v) = S_u \cap S_v$$

Envelope points

$$\{\, f, \bar{f} \,\} = L(u,v) \cap S(u,v)$$

Sphere Congruences and Line Congruences



$$\begin{cases} S: \quad \|x - c(u,v)\|^2 - r(u,v)^2 = 0 \\ S_u: \langle x - c, c_u \rangle + r\, r_u = 0 \\ S_v: \langle x - c, c_v \rangle + r\, r_v = 0 \end{cases}$$

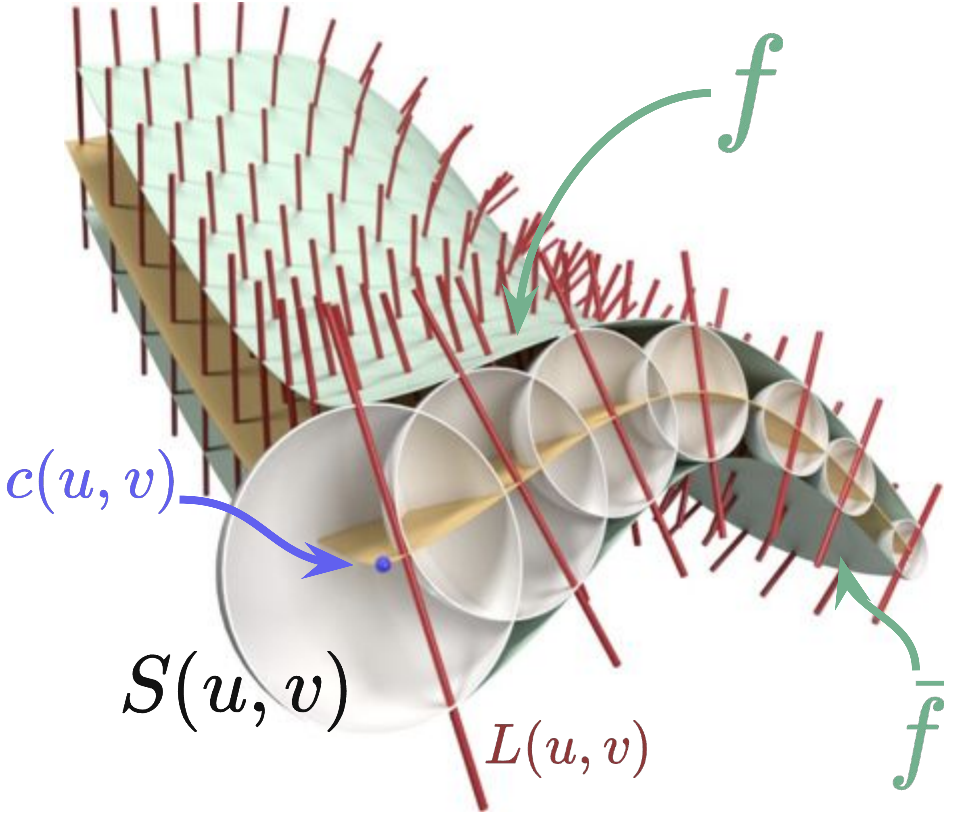
Associated line congruence

$$L(u,v) = S_u \cap S_v$$

Envelope point by construction

$$f = L(u,v) \cap S(u,v)$$

Sphere Congruences and Line Congruences



- Sphere congruence

  $$S(u, v)$$

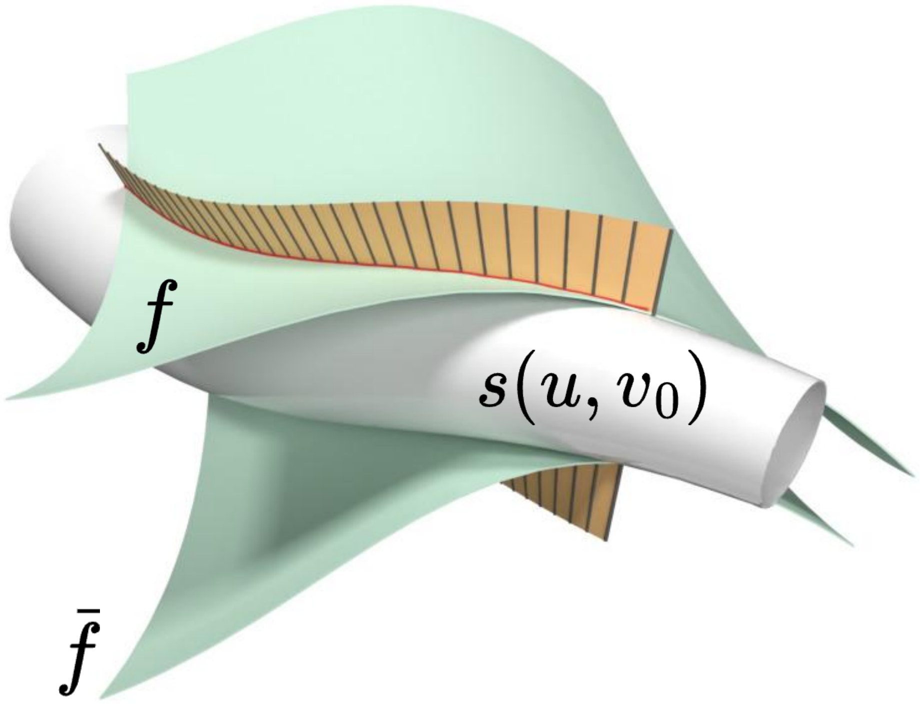
- Line Congruence [Support Structure]

  $$L(u, v)$$

- Envelopes

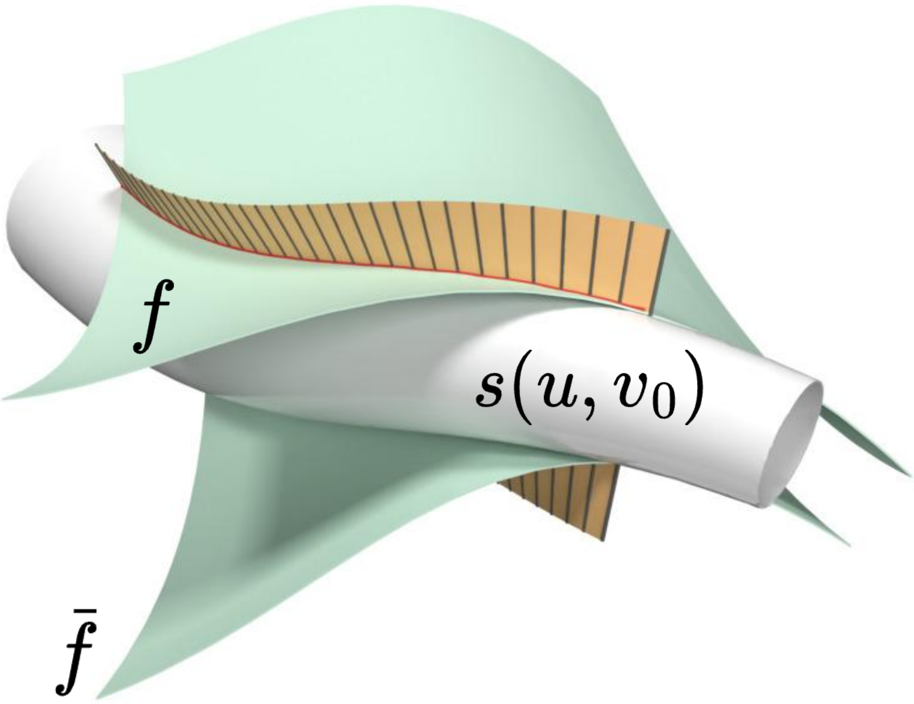
  $$f \quad \bar{f}$$

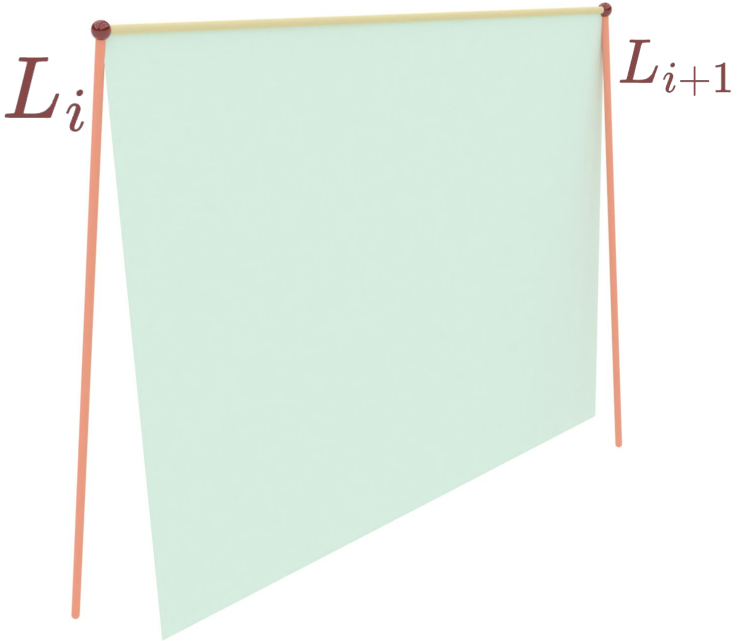
Torsal Parametrization of LC



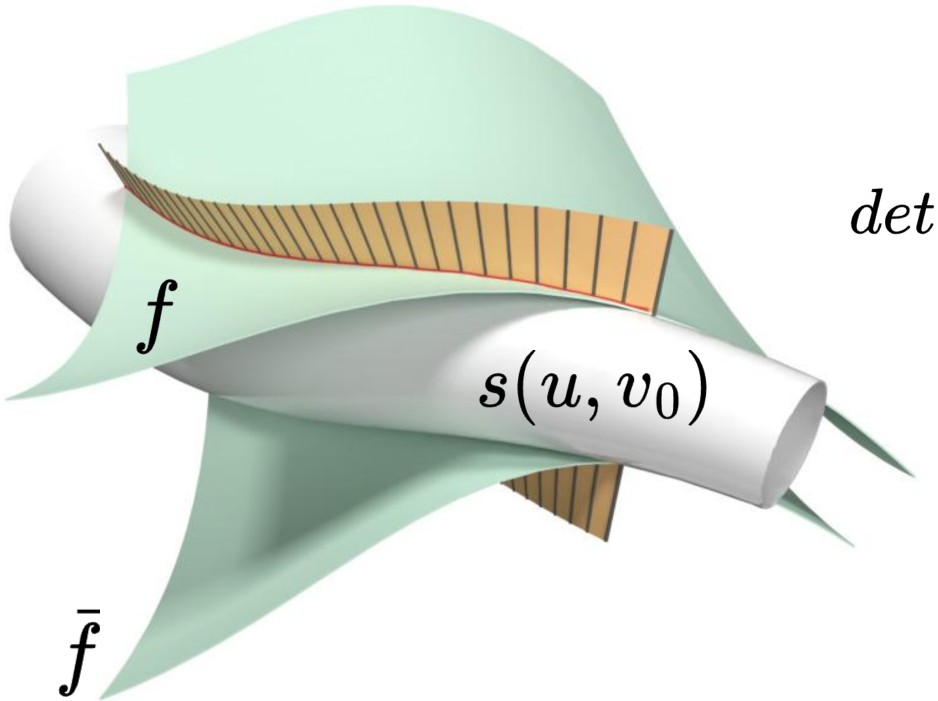
- Channel surface intersect both envelopes in two curves that form a **ruled surface**
- We are interested in **torsal ruled surfaces [developable surface]** which are related to **torsion free** support structures

Torsal Parametrization of LC



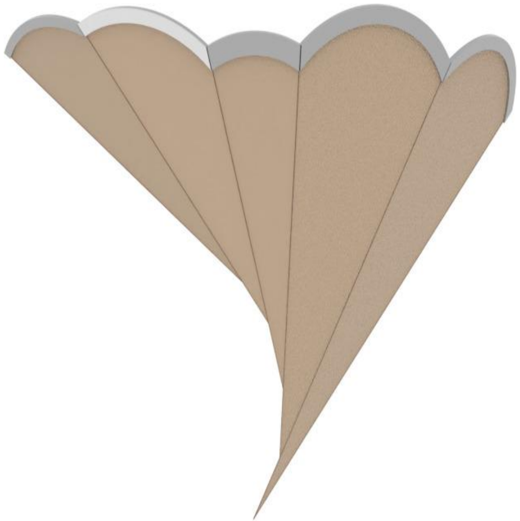


Torsal Parametrization of LC

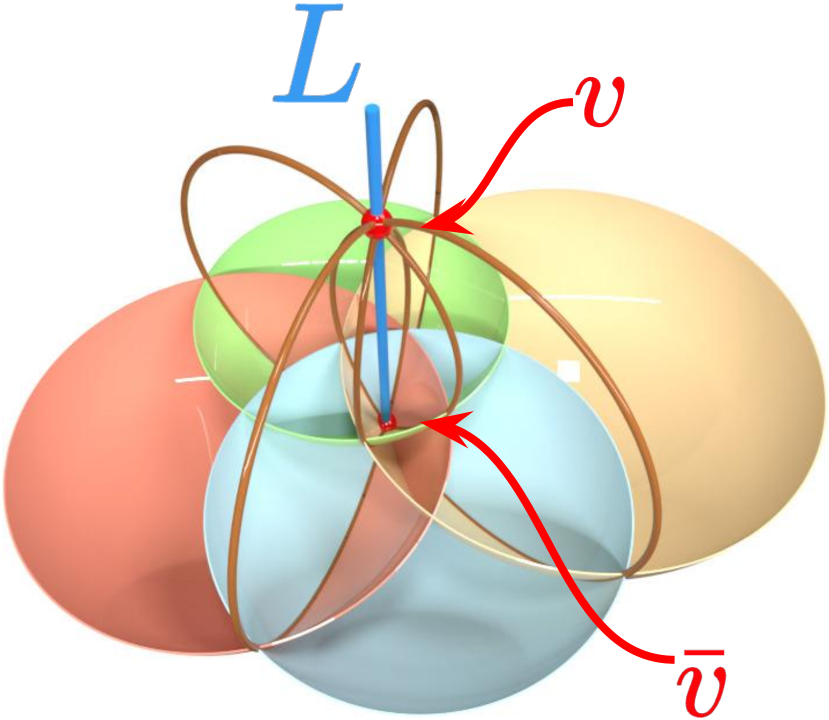


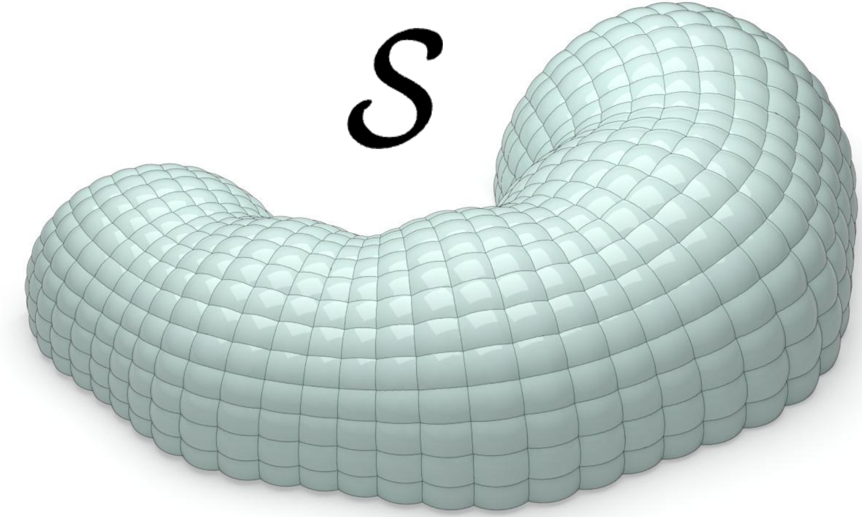
- Torsal direction $(\dot{u}, \dot{v})$ given a line congruence $L(u, v)$

$$det\left(\dot{u}f_u + \dot{v}f_v,\ l,\ \dot{u}l_u + \dot{v}l_v\right) = 0$$



Discrete Sphere Congruence





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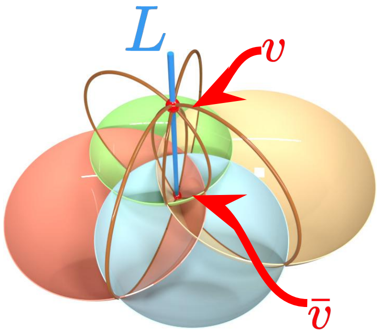
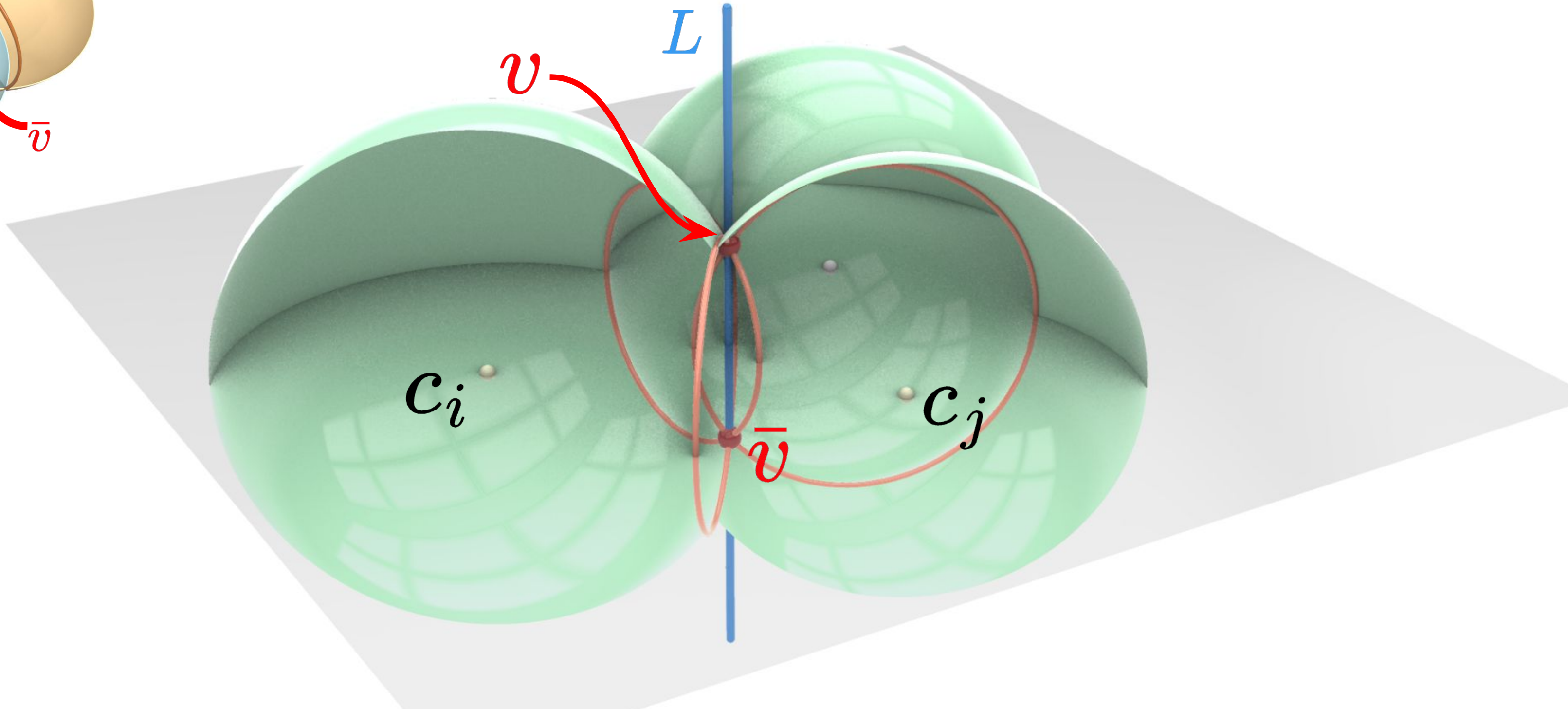
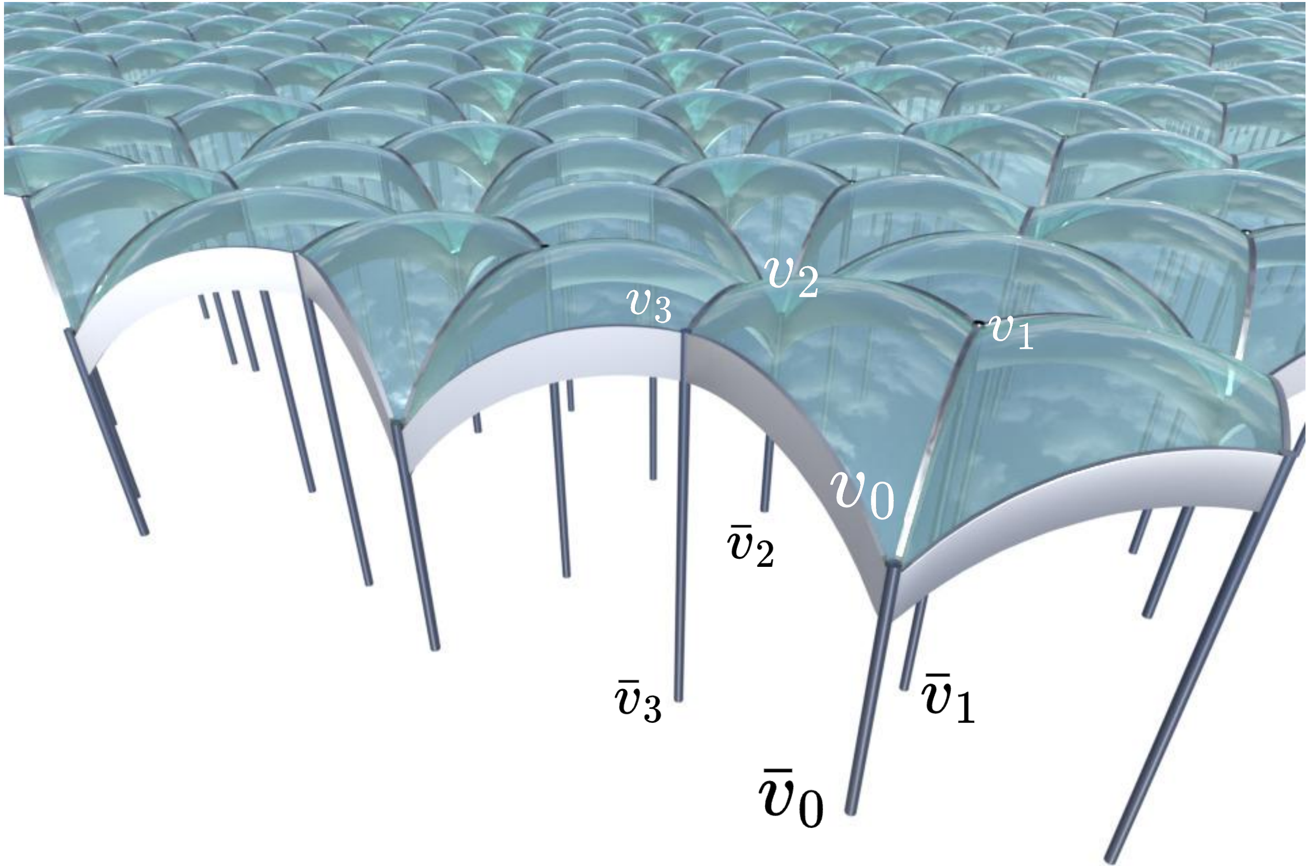


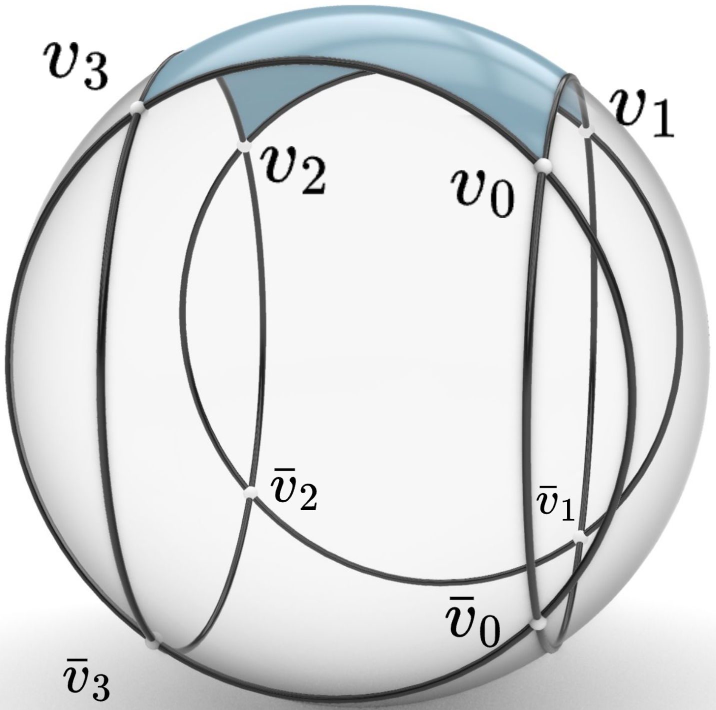


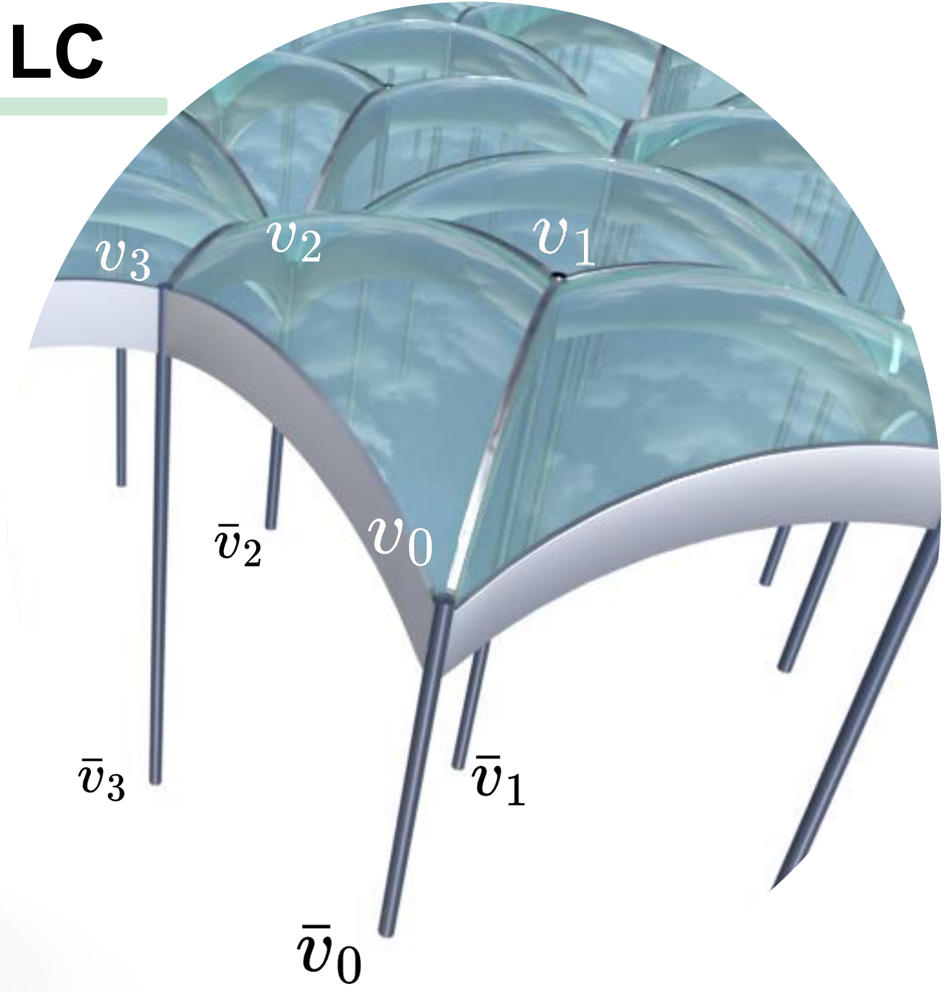
Torsal Parametrization of LC

- Torsal parametrization of the line congruence results in two families of developable surfaces



Torsal Parametrization of LC

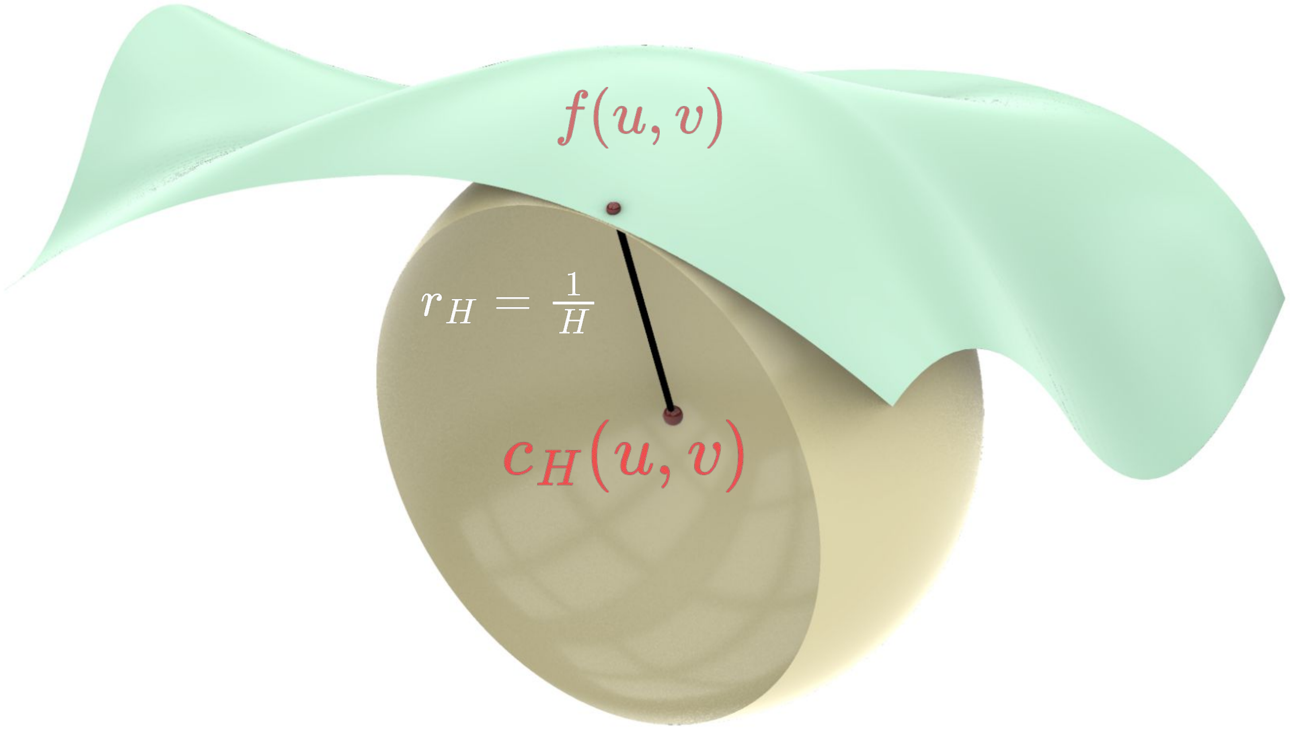





Sphere Congruence
Requirements

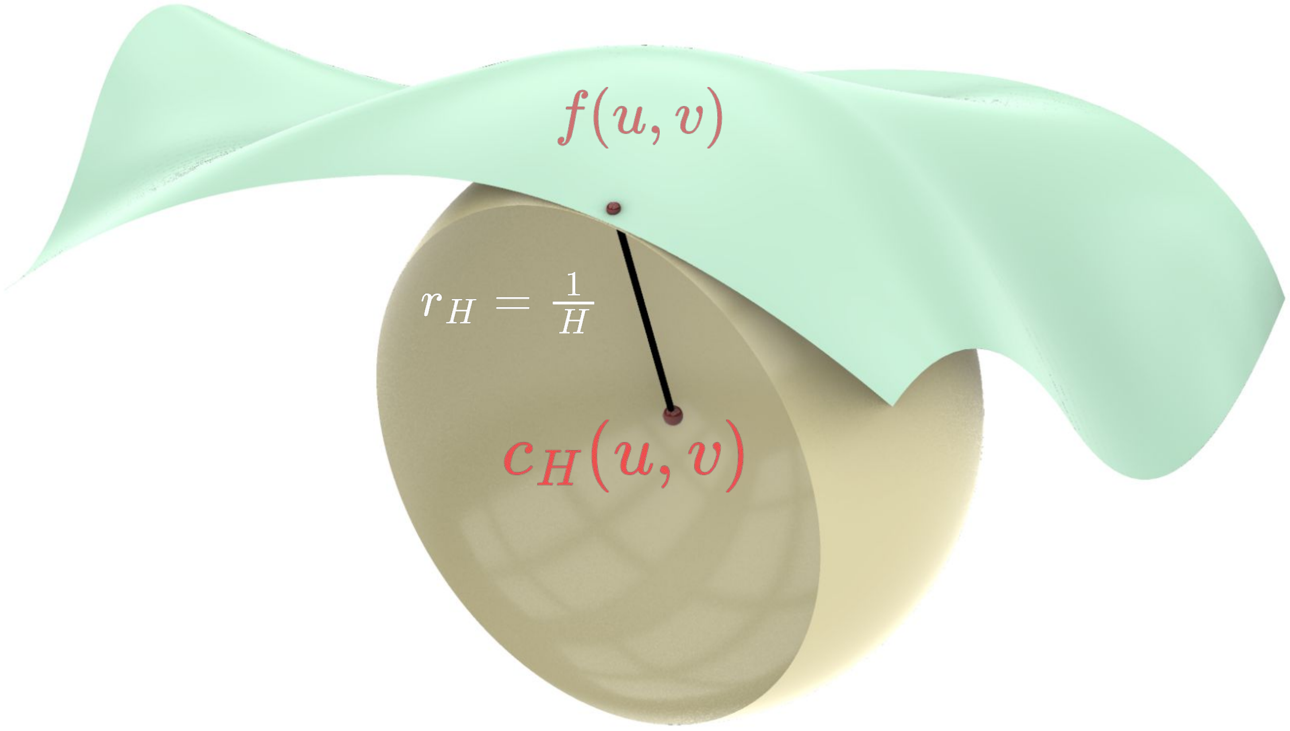
Requirements: A) Local Approximation

- Central spheres
- Mean Curvature
- Möbius Invariant

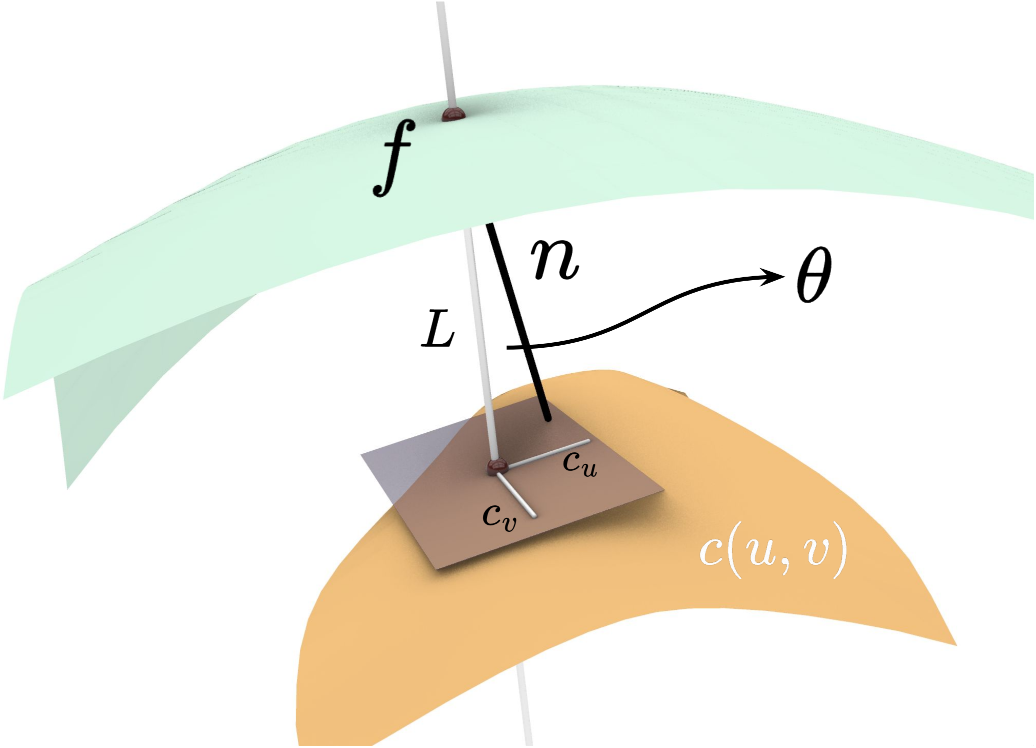


Requirements: A) Local Approximation

- Central spheres
- Mean Curvature
- Möbius Invariant
- Surfaces without change in sign of **H**



Requirements: B) Line angle with normal

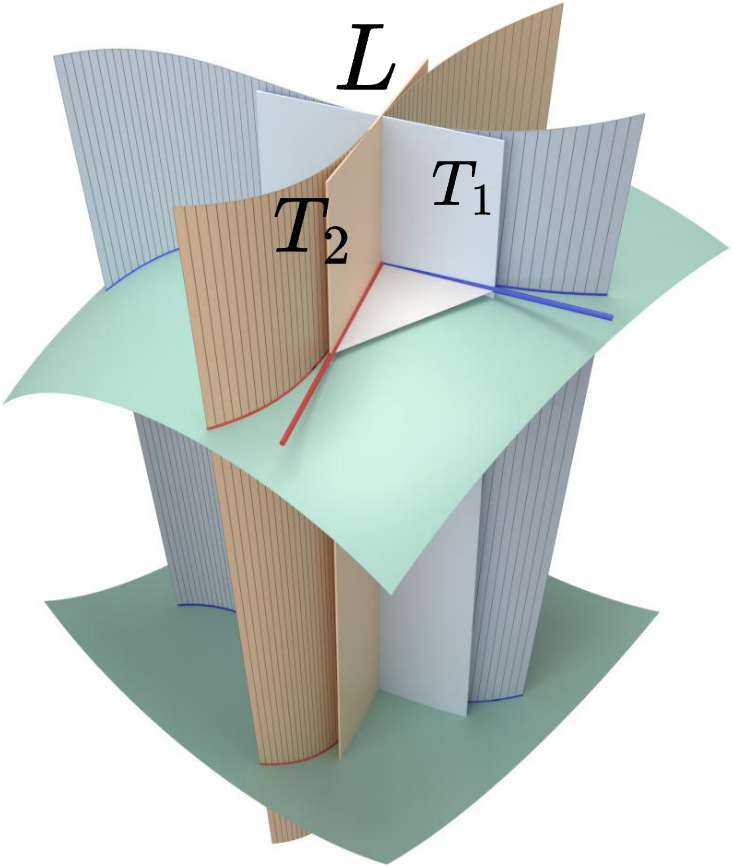


- Sphere intersection lies on planes that should have an angle within a threshold angle θ

- We impose a restriction of the node axis directions L

$$\angle(L, n) \leq \theta$$

Requirements: C) Torsal planes Angles



- The angle between the developable surfaces at the node L should not exceed a threshold angle

$$\frac{\pi}{4} \leq \angle({\color{blue}n_{t_1}}, {\color{red}n_{t_2}}) \leq \frac{\pi}{2}$$

ST-Meshes
Adaptive Face Size

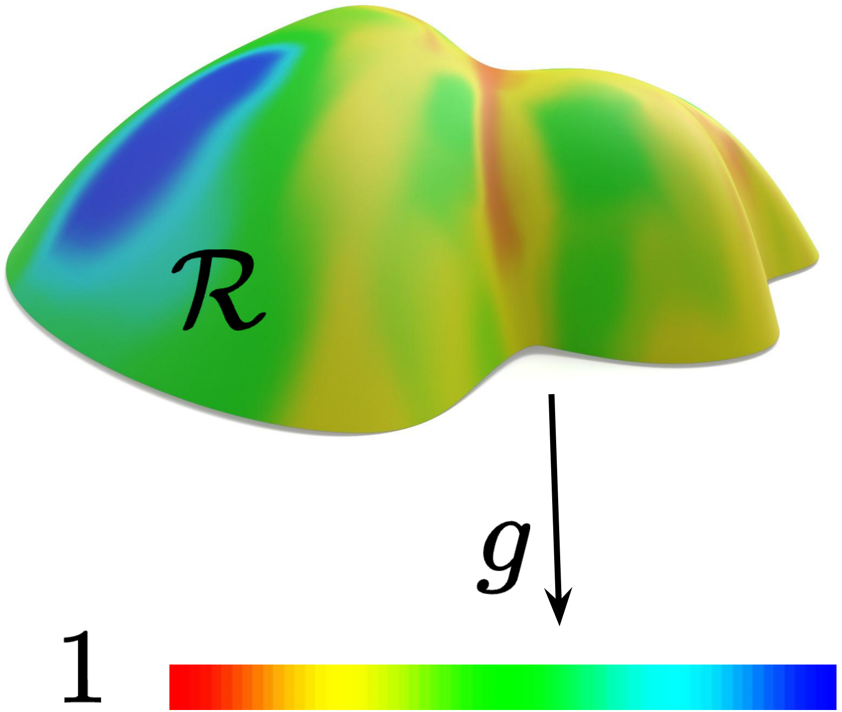
Meshes with Adapted Faces Size: Pliant Remeshing

- Given a function

  $$g : \mathcal{R} \rightarrow \mathbb{R}^+$$

  that determines the desired target edge lengths

- $\pi(e)$ denotes the projection of the edge midpoint onto $\mathcal{R}$

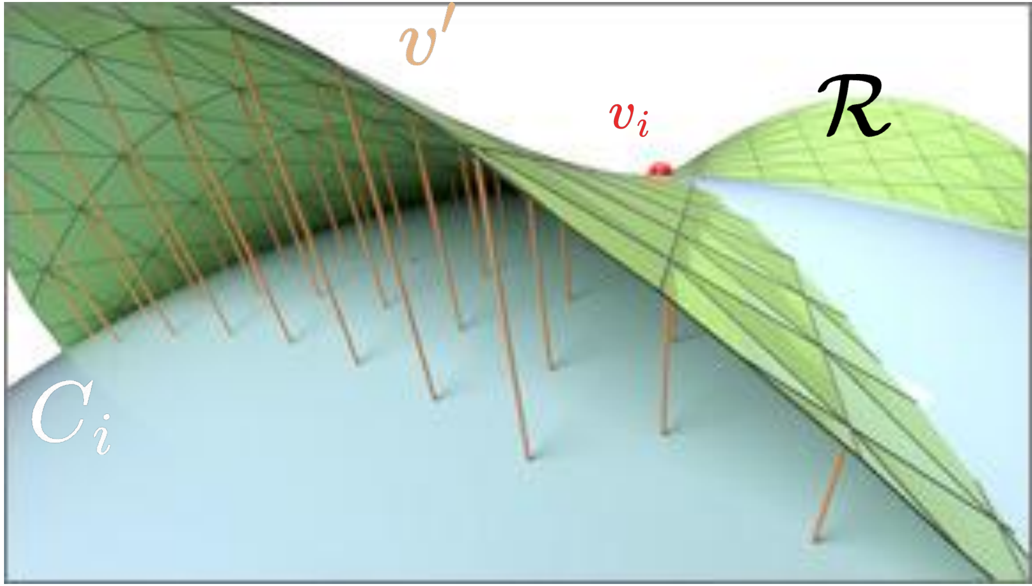


$$l(e) := \frac{|e|}{g(\pi(e))} \longrightarrow 1$$

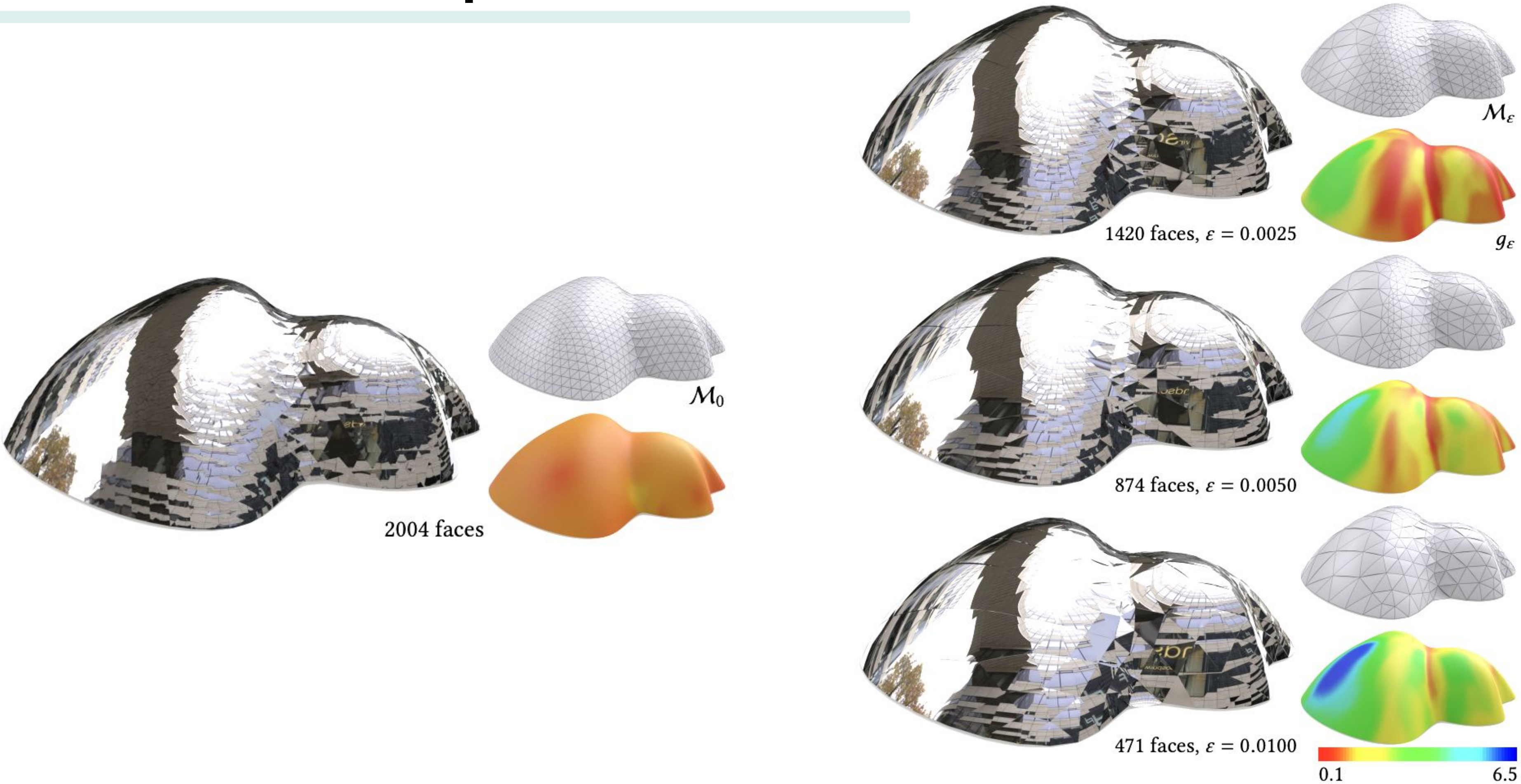
Leif P. Kobbelt, Thilo Bareuther, and Hans-Peter Seidel. 2000. Multiresolution Shape Deformations for Meshes with Dynamic Vertex Connectivity. *Computer Graphics Forum* 19, 3 (2000), 249–260.

Meshes with Adapted Faces Size: Influence Zone

- To approximate with a small number of spheres we compute a region around a vertex v_i
- We collect vertices v' in a breadth-first manner as long as the distance with C_i is within tolerance value ε

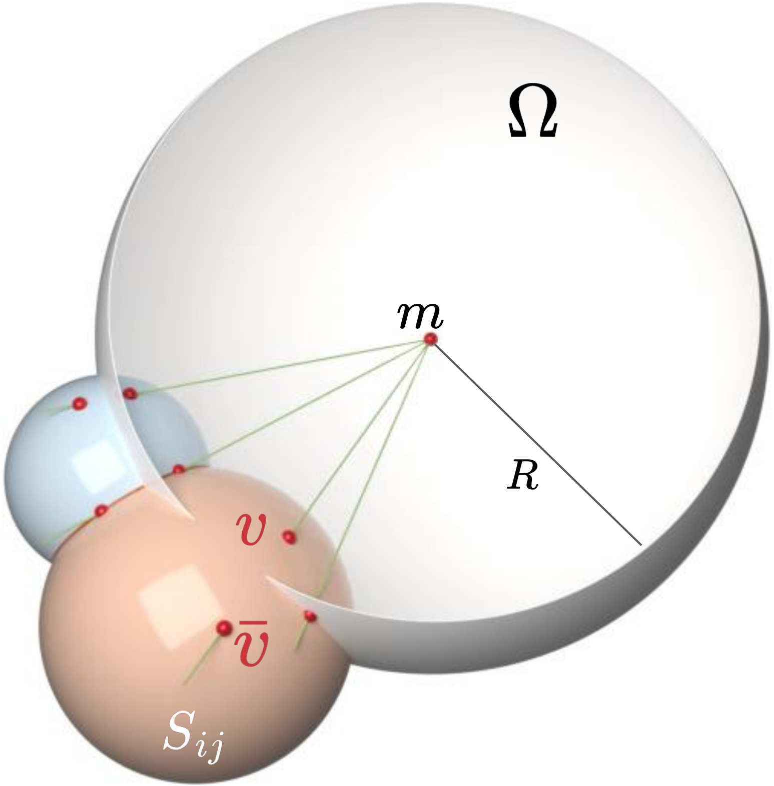


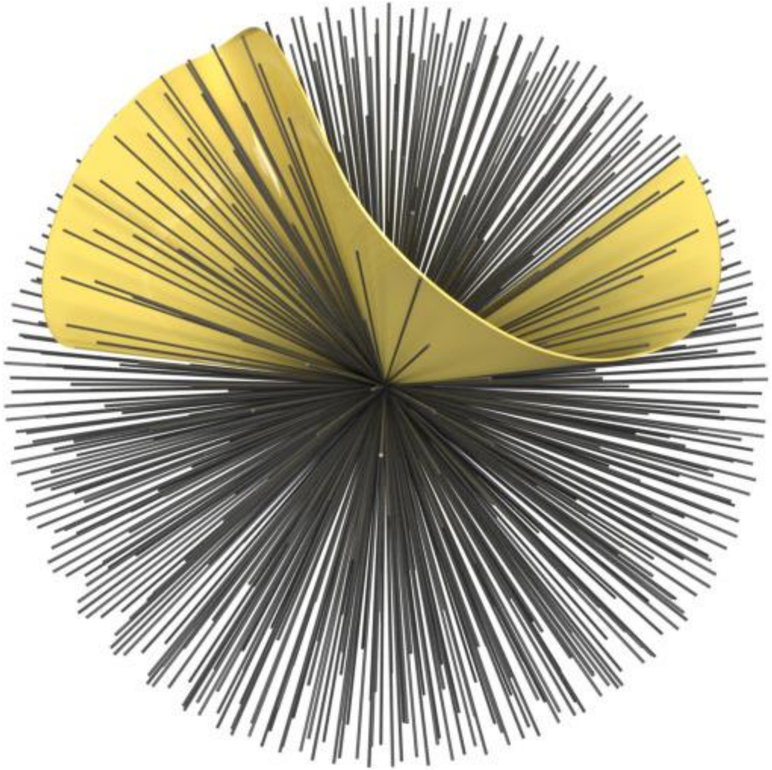
Meshes with Adapted Faces Size



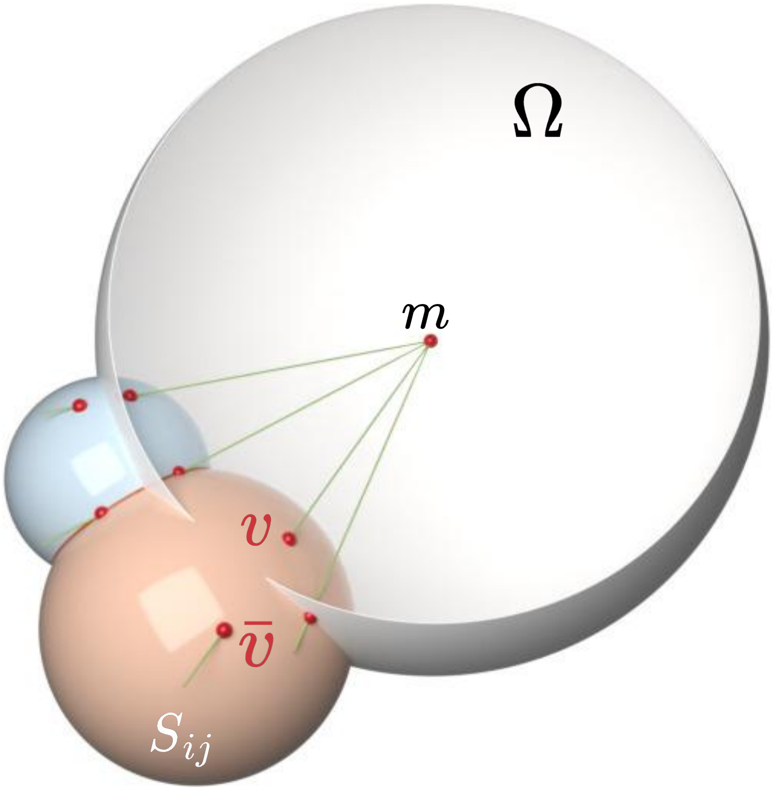
ST-Meshes
Support Structure

ST-Meshes and Sphere Linear Complexes



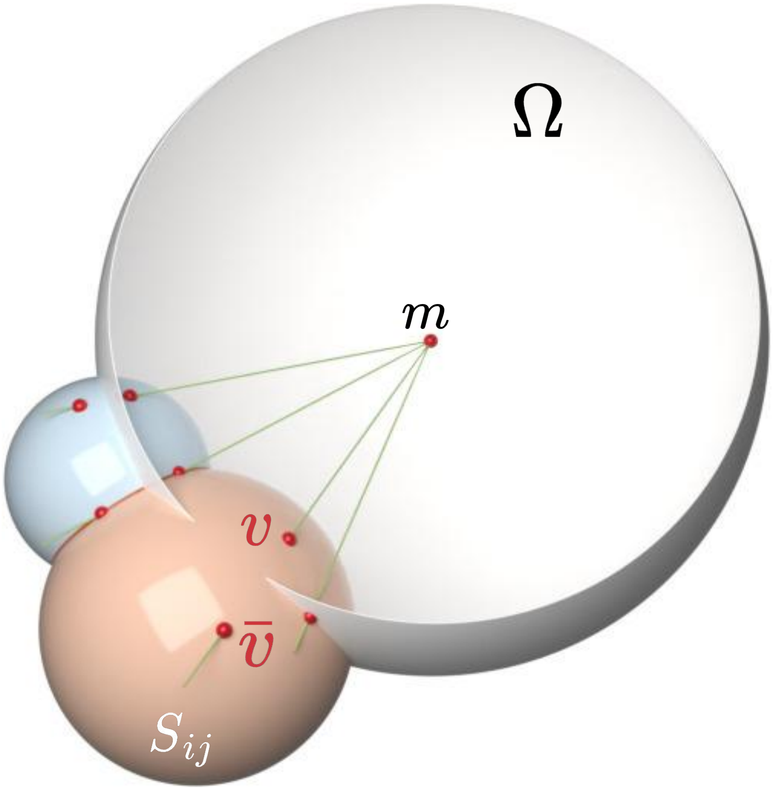


Meshes with Support Structure



- The **absolute sphere** Ω can have real, imaginary or ∞ (absolute plane) radius.
- The spheres S_{ij} lie in a linear sphere complex and are orthogonal to Ω

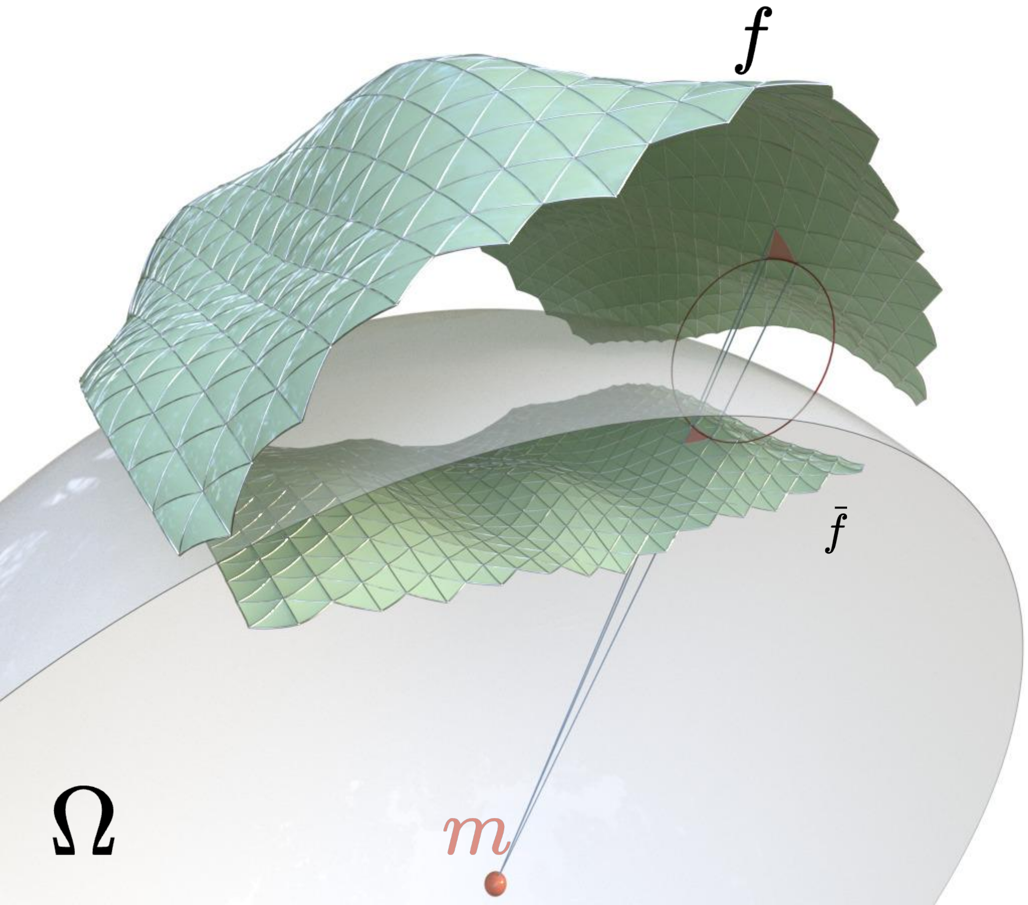
Meshes with Support Structure



- The relation between the envelope points is given by

$$R^2 = mv \cdot m\bar{v}$$

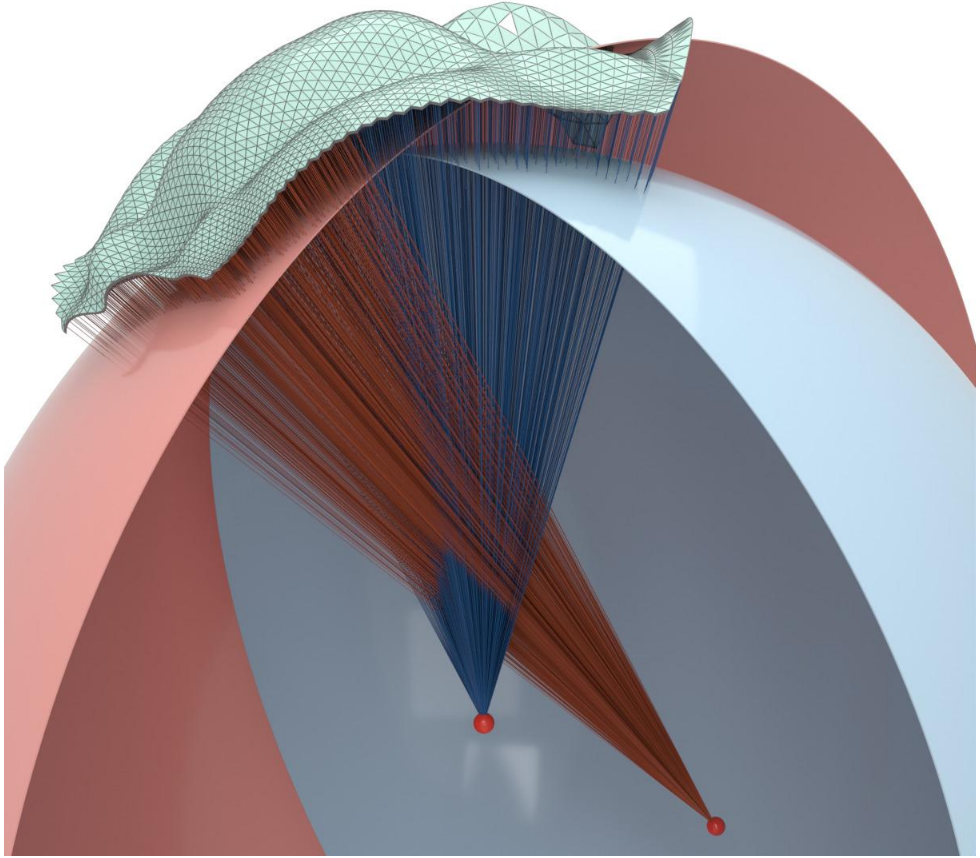
Meshes with Support Structure



- Both $f, \bar{f}$ are ST-meshes with support structure.
- $f, \bar{f}$ are related by a sphere inversion.

ST-Meshes with Support Structure

- In applications with larger meshes we can use more sphere meshes

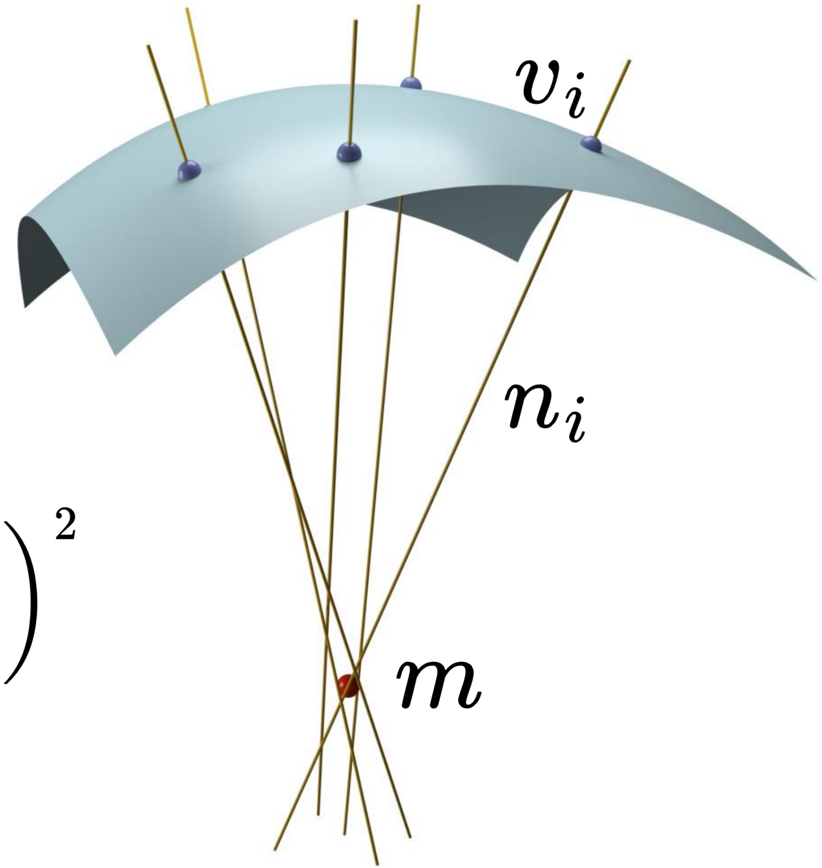


ST-Meshes: Algorithm

- **Estimating the center** m of the absolute sphere

- Minimize the angle between the line congruence (support structure) with the normals of the reference surface

$$E_{center} = \sum_{v_i \in V} \left(1 - \frac{\langle v_i - m,\ n_i \rangle}{||v_i - m||^2}\right)^2$$

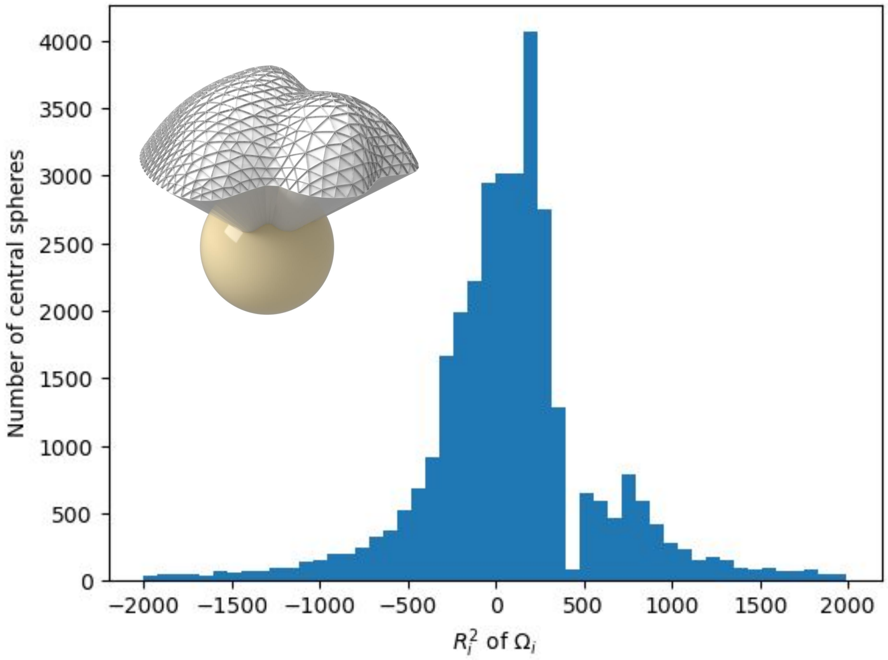


ST-Meshes: Algorithm

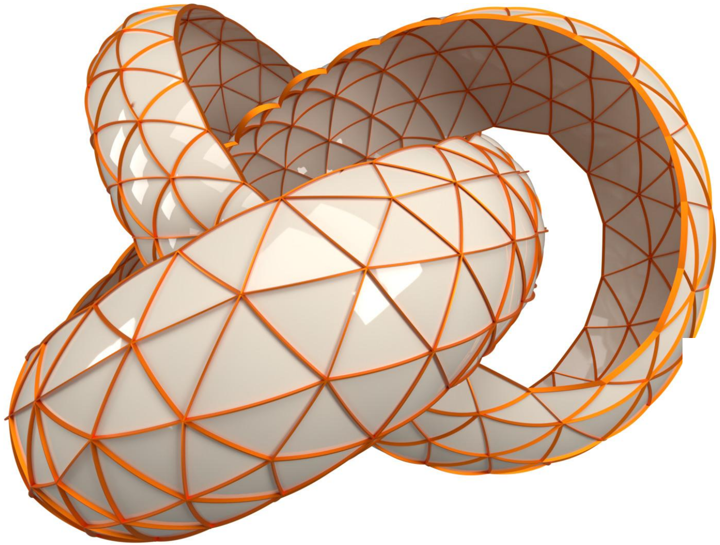
- **Estimating the radius** R

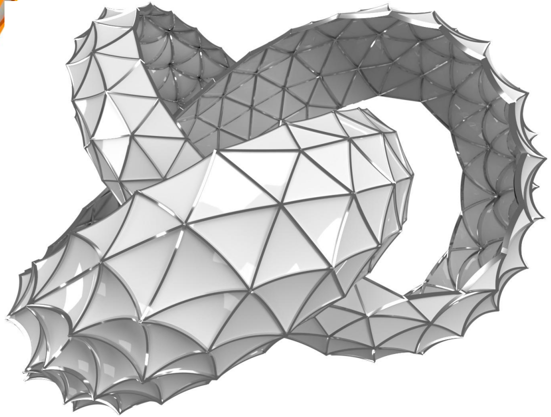
- For each central sphere C_i we compute a radius R_i of an absolute sphere Ω_i

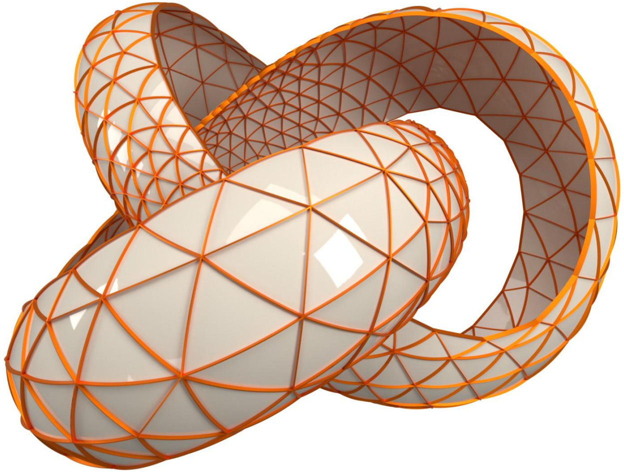
- We take R as the mean over all R_i



ST-Meshes: Algorithm

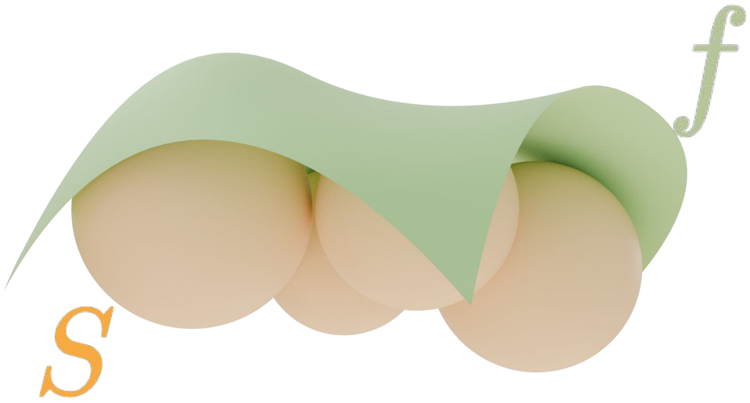






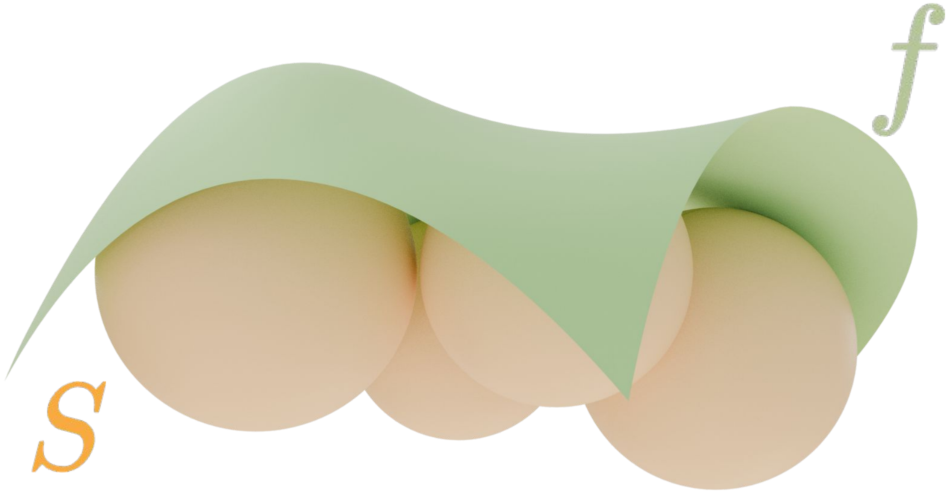
SQ-Meshes

SQ Meshes: Setup



1. Consider a **grid** over a **B-Spline** surface.
2. Use **central spheres** as the initial sphere congruence guess.
3. Optimize **line congruence** to satisfy angle constraints.
4. Optimize **torsal directions** to satisfy angle constraints.
5. Obtain a **torsal direction field** over the grid.
6. **Remesh** following the torsal direction.
7. Perform **post-optimization** to fit spheres, support structure, and planarity of torsal planes.

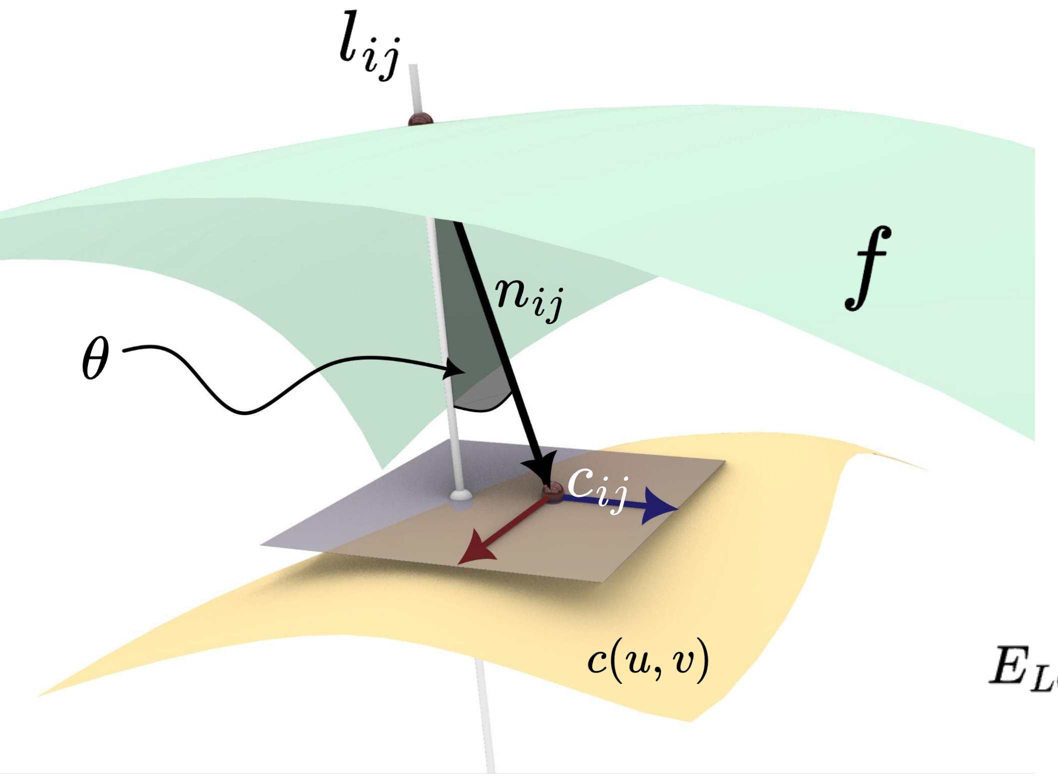
SQ Meshes: Setup



- We consider our input **reference** surface $f(u,v)$ as a **B-Spline** surface with degree at least 5 on each parameter u, v
- Initialize $S(u,v)$ as a central sphere congruence
- We use a radius function represented in B-Spline form

$$r(u,v) = \sum_{kl} N_k^m(u) N_l^n(v)\, r_{kl}$$

SQ Meshes: Line Congruence Optimization



- Variables: $r_{kl},\ l_{ij}$
- Line congruence condition

$$l = c_u \times c_v$$

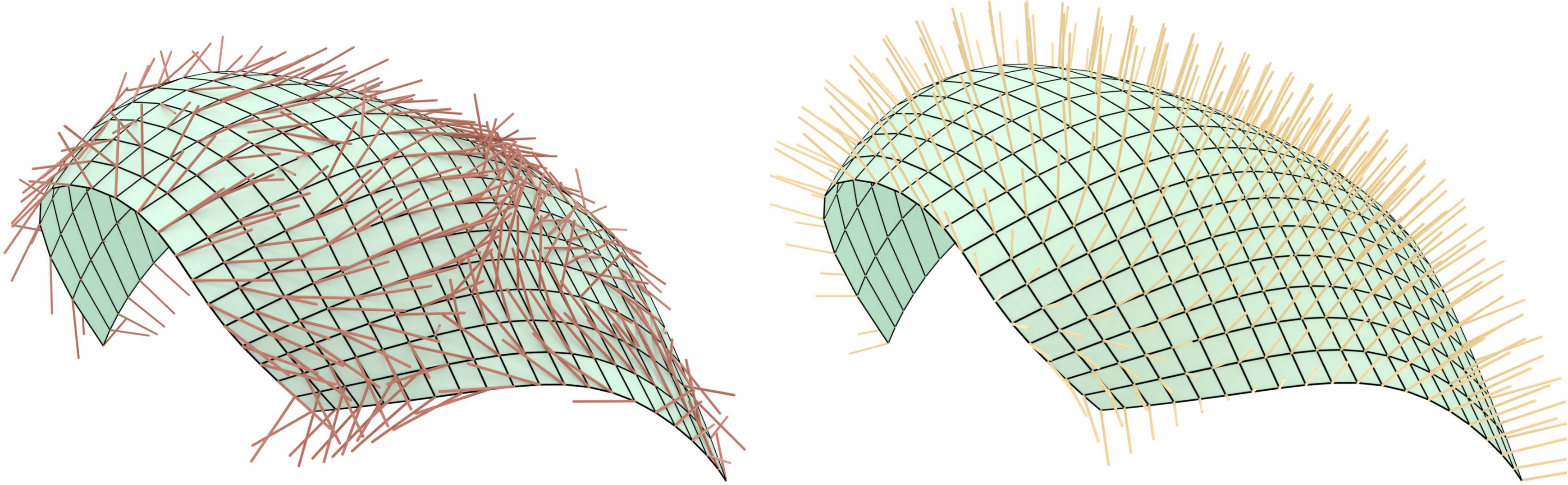
$$E_{LC} = \sum_{i,j \in G} \langle l_{ij},\ c_{ij,u} \rangle^2 + \langle l_{ij},\ c_{ij,v} \rangle^2$$

- Angle constraint

$$\langle l_{ij}, n_{ij} \rangle \leq \theta$$

$$E_{LC_{orth}} = \sum_{i,j \in G} \left(\langle l_{ij},\ n_{ij} \rangle^2 - \theta^2 - \mu_{ij} \right)^2$$

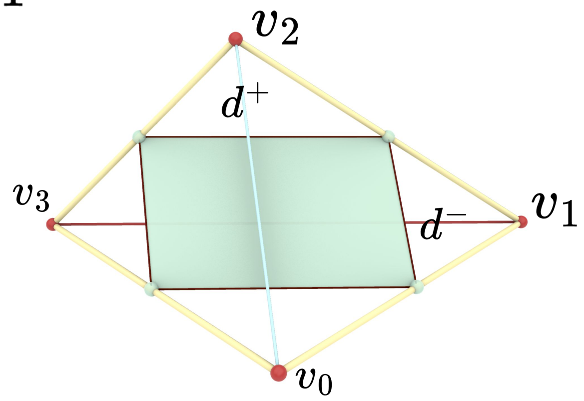
SQ Meshes: Line Congruence

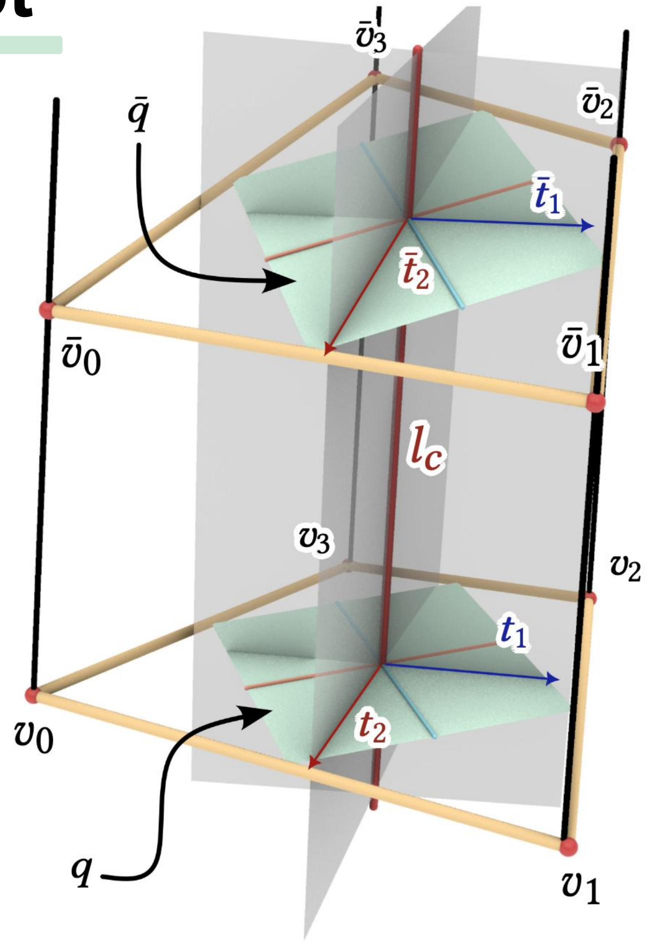


SQ Meshes: Torsal Directions Opt

- Checkerboard Pattern approach
- Approximate derivatives by diagonal directions of the quads

$$t = \mu_0 d^+ + \ \mu_1 d^-$$
$$l_t = \mu_0 l^+ + \ \mu_1 l^-$$



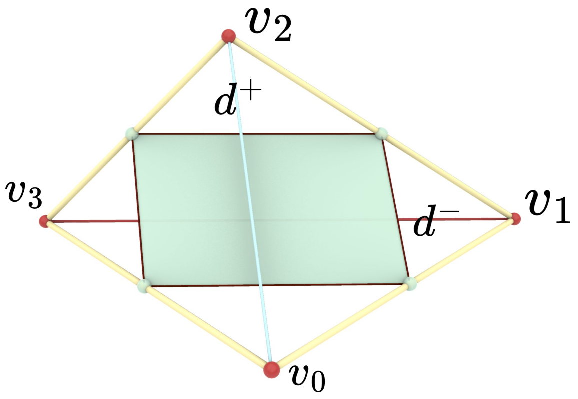


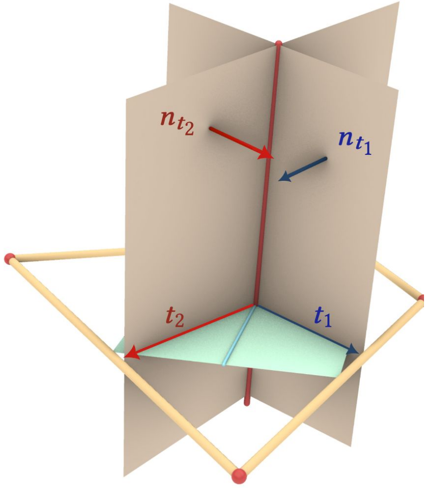
SQ Meshes: Torsal Directions Opt

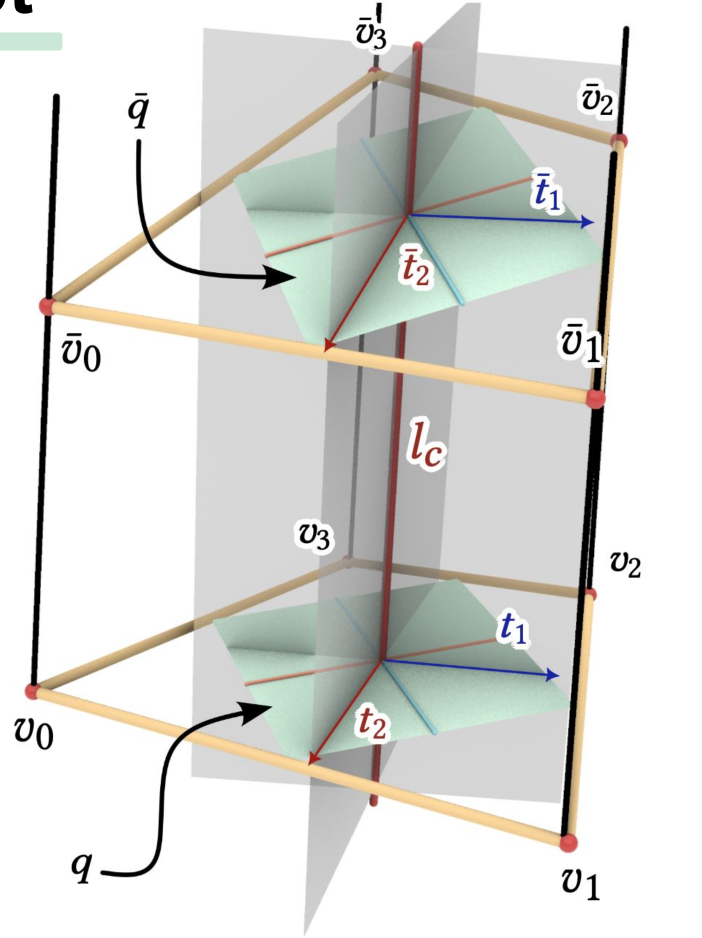
- Torsal directions = Planarity optimization

$$det(l_t, l_c, t) = 0$$

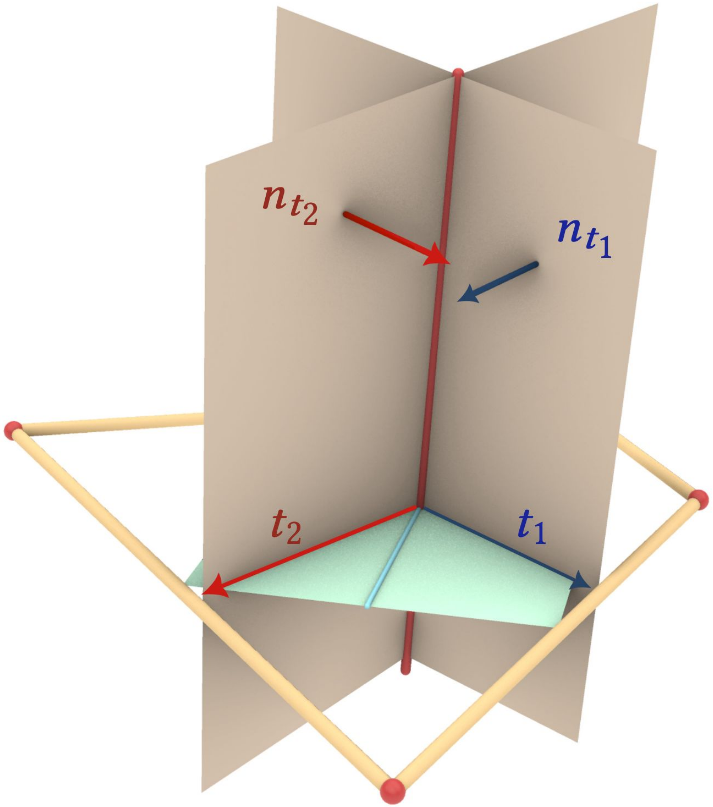
$$E_{torsal} = \sum_{q \in Q} \left\langle \frac{l_t}{||l_t||}, n_t \right\rangle^2 + \left\langle \frac{l_c}{||l_c||}, n_t \right\rangle^2 + \langle t, n_t \rangle^2$$







SQ Meshes: Torsal Directions Opt



- Angle constraint

$$\langle n_{t_1}, n_{t_2} \rangle \leq \cos(\alpha)$$

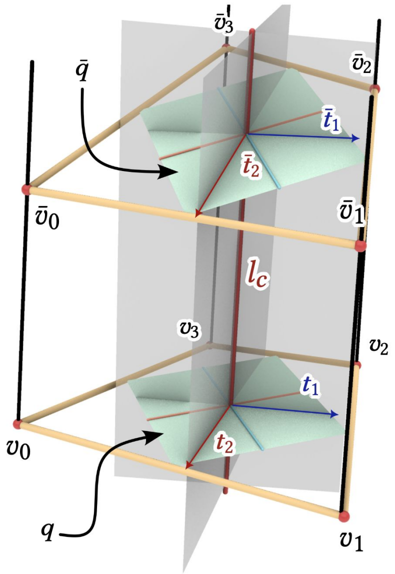
$$E_{torsal} = \sum_{q \in Q} \left(\langle n_{t_1}, n_{t_2} \rangle^2 - \cos^2(\alpha) + \nu_q^2 \right)^2$$

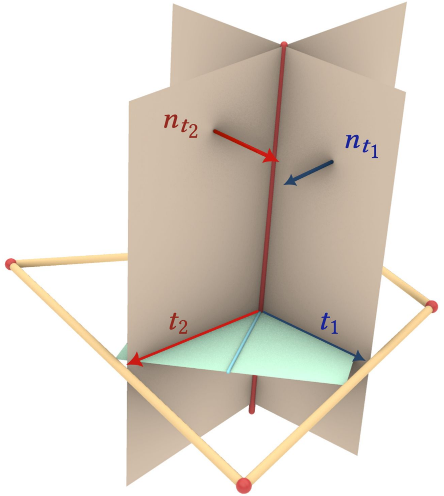
SQ Meshes: Torsal Directions Opt

Variables: r_{ij}, l_{ij}, μ_t, ν_t, n_t, μ_{ij}, ν_q

- Planarity of torsal planes

$$E_{torsal} = \sum_{q \in Q} \left\langle \frac{l_t}{||l_t||}, n_t \right\rangle^2 + \left\langle \frac{l_c}{||l_c||}, n_t \right\rangle^2 + \langle t, n_t \rangle^2$$

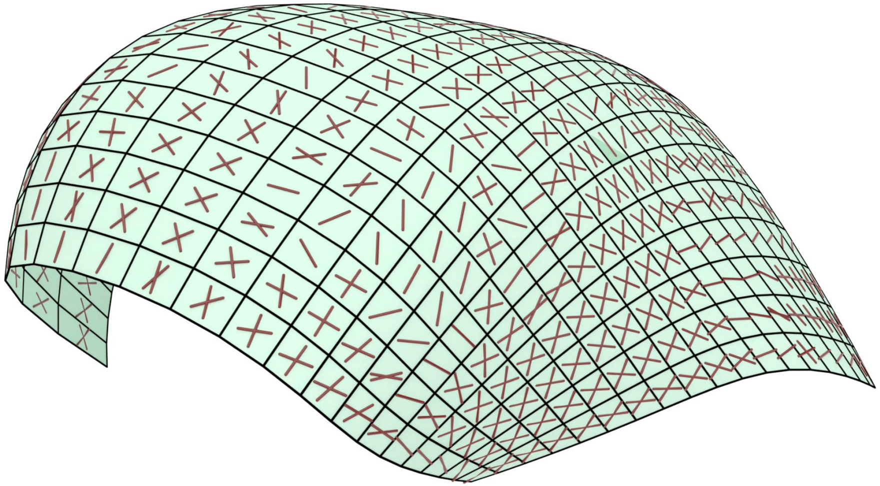


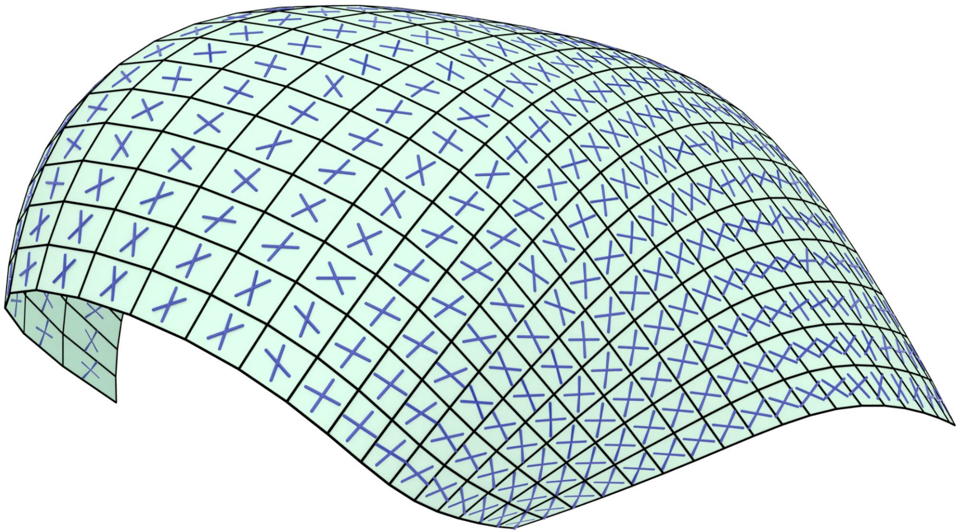


- Torsal planes angles

$$E_{t_{angle}} = \sum_{q \in Q} \left(\langle n_{t_1}, n_{t_2} \rangle^2 - \cos^2(\alpha) + \nu_q^2 \right)^2$$

SQ Meshes: Torsal Directions

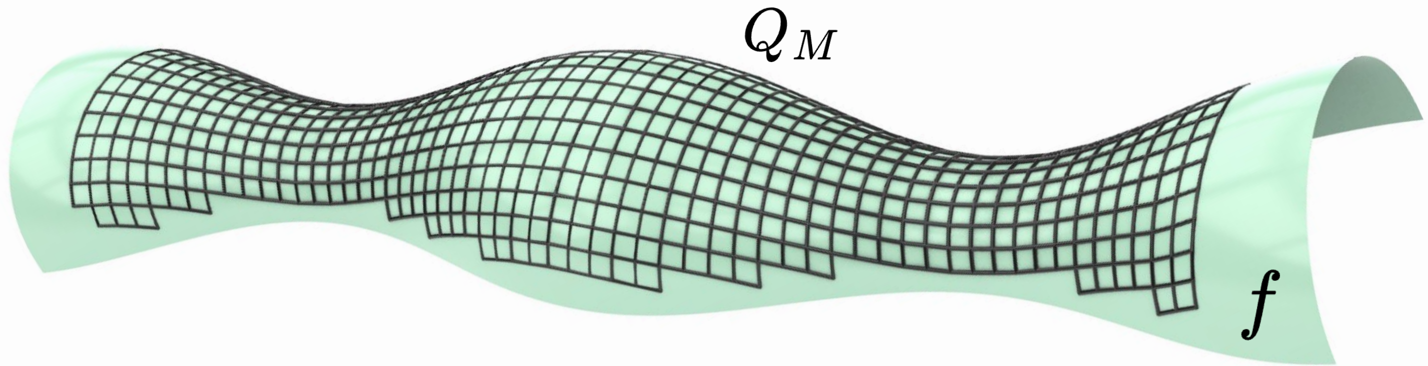




SQ Meshes: Remeshing

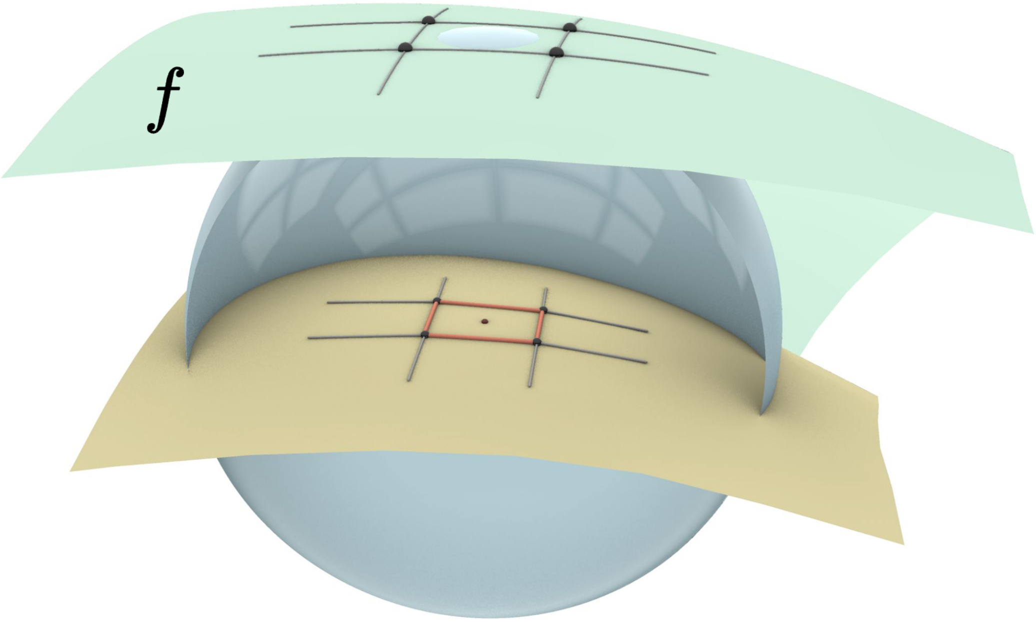
- We use IGL **Anisotropic Remeshing** [Panozzo et al., 2014] and **LibQEx** [Ebke et al., 2013] to obtain a **quad mesh** aligned with the torsal field obtained from the optimization.

- Q_M Remeshed Mesh
- f Reference surface



- Daniele Panozzo, Enrico Puppo, Marco Tarini, and Olga Sorkine-Hornung. 2014. Frame fields: anisotropic and non-orthogonal cross fields. ACM Trans. Graph. 33, 4, Article 134 (2014), 11 pages.
- Hans-Christian Ebke, David Bommes, Marcel Campen, and Leif Kobbelt. 2013. QEx: Robust quad mesh extraction. ACM Trans. Graph. 32, 6 (2013), 1–10.

SQ Meshes: Post Processing



- Implicit sphere equation

$$\psi(x) := Ax^2 - \langle B, x \rangle + C = 0$$

- Normalization factor

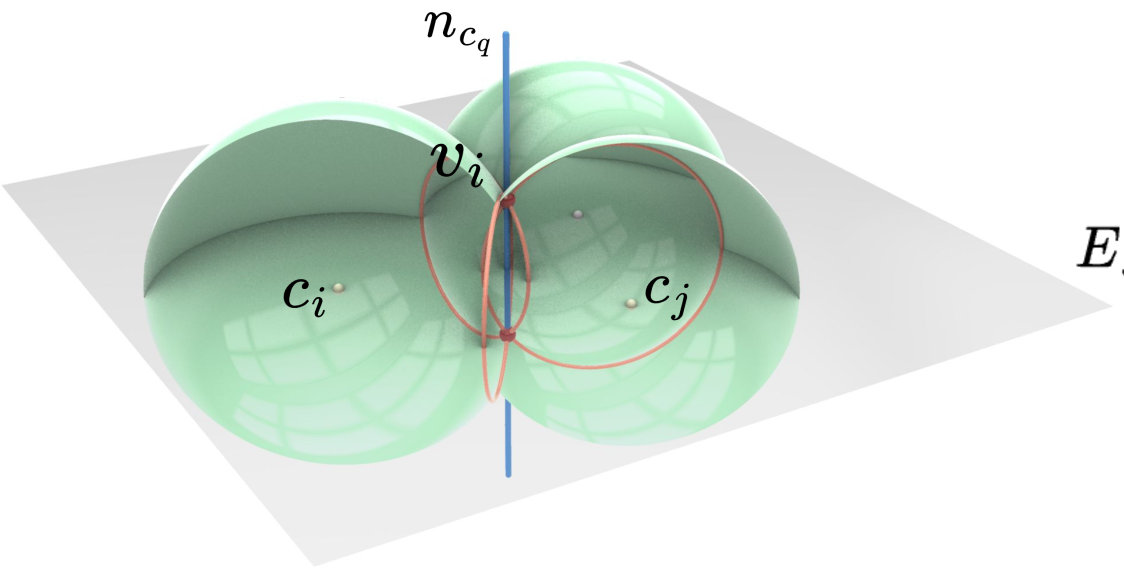
$$\varphi(f) := B_f^T B_f - 4A_f C_f - 1$$

- Sphere fitting

$$E_{sphere} = \sum_{f \in F} \sum_{q_i \in f} \psi(q_i)^2$$

$$E_{unit} = \sum_{f \in F} \varphi^2(f)$$

SQ Meshes: Post Processing

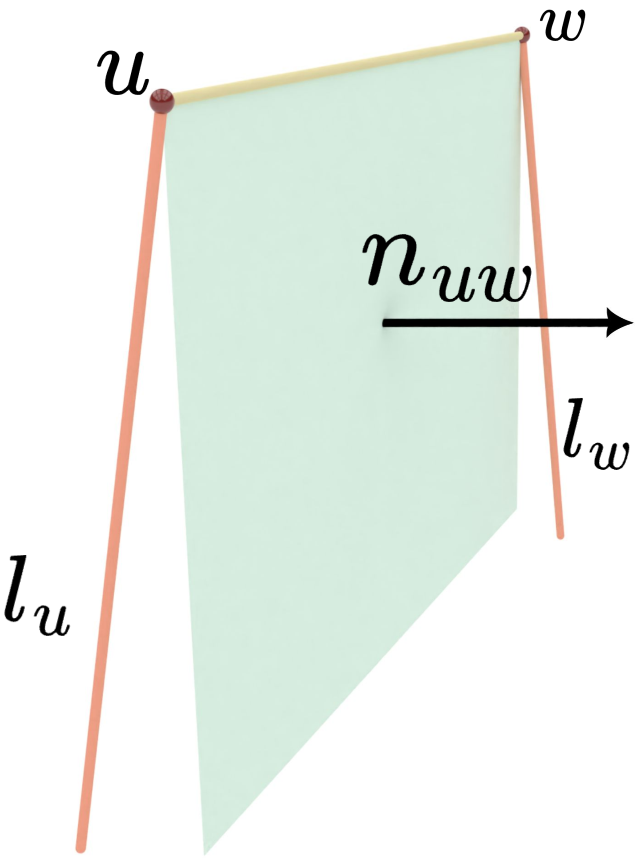


- Energy Support Structure

  Initial $n_{c_q} = l_i$

$$E_{supp} = \sum_{c_q} \sum_{c_i,\, c_j \in c_q} \langle c_i - c_j, n_{c_q} \rangle$$

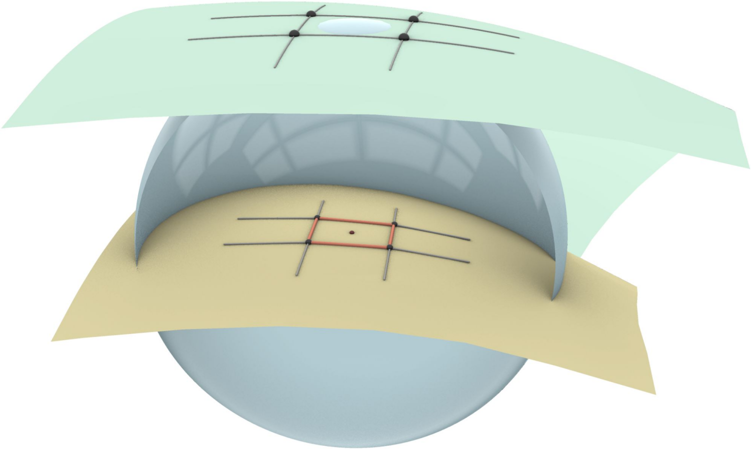
SQ Meshes: Post Processing



- Energy Planarity of Torsal planes

$$E_{\text{torsal plane}} = \sum_{uw \in edges} \langle l_u, n_{uw} \rangle^2 + \langle l_w, n_{uw} \rangle^2 + \left\langle \frac{w-u}{||w-u||}, n_{uw} \right\rangle$$

SQ Meshes: Post Processing



- Energy Support Structure

$$E_{supp} = \sum_{c_q} \sum_{c_i,\, c_j \in c_q} \langle c_i - c_j, n_{c_q} \rangle$$

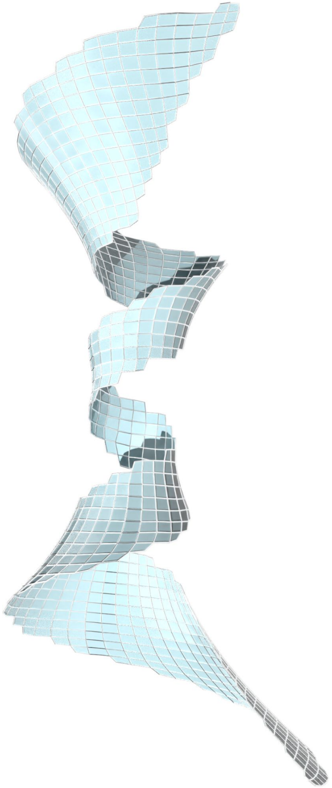
- Planarity of Torsal planes

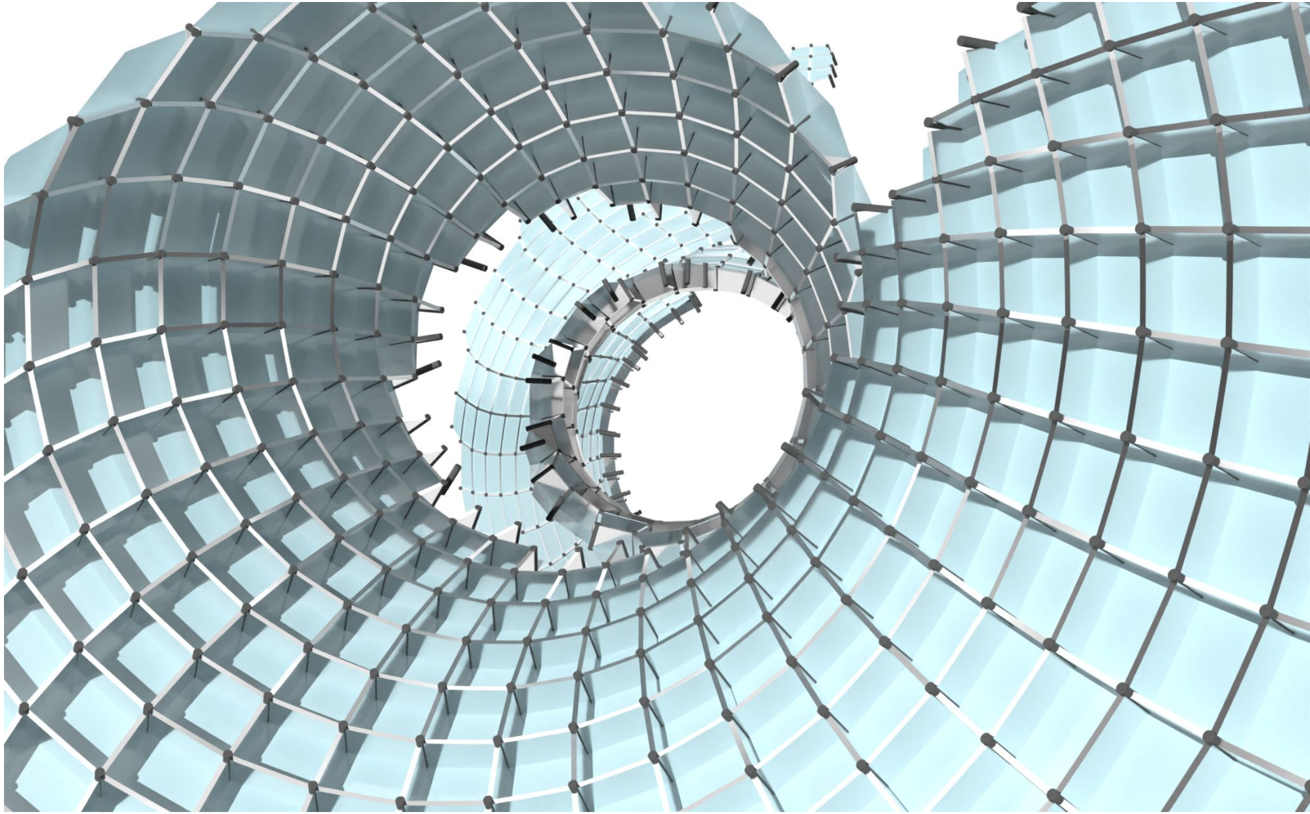
$$E_{\text{torsal plane}} = \sum_{uw \in edges} \langle l_u, n_{uw} \rangle^2 + \langle l_w, n_{uw} \rangle^2 + \left\langle \tfrac{w-u}{||w-u||}, n_{uw} \right\rangle$$

- Sphere fitting

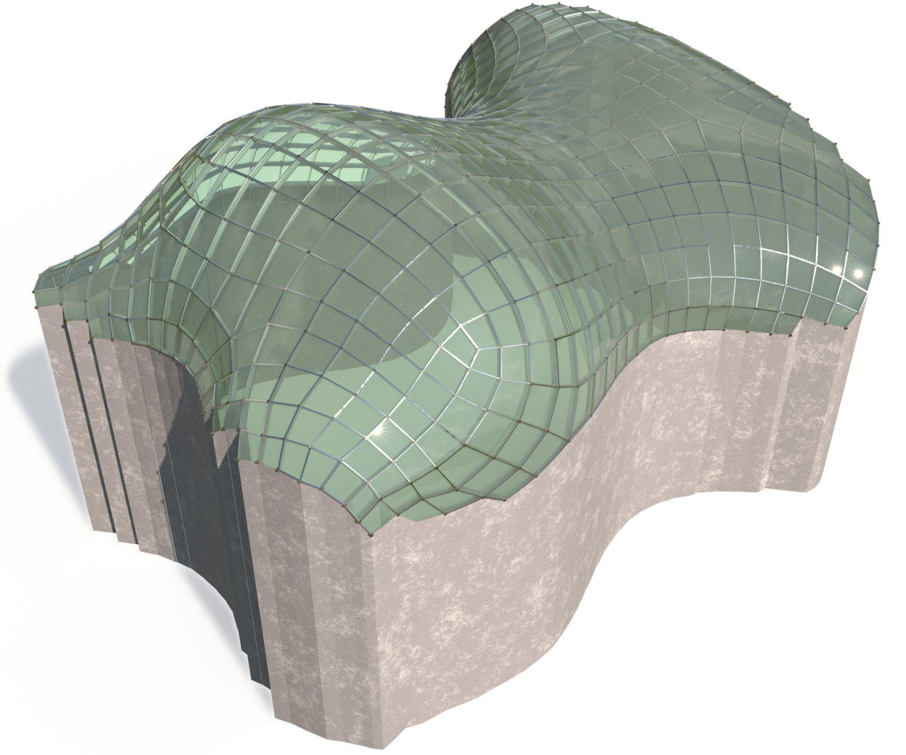
$$E_{sphere} = \sum_{f \in F} \sum_{q_i \in f} \psi(q_i)^2 \quad E_{unit} = \sum_{f \in F} \varphi^2(f)$$

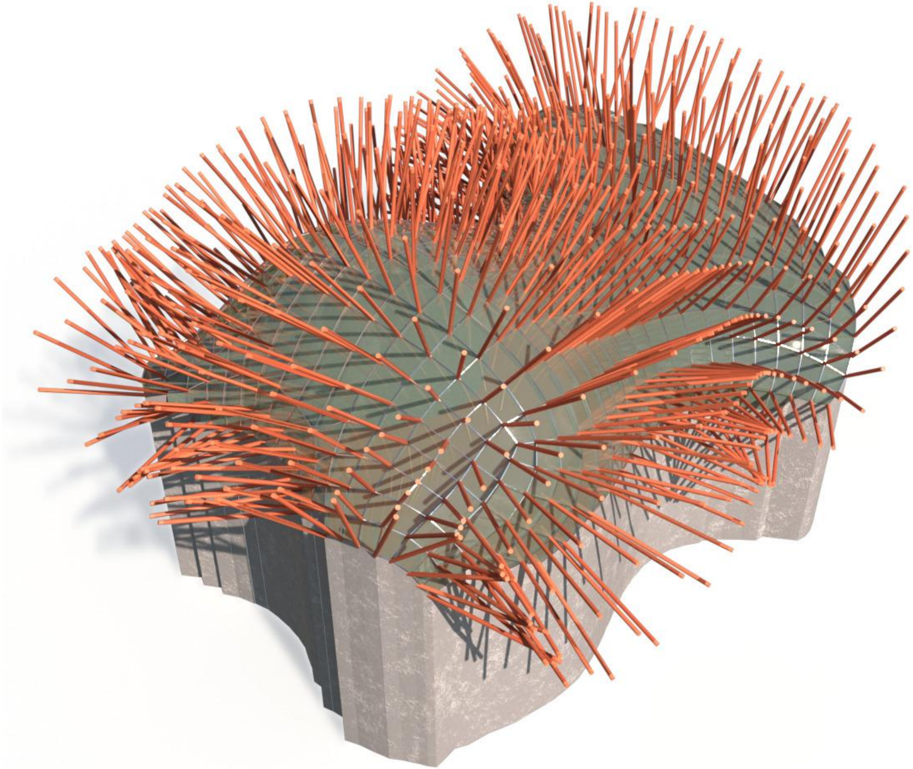
SQ Meshes

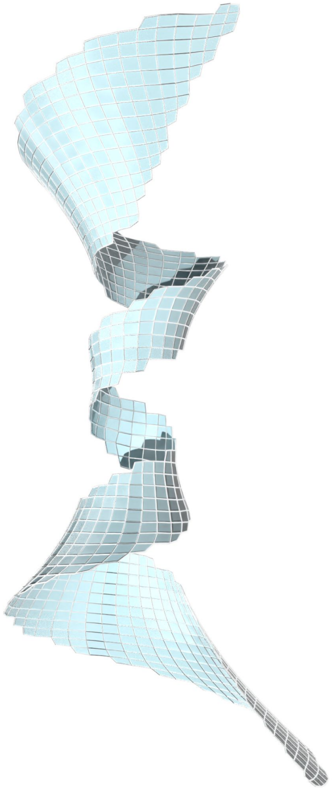
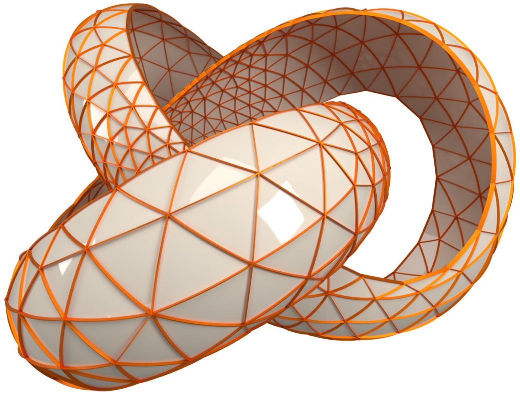
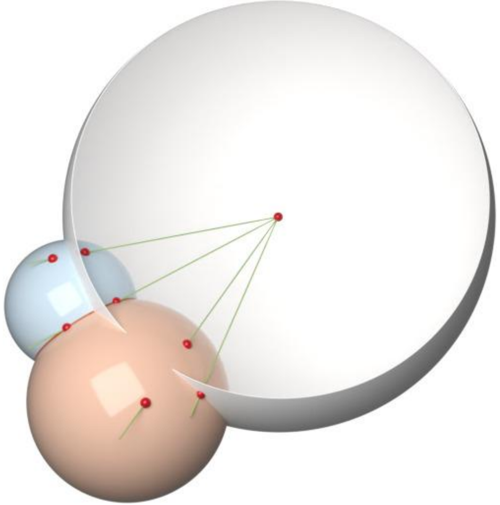




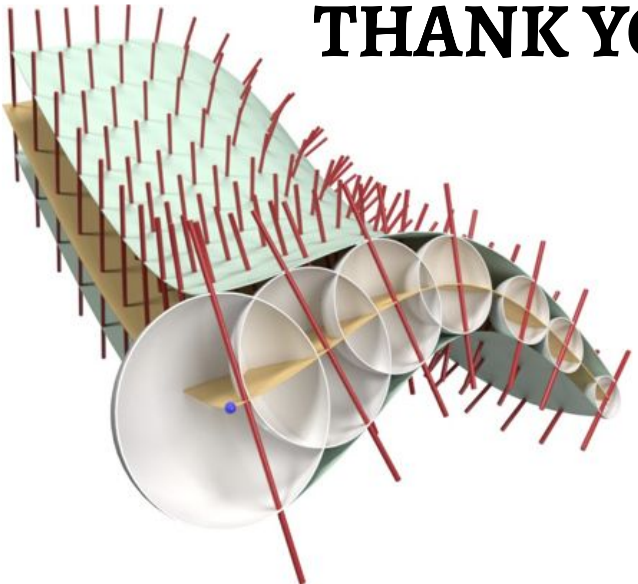
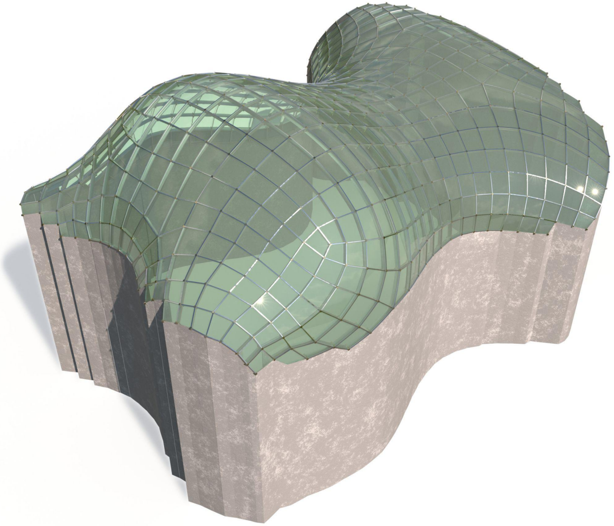
SQ Meshes

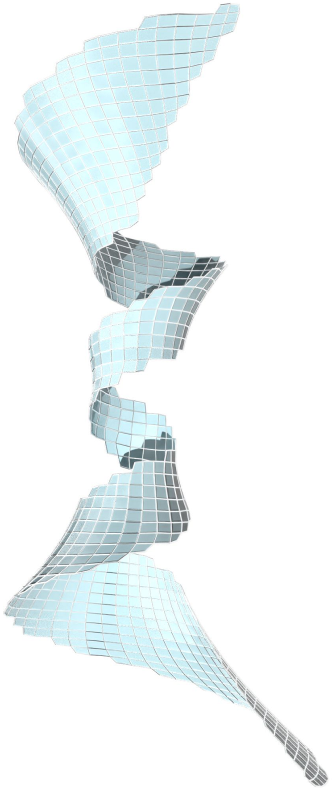


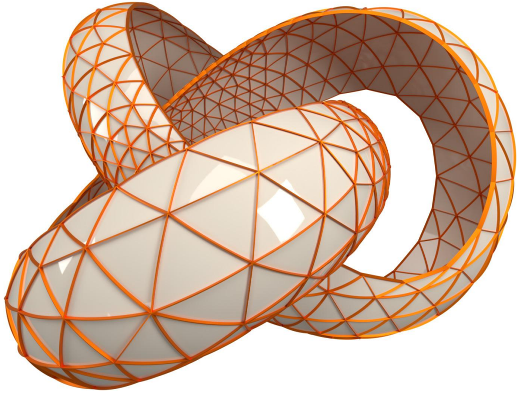


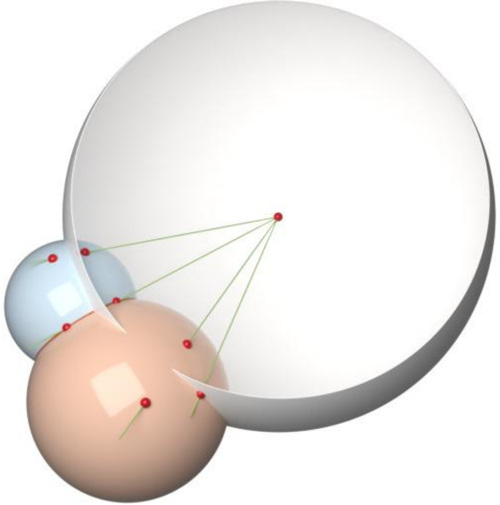




THANK YOU !

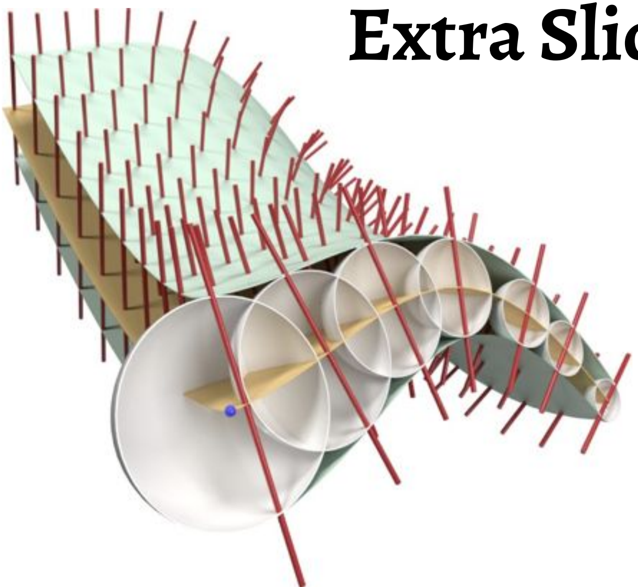



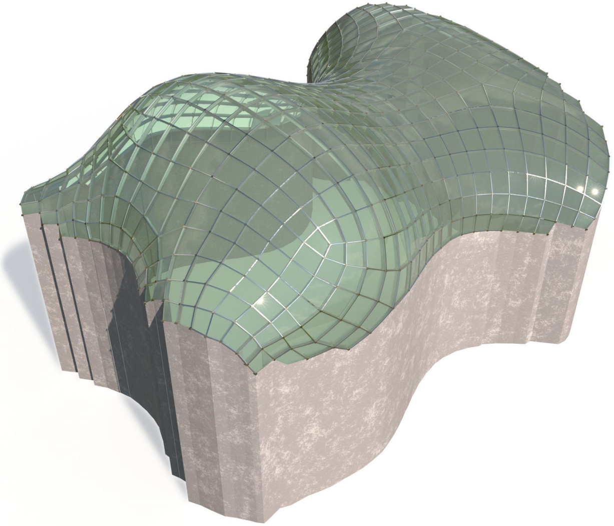




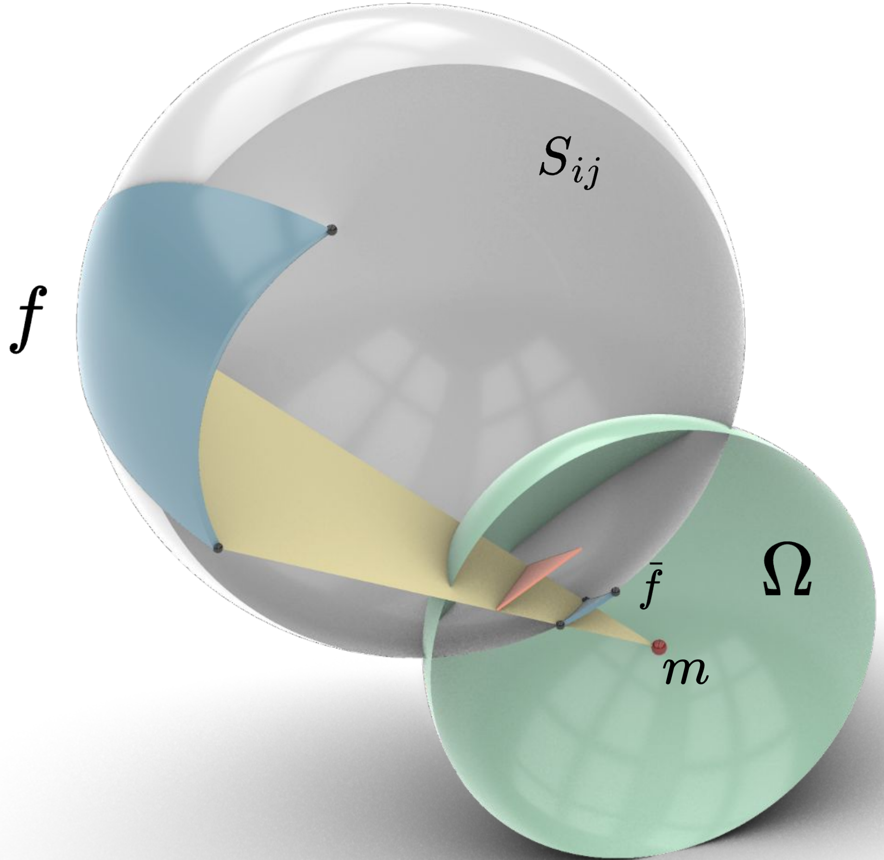


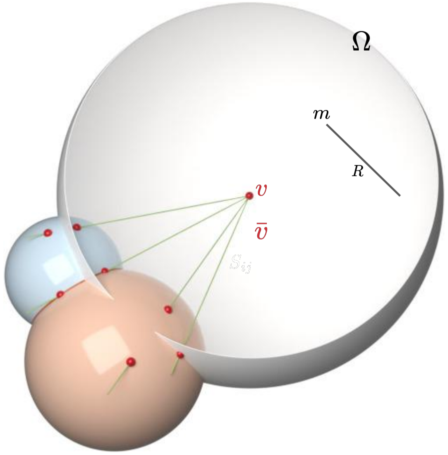
Extra Slides



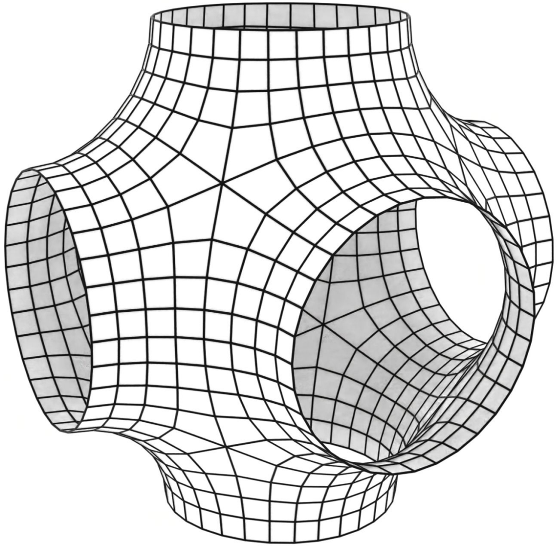


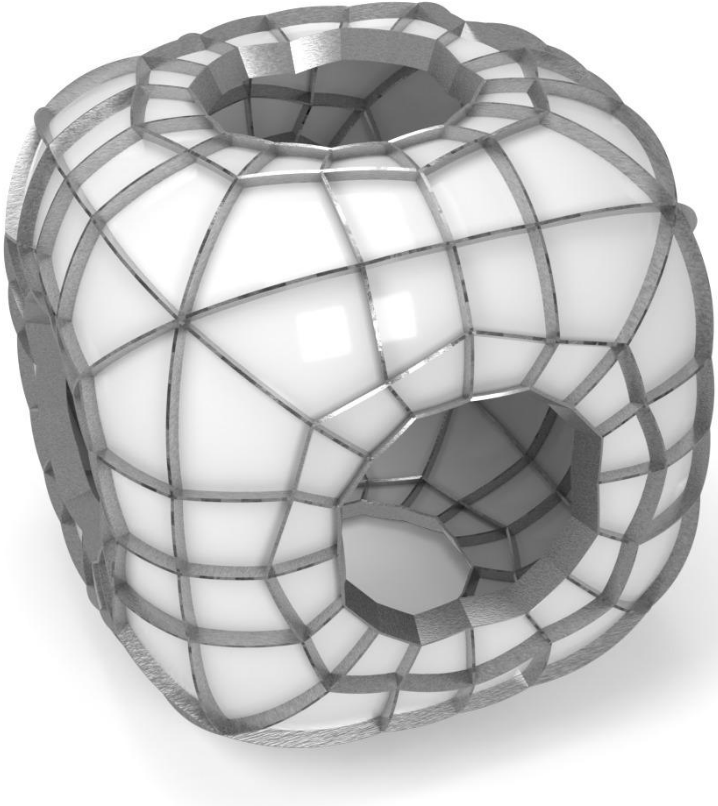
ST-Meshes and Sphere Linear Complexes



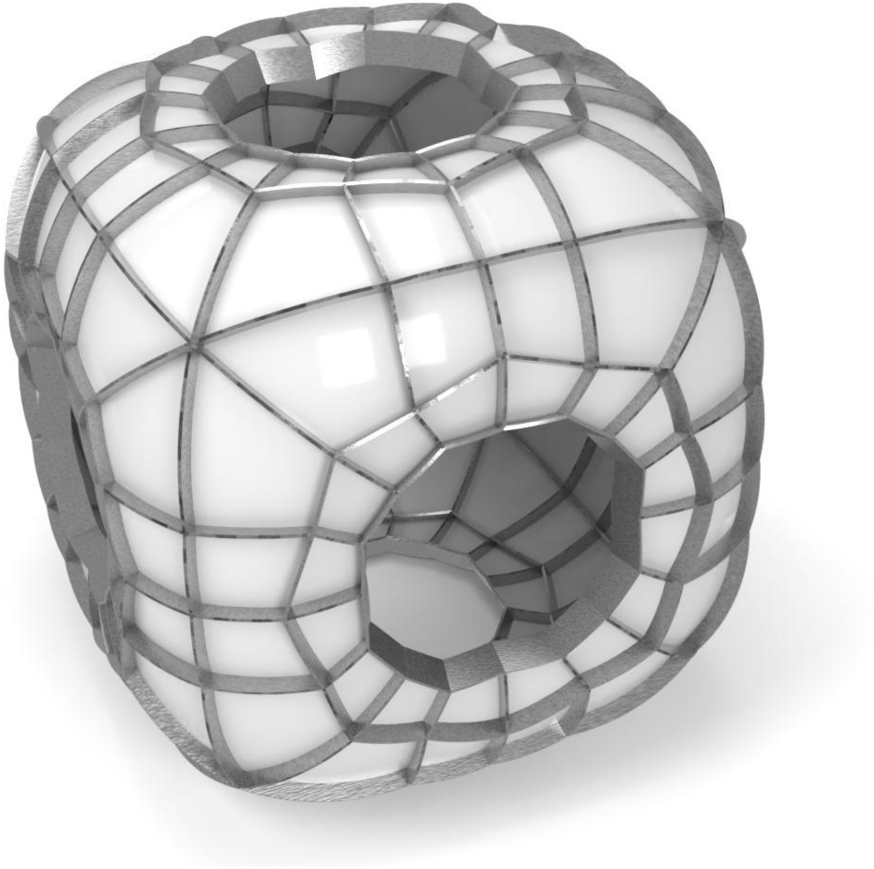


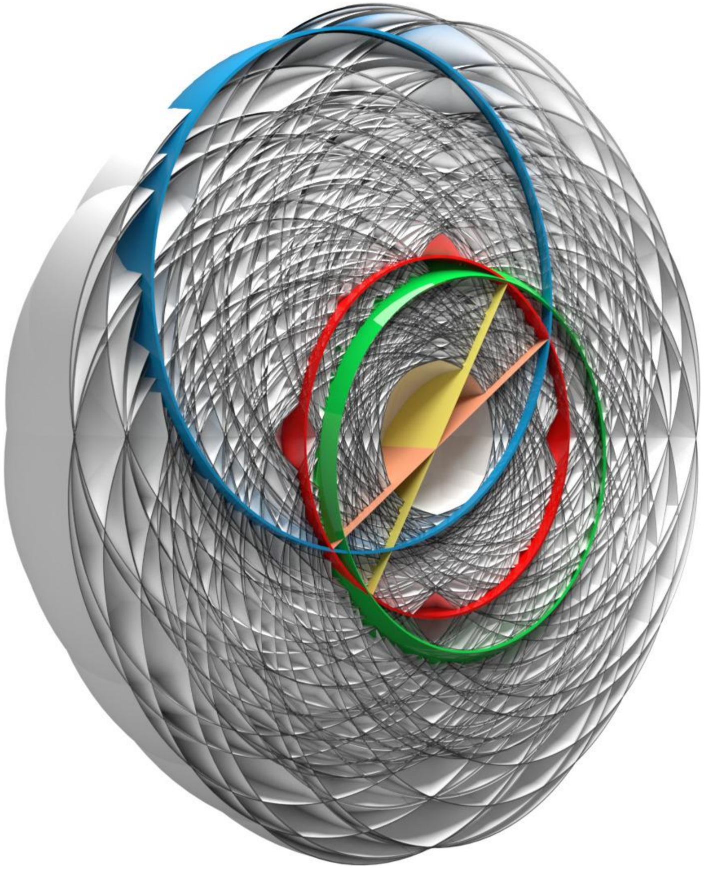
Extra slides: Non-Euclidean Minimal Surfaces





Extra slides: Non-Euclidean Minimal Surfaces





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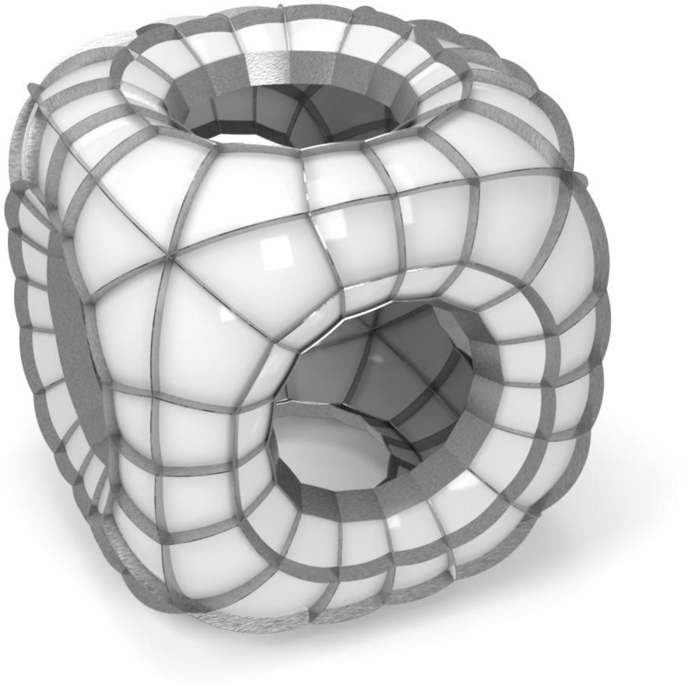

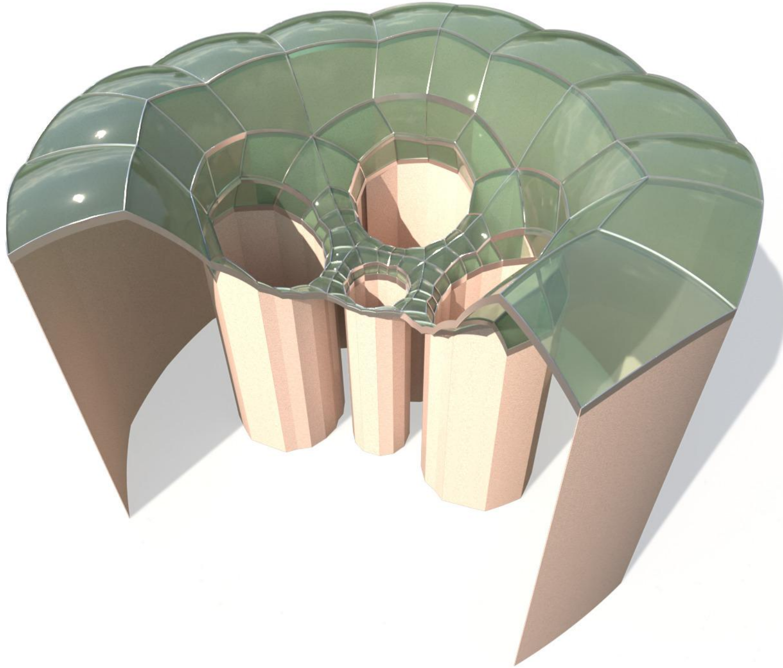

Meshes with Support Structure

$S \perp \Omega$ | $\mathrm{Opt}\ R$

Pliant Remeshing with central spheres

Least squares spheres orthogonal to Ω