Kœnigs nets and autoconjugate curves

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Discrete Geometric Structures Workshop, Vienna

Technische Universität Berlin

Motivation

The aim is to discretise classical results concerning Laplace sequences of Kœnigs nets.

Collaboration with Niklas Affolter.

Outline

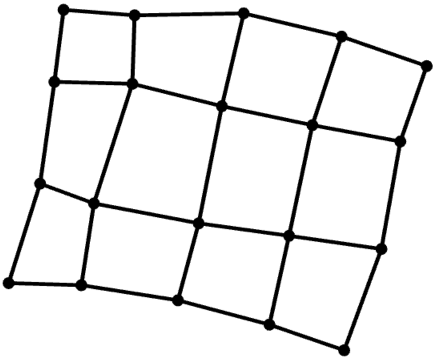
- Laplace sequences
- Two discretisations of Kœnigs nets
- Kœnigs nets with d-dim parameter lines
- Autoconjugate curves of quadrics

Definition

A function $P : \mathbb{Z}^2 \to \mathbb{RP}^n$ is a **Q-net** if $\forall (i,j) \in \mathbb{Z}^2$

$$P(i,j), P(i+1,j), P(i+1,j+1), P(i,j+1)$$

are coplanar points.



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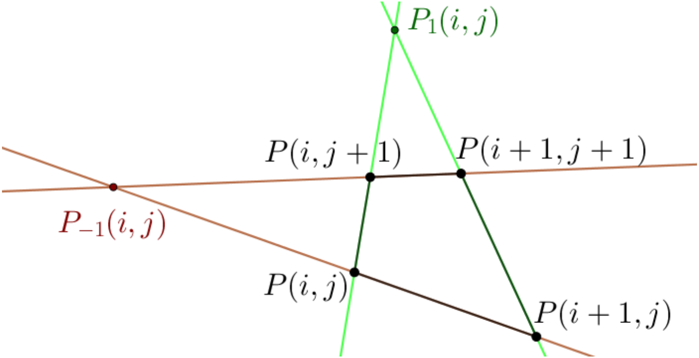
are coplanar points.

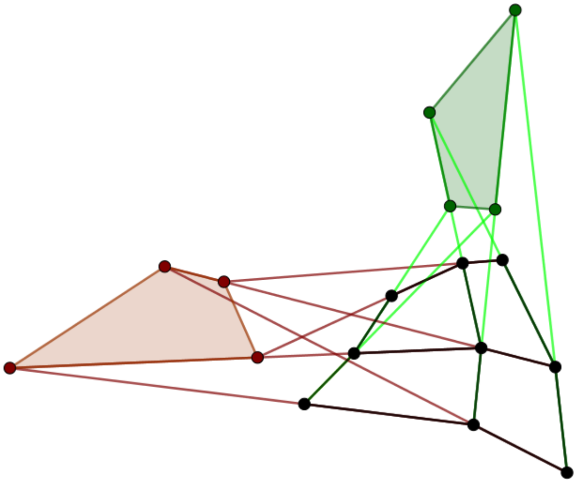
Q-nets were introduced by R. Sauer in the 1930s.

Q-nets provide a discretisation of *conjugate nets*.

Any Q-net $P : \mathbb{Z}^2 \to \mathbb{RP}^n$ has two Laplace transforms P_1 and P_{-1}



The Laplace transforms are also Q-nets.



So, Laplace transformations can be iterated.

The iterated Laplace transformations determine a Laplace sequence.

$$\ldots \leftarrow P_{-3} \leftarrow P_{-2} \leftarrow P_{-1} \leftarrow P \rightarrow P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow \ldots$$

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[A. Doliwa. *Geometric discretization of the Toda system*, 1997.]

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Definition

The Laplace sequence terminates if an iterated Laplace transform degenerates to a discrete curve.

There are two types of terminations.

Definition

Let P be a Q-net with Laplace transforms P_1 and P_{-1}

P_1 is Laplace degenerate if $P_1(i,j)$ is independent of i

P_1 is Goursat degenerate if $P_1(i,j)$ is independent of j

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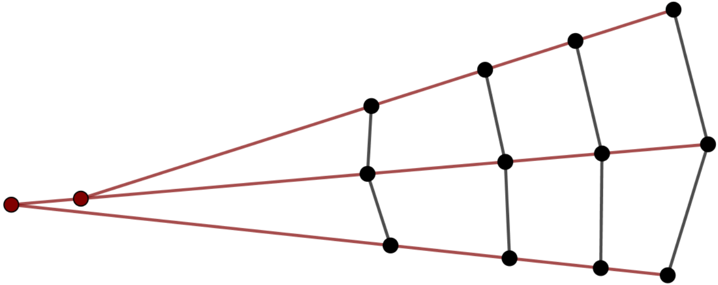
Example



Q-net with a Laplace transform that is Goursat degenerate.

It has one family of parameter lines that are straight lines.

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Definition

Let $P\colon \mathbb{Z}^2 \to \mathbb{RP}^n$ be a Q-net. The Laplace invariants $H : \mathbb{Z}^2 \to \mathbb{R}$ and $K : \mathbb{Z}^2 \to \mathbb{R}$ are defined as

$$H(i,j) := \mathrm{cr}(P(i,j), P_1(i,j), P(i,j+1), P_1(i-1,j)),$$

$$K(i,j) := \mathrm{cr}(P(i,j), P_{-1}(i,j), P(i+1,j), P_{-1}(i,j-1)).$$

[A. Doliwa. *Geometric discretization of the Toda system*, 1997.]

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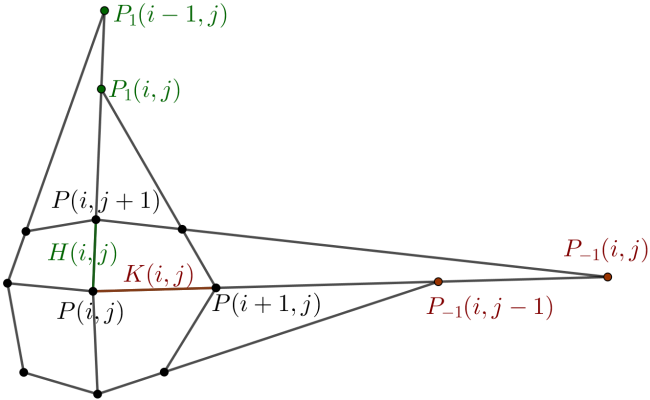
We would like to define discrete Koenigs nets as Q-nets with equal Laplace invariants. There are two ways to do this.



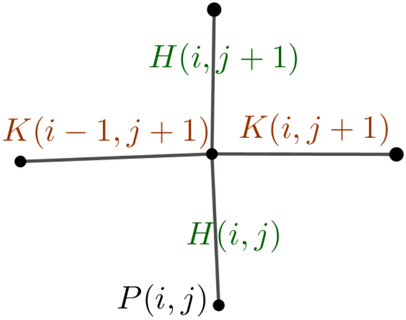
H is assigned to the vertical edges of $\mathbb{Z}^2$

K is assigned to the horizontal edges of $\mathbb{Z}^2$

Definition

Let $P\colon \mathbb{Z}^2 \to \mathbb{RP}^n$ be a Q-net. It is a **BS-Kœnigs net** if $\forall (i,j) \in \mathbb{Z}^2$

$$H(i,j) \cdot H(i,j+1) = K(i,j+1) \cdot K(i-1,j+1).$$



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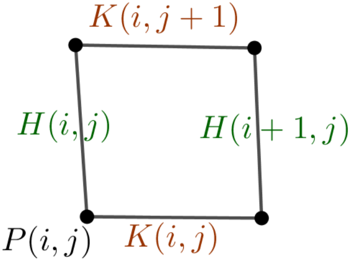
Equivalent to the Kœnigs nets of Bobenko and Suris.

[A. I. Bobenko and Y. B. Suris. *Discrete Koenigs Nets and Discrete Isothermic Surfaces*, 2009.]

Definition

Let $P\colon \mathbb{Z}^2 \to \mathbb{RP}^n$ be a Q-net. It is a D-Kœnigs net if $\forall (i,j) \in \mathbb{Z}^2$

$$H(i,j) \cdot H(i+1,j) = K(i,j) \cdot K(i,j+1)$$



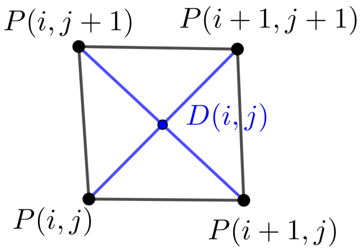
Definition

Let $P\colon \mathbb{Z}^2 \to \mathbb{RP}^n$ be a Q-net. It is a D-Kœnigs net if $\forall (i,j) \in \mathbb{Z}^2$

$$H(i,j) \cdot H(i+1,j) = K(i,j) \cdot K(i,j+1)$$

Equivalent to the discrete Kœnigs nets introduced by Doliwa.

[A. Doliwa. *Geometric discretization of the Koenigs nets*, 2003.]

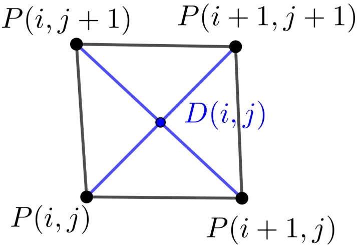


Definition

Let $P : \mathbb{Z}^2 \to \mathbb{RP}^n$ be a Q-net.

Define the diagonal intersection net $D : \mathbb{Z}^2 \to \mathbb{RP}^n$ such that

$$D(i,j) := (P(i,j) \vee P(i+1,j+1)) \cap (P(i+1,j) \vee P(i,j+1)).$$



If $P : \mathbb{Z}^2 \to \mathbb{RP}^n$ is a BS-Kœnigs net, then the diagonal intersection net $D : \mathbb{Z}^2 \to \mathbb{RP}^n$ is a D-Kœnigs net.

[A. I. Bobenko, Y. B. Suris. *Discrete Differential Geometry: Integrable Structure*, 2008.]

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- ▶ Autoconjugate curves of quadrics

Definition

Let $P : \mathbb{Z}^2 \to \mathbb{RP}^n$ be a net.

It has *d-dimensional parameter lines* if $\forall i, j \in \mathbb{Z}$

$$U(j) := \mathrm{join}\{P(i,j) \mid i \in \mathbb{Z}\}$$

$$V(i) := \mathrm{join}\{P(i,j) \mid j \in \mathbb{Z}\}$$

are d-dimensional projective subspaces.

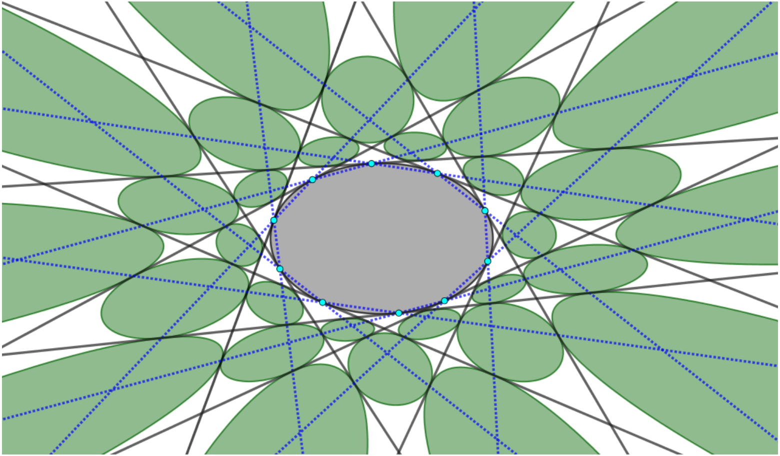
Theorem

Let $P : \mathbb{Z}^2 \to \mathbb{RP}^{2d}$ be a BS-Kœnigs net with d-dimensional parameter lines, $d \geq 1$. Then,

(i) there is a quadric $\mathcal{Q}$ such that each d-space $V(i)$ and each d-space $U(j)$ is tangent to $\mathcal{Q}$ along an isotropic $(d-1)$-space of $\mathcal{Q}$,

(ii) the diagonal intersection D-Kœnigs net $D : \mathbb{Z}^2 \to \mathbb{RP}^{2d}$ has d-dimensional parameter lines and the vertices of the Laplace transforms $D_{\pm 1}, \ldots, D_{\pm d}$ are contained in $\mathcal{Q}$,

(iii) $P_{\pm d}$ and $D_{\pm d}$ are Goursat degenerate.



[A. I. Bobenko and A. Y. Fairley. *Nets of Lines with the Combinatorics of the Square Grid and with Touching Inscribed Conics*, 2021.]

The case $d = 1$ of the theorem.

The theorem is a discretisation of a completely analogous theorem

for smooth Kœnigs nets with d-dimensional parameter lines.

[G. Tzitzéica. *Géométrie différentielle projective des réseaux*, 1924.]

[E. Bompiani. *Risoluzione geometrica del problema di Moutard sulla costruzione delle equazioni di Laplace ad integrale esplicito*, 1915.]

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Definition

Let $[\gamma]$ be a curve in $\mathbb{RP}^n$ represented by a smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^{n+1}$. For any $d \in \mathbb{N}$, the osculating d-space at $[\gamma(t)]$ is the projective subspace

$$[\mathrm{span}\{\gamma(t), \dot{\gamma}(t), \ldots, \gamma^{(d)}(t)\}].$$

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$$[\mathrm{span}\{\gamma(t), \dot{\gamma}(t), \ldots, \gamma^{(d)}(t)\}].$$

Geometric interpretation: limit of the space spanned by $d+1$ points on the curve near $[\gamma(t)]$.

In general, the osculating d-space is d-dimensional provided $d \leq n$.

Definition

A curve in $\mathbb{RP}^n$ is an autoconjugate curve of a quadric $\mathcal{Q}$ if for every point P of the curve, the polar $P^\perp$ equals the osculating hyperplane of the curve at P.

Definition

A curve in $\mathbb{R}\mathrm{P}^n$ is an autoconjugate curve of a quadric $\mathcal{Q}$ if for every point P of the curve, the polar $P^\perp$ equals the osculating hyperplane of the curve at P.

In fact, it suffices to consider autoconjugate curves of quadrics in $\mathbb{R}\mathrm{P}^n$ for *even* n.

Equivalent definition (which is easier to discretise)

A curve in $\mathbb{R}\mathrm{P}^{2n}$ is an autoconjugate curve of a quadric $\mathcal{Q}$ if every osculating $(n-1)$-space is contained in $\mathcal{Q}$.

[G. Tzitzéica. *Géométrie différentielle projective des réseaux*, 1924.]

Example: an autoconjugate curve of a non-degenerate conic $\mathcal{C} \subset \mathbb{R}\mathrm{P}^2$ is just a piece of $\mathcal{C}$.

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Example: an autoconjugate curve of a quadric $\mathcal{Q} \subset \mathbb{R}\mathrm{P}^4$ of signature $(+ + + - -)$ is a curve in $\mathcal{Q}$ with tangent lines that are contained in $\mathcal{Q}$.

Definition

A function $\alpha\colon \mathbb{Z} \to \mathbb{RP}^{2n}$ is a discrete autoconjugate curve of a quadric $\mathcal{Q}$ if the join of any n consecutive points is a $(n-1)$-dimensional space that is contained in $\mathcal{Q}$.

Definition

A function $\alpha\colon \mathbb{Z} \to \mathbb{RP}^{2n}$ is a discrete autoconjugate curve of a quadric $\mathcal{Q}$ if the join of any n consecutive points is a $(n-1)$-dimensional space that is contained in $\mathcal{Q}$.

In fact, we have already seen examples of discrete autoconjugate curves.

Theorem

Let $P : \mathbb{Z}^2 \to \mathbb{RP}^{2d}$ be a BS-Kœnigs net with d-dimensional parameter lines, $d \geq 1$. Then,

(i) there is a quadric $\mathcal{Q}$ such that each d-space $V(i)$ and each d-space $U(j)$ is tangent to $\mathcal{Q}$ along an isotropic $(d-1)$-space of $\mathcal{Q}$,

(ii) the diagonal intersection D-Kœnigs net $D : \mathbb{Z}^2 \to \mathbb{RP}^{2d}$ has d-dimensional parameter lines and the vertices of the Laplace transforms $D_{\pm 1}, \ldots, D_{\pm d}$ are contained in $\mathcal{Q}$,

(iii) $P_{\pm d}$ and $D_{\pm d}$ are Goursat degenerate,

$D_{\pm d}$ are discrete autoconjugate curves.

Actually, there is a correspondence between pairs of discrete autoconjugate curves of quadrics and Kœnigs nets with d-dimensional parameter lines.

This is in complete agreement with the smooth theory.

[G. Tzitzéica. *Géométrie différentielle projective des réseaux*, 1924.]

Theorem

Let $\sigma, \tau : \mathbb{Z} \to \mathbb{RP}^{2d}$ be two discrete autoconjugate curves of a non-degenerate quadric hypersurface $\mathcal{Q} \subset \mathbb{RP}^{2d}$. For all $i, j \in \mathbb{Z}$, consider the osculating $(d-1)$-spaces

$$S(j) := \sigma(j) \vee \ldots \vee \sigma(j+d-1),$$

$$T(i) := \tau(i) \vee \ldots \vee \tau(i+d-1).$$

Let $S^{\perp}(j)$ and $T^{\perp}(i)$ be the polars. Define $P : \mathbb{Z}^2 \to \mathbb{RP}^{2d}$ such that $P(i,j) := S^{\perp}(j) \cap T^{\perp}(i)$. Then, P is a BS-Kœnigs net with d-dimensional parameter lines.

Thanks for your attention!