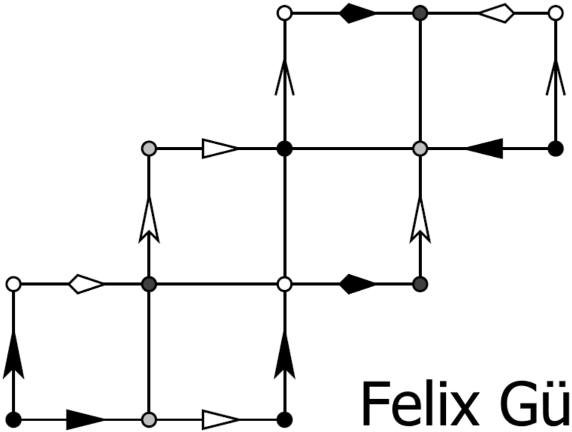
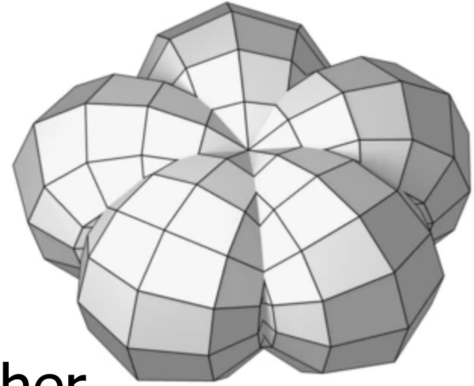
Convergence and Symmetry of Discrete Period Matrices

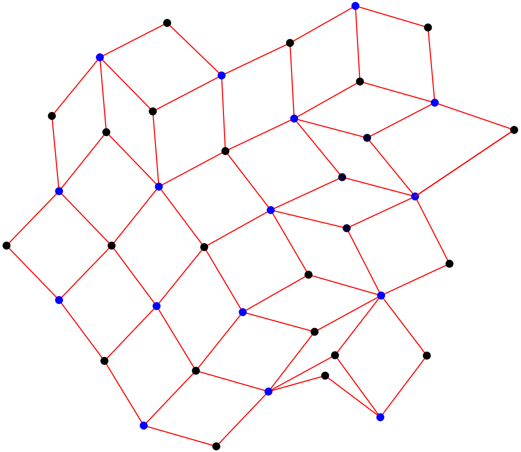




Felix Günther

With the support of the Marie Curie Alumni Association, through the MCAA-New-Horizon Project.

Discrete Riemann surface (Σ, Λ)



Λ strongly regular finite bipartite quad-graph on closed polyhedral surface Σ (Riemann surface!)

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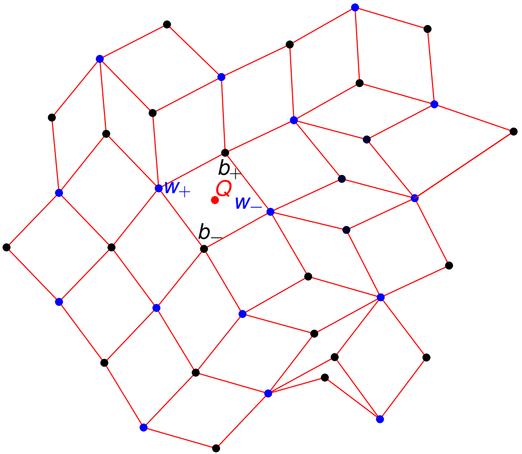

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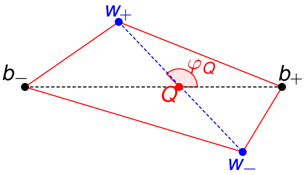

discrete complex structure

$$\rho_Q := -i\frac{z_Q(w_+) - z_Q(w_-)}{z_Q(b_+) - z_Q(b_-)}$$

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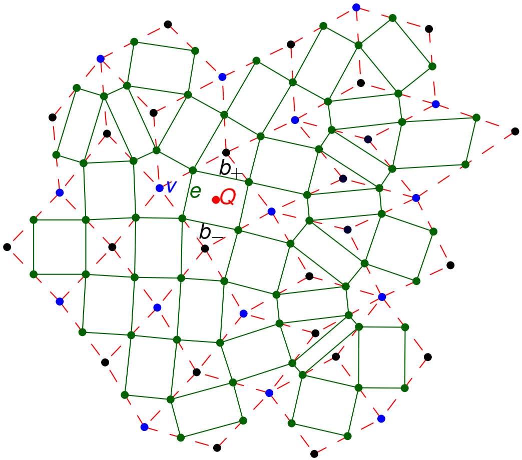

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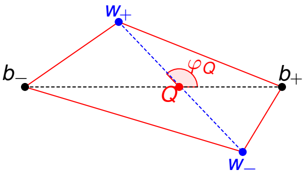

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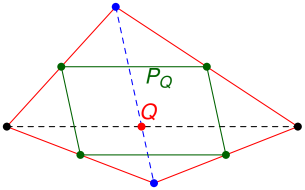
X medial graph of $\Lambda \cong \diamond^*$

Discrete derivatives

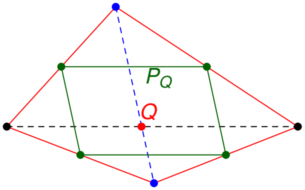


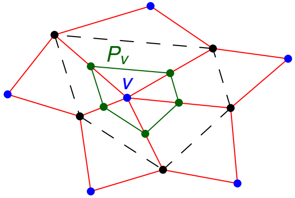
$$\partial_\Lambda f(Q) = \frac{-1}{2i\mathrm{ar}(F_Q)} \oint_{P_Q} f d\bar{z},$$

$$\bar{\partial}_\Lambda f(Q) = \frac{1}{2i\mathrm{ar}(F_Q)} \oint_{P_Q} f dz.$$

Definition: $\mathrm{ar}(F) := 2\mathrm{area}(F)$ for $F \in F(X)$.

Discrete derivatives





$$\partial_\Lambda f(Q) = \frac{-1}{2i\operatorname{ar}(F_Q)} \oint_{P_Q} f d\bar{z}, \qquad \partial_\diamond h(v) := \frac{-1}{2i\operatorname{ar}(F_v)} \oint_{P_v} h d\bar{z},$$

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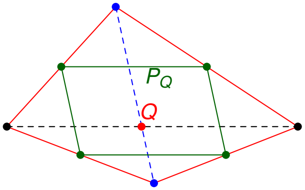

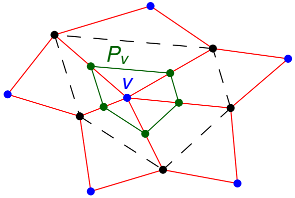

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ω of *type* $\diamond$ if $\omega = pdz + qd\bar{z}$ for $p, q : V(\diamond) \to \mathbb{C}$

Discrete exterior calculus

Let $f : V(\Lambda) \to \mathbb{C}$, $\omega = pdz + qd\bar{z}$ discrete one-form. Discrete exterior derivatives are defined by:

$$df := \partial_\Lambda f dz + \bar{\partial}_\Lambda f d\bar{z}; d\omega|_{F_v} := \left(\partial_\diamond q - \bar{\partial}_\diamond p\right) \Omega_\Lambda, d\omega|_{F_Q} := \left(\partial_\Lambda q - \bar{\partial}_\Lambda p\right) \Omega_\diamond$$

DISCRETE EXTERIOR CALCULUS

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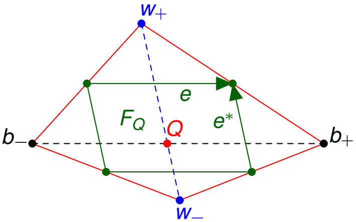
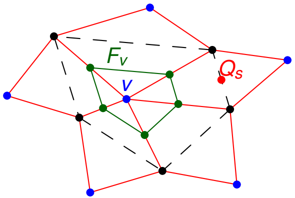
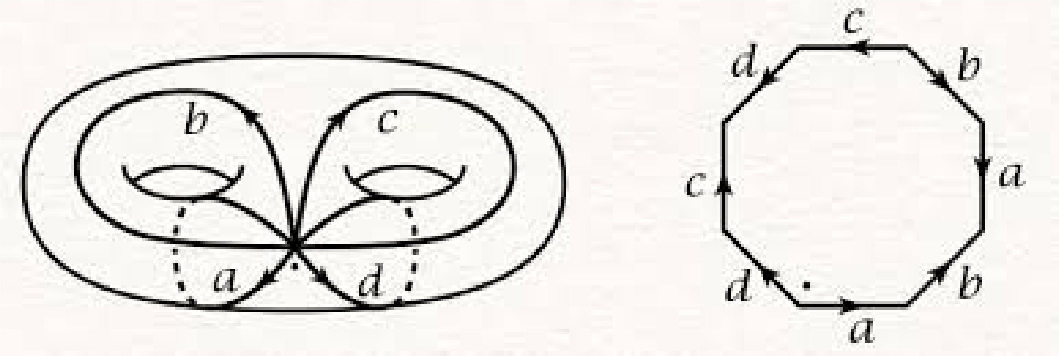




Stokes' theorem: $\int_e df = \frac{f(w_+)+f(b_+)}{2} - \frac{f(w_+)+f(b_-)}{2}$ and $\iint_F d\omega = \int_{\partial F} \omega$.

Canonical homology basis and fundamental $4g$-gon



Canonical homology basis $\{a_1, b_1, \ldots, a_g, b_g\}$ defined by $a_i \circ b_j = \delta_{ij}$, $a_i \circ a_j = 0 = b_i \circ b_j$.

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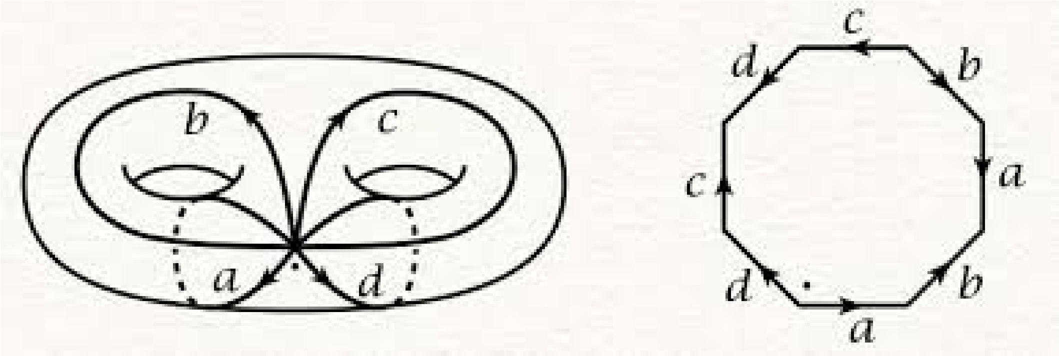

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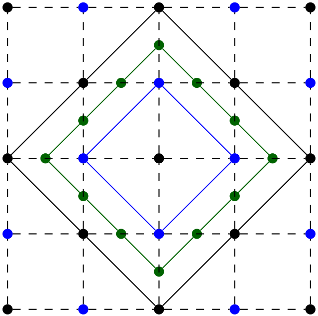
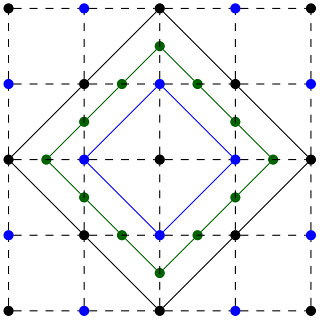
Dual canonical basis of (smooth) holomorphic differentials $\{\omega_1, \ldots, \omega_g\}$ defined by $\int_{a_i} \omega_j = \delta_{ij}$.

Discrete periods



Cycles P on X, BP on Γ, and WP on Γ^*

Discrete periods



Cycles P on X, BP on Γ, and WP on Γ^*

Black and *white* periods of closed discrete one-form ω satisfy

$$A_k^B + A_k^W = 2\int_{B\alpha_k} \omega + 2\int_{W\alpha_k} \omega = 2\oint_{\alpha_k} \omega = 2A_k \quad \text{and} \quad B_k^B + B_k^W = 2B_k.$$

Discrete period matrices

ω_k^B discrete holomorphic differential with black a_j-period δ_{jk} and white a-periods 0.

$$\Pi_{jk}^{B,B} := 2\int_{Bb_j} \omega_k^B; \quad \Pi_{jk}^{W,B} := 2\int_{Wb_j} \omega_k^B.$$

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Theorem

$\Pi = (\Pi^{B,W} + \Pi^{B,B} + \Pi^{W,W} + \Pi^{W,B})/2$ and $\widetilde{\Pi} := \begin{pmatrix} \Pi^{B,W} & \Pi^{B,B} \\ \Pi^{W,W} & \Pi^{W,B} \end{pmatrix}$ are symmetric and their imaginary parts are positive definite.

Convergence of discrete Dirichlet energy

$U := \int \mathrm{Re}(\Omega)$ and $u := \int \mathrm{Re}(\omega)$ multi-valued (discrete) harmonic functions.

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Second Lemma of Strang:

$$\|U - u\| \leq 2 \inf_{v \in S_P} \|U - v\| + \sup_{f \in S_0} \frac{|a(U,f)|}{\|f\|}$$

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$\sup\limits_{f \in S_0} \frac{|a(U,f)|}{\|f\|}$ *consistency error*: use method of nonconforming Crouzeix-Raviart elements

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Second Lemma of Strang:

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$$\left| \|U\|^2 - \|u\|^2 \right| \leq \mathrm{Const}(P, \Sigma, \phi) \begin{cases} h & \text{if } \gamma_\Sigma > 1/2, \\ h\,|\log h| & \text{if } \gamma_\Sigma = 1/2, \\ h^{2\gamma_\Sigma} & \text{if } \gamma_\Sigma < 1/2, \end{cases}$$

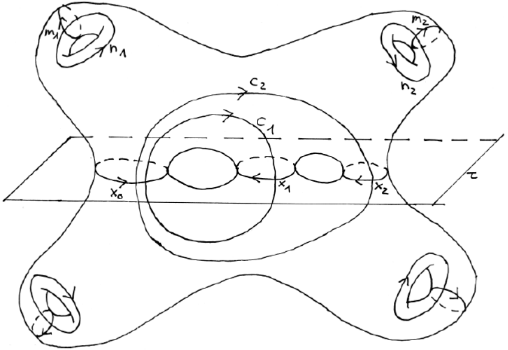
if maximum edge length $h \leq \mathrm{const}(\Sigma)$. γ_Σ smallest (conical) singularity index.

Convergence of discrete period matrices

Lemma

$M := \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ symmetric positive definite matrix with $g \times g$-blocks $A = A^T, B, C = C^T$. Let $L := \begin{pmatrix} 1_g & 1_g \end{pmatrix}$.
Then, $LML^T - 4\left(LM^{-1}L^T\right)^{-1}$ is positive semidefinite. Moreover, if this matrix approaches zero and M remains bounded, then $A - C$ and $B - B^T$ also approach zero.

Real Riemann surfaces (Σ, τ)



$0 \le k \le g+1$ real ovals; dividing or non-dividing.
(Σ, τ) dividing: $k \equiv g+1 \mod 2$.

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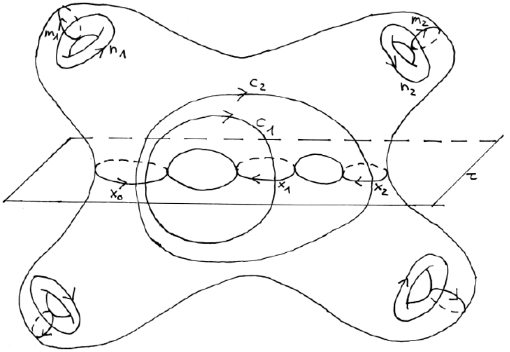

$$\Pi = iT + \frac{1}{2}H$$

T real positive definite, $H \in \mathbb{Z}_2^{g \times g}$.

$k \neq 0$: $\mathrm{rank}(H) = g + 1 - k$ and $\mathrm{diag}(H) = \delta_{\text{non-dividing}}$.

$k = 0$: $\mathrm{diag}(H) = 0$ and $\mathrm{rank}(H) = 2\lfloor \frac{g}{2} \rfloor$.

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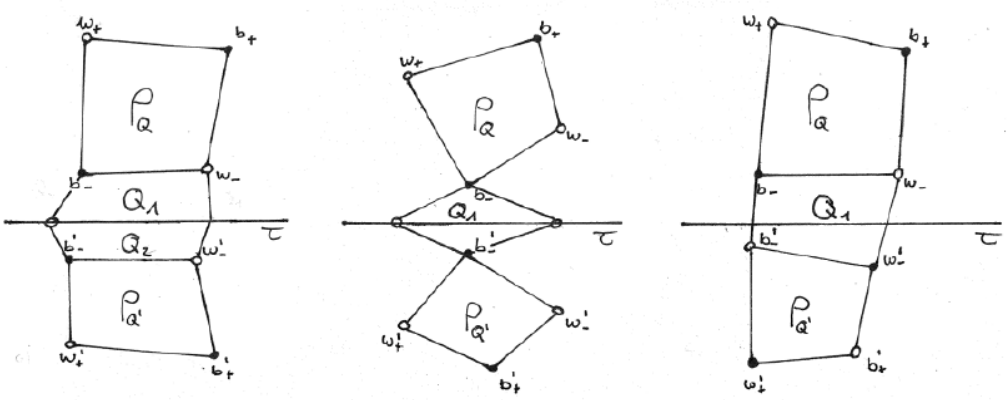
DISCRETE REAL RIEMANN SURFACES



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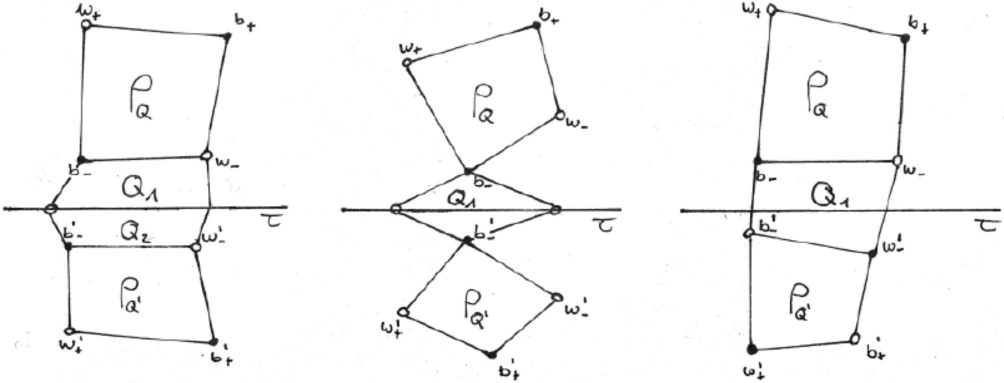
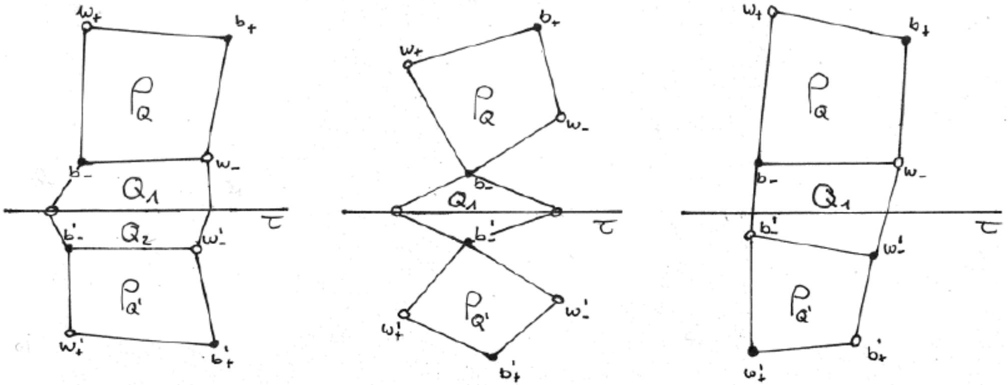


$\tau(\rho_Q) = \overline{\rho_{\tau(Q)}}$ if colors stay, $\tau(\rho_Q) = \overline{\rho_{\tau(Q)}}^{-1}$ if colors switch.

Discrete real Riemann surfaces



$$\tilde{\Pi} = \begin{pmatrix} i\Pi_1 & i\Pi_2 + \frac{1}{2}H \\ i\Pi_2^T + \frac{1}{2}H & i\Pi_3 \end{pmatrix}$$

if colors stay; else if colors switch: $\Pi^{B,B} + \bar{\Pi}^{W,W} = H$ and $\Pi^{B,W} + \bar{\Pi}^{W,B} = 0$

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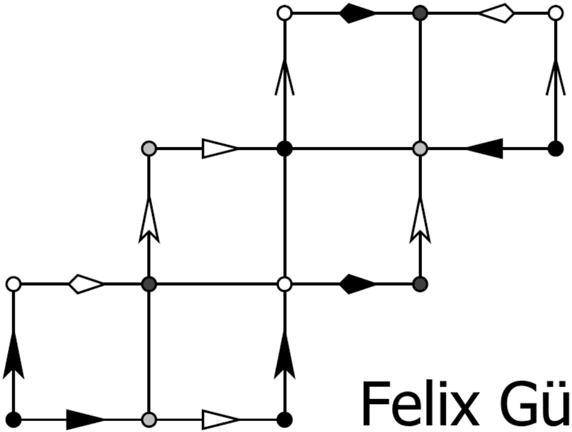

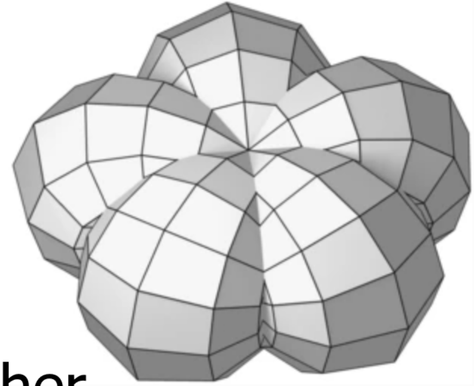

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