Discrete hyperbolic Laplacian

Wai Yeung Lam

University of Luxembourg

4 September 2024

Workshop on Discrete Geometric Structures

Joint work with Ivan Izmestiev (TU Wien)

Graph Laplacian

Definition

Given

1. Graph $G = (V, E)$
2. Edge weights $c : E \to \mathbb{R}_{>0}$, where $c_{ij} = c_{ji}$.
3. Vertex weights $d : V \to \mathbb{R}_{>0}$.

The (normalized) graph Laplacian $\triangle : \mathbb{R}^V \to \mathbb{R}^V$ is defined such that $\forall u : V \to \mathbb{R}$

$$(\triangle u)_i := \frac{1}{d_i}\Big(\sum_j c_{ij}(u_j - u_i)\Big) \quad \forall i \in V$$

We call u harmonic if $\triangle u \equiv 0$.

Dictionary in electric network (Random walk, Spanning tree models in stat. mech.):

1. Edge weight $c \iff$ conductance
2. Harmonic function $u \iff$ voltage
3. $c_{ij}(u_j - u_i) \iff$ current from vertex i to j
4. $(\triangle u)_i = 0 \iff$ zero net current at vertex i.

Graph Laplacian

Definition

Given

1. Graph $G = (V, E)$
2. Edge weights $c : E \to \mathbb{R}_{>0}$
3. Vertex weights $d : V \to \mathbb{R}_{>0}$.

The (normalized) graph Laplacian $\triangle : \mathbb{R}^V \to \mathbb{R}^V$ is defined such that $\forall u : V \to \mathbb{R}$

$$(\triangle u)_i := \frac{1}{d_i}\Big(\sum_j c_{ij}(u_j - u_i)\Big) \quad \forall i \in V$$

We call u harmonic if $\triangle u \equiv 0$.

Example (Combinatorial Laplacian): $c \equiv 1$ and $d_i =$ vertex degree.

$\Longrightarrow$ Study graphs via spectrum, first non-zero eigenvalues

E.g. Cayley graphs from groups

Graph Laplacian

Geometric Laplacian?

Given a geodesic triangulation of a hyperbolic surface, a graph Laplacian with edge weights adapted to the geometry.

Goal: Geometry $\Longleftrightarrow$ Edge weights

Euclidean Laplacian $\triangle_E$

Given a geodesic triangulation (V, E, F) in Euclidean plane $\mathbb{R}^2$, take

$$c_{ij} := \frac{\cot \angle jki + \cot \angle ilj}{2}$$

$\to$ "cotangent weight" (Pinkall-Polthier 1993)

1. Delaunay triangulation $\iff c_{ij} \geq 0$.
2. Vertex position $f : V \to \mathbb{C}$ and circumcenters $f^\dagger : F \to \mathbb{C}$

$$f^\dagger_{ij,l} - f^\dagger_{ij,r} = \sqrt{-1} c_{ij} (f_j - f_i) \implies \triangle_E f \equiv 0.$$

3. Generalized definition for weighted Delaunay decomposition

$$c_{ij} := \frac{|f^\dagger_{ij,l} - f^\dagger_{ij,r}|}{|f_j - f_i|} = \frac{\text{dual edge length}}{\text{primal edge length}}$$

4. **Case: Triangulation of an Euclidean torus** $f : V \to S$.
   Developing maps of universal cover $\tilde{f} : \tilde{V} \to \mathbb{C}$, $\tilde{f}^\dagger : \tilde{F} \to \mathbb{C}$
   Write $\gamma_1, \gamma_2 \in \pi_1$ generators of fundamental group. $\exists$ translation $\tau_1, \tau_2 \in \mathbb{C}$ s.t.

$$\tilde{f} \circ \gamma_r - \tilde{f} = \tau_r = \tilde{f}^\dagger \circ \gamma_r - \tilde{f}^\dagger$$

Maxwell-Cremona: $\tilde{f}, \tilde{f}^\dagger$ reciprocal perpendicular diagrams with same translation periods

Hyperbolic Laplacian $\triangle_H$

Given a geodesic triangulation in hyperbolic plane, take

$$c_{ij} := \frac{\tan\left(\frac{\angle kij + \angle ijk - \angle jki}{2}\right) + \tan\left(\frac{\angle jil + \angle lji - \angle ilj}{2}\right)}{2\cosh^2\frac{\ell_{ij}}{2}}$$

$$d_i := \sum_j c_{ij} \sinh^2 \frac{\ell_{ij}}{2}$$

1. Delaunay $\iff$ $c_{ij} \geq 0$.
2. Consider the hyperboloid model in Minkowski space $\mathbb{H} \subset \mathbb{R}^{2,1}$.
   Vertex position $f : V \to \mathbb{H} \subset \mathbb{R}^{2,1}$ and $f^\dagger : F \to \mathbb{R}^{2,1}$ polar dual.
   f (convex) polyhedral surface in $\mathbb{R}^{2,1}$ with vertices on $\mathbb{H}$
   $f^\dagger$ (convex) polyhedral surface with faces tangent to $\mathbb{H}$, where d is face area

$$f^\dagger_{ij,l} - f^\dagger_{ij,r} = c_{ij}(f_i \times f_j) \implies \triangle_H f = 2f$$

3. **Case: Triangulation of a closed hyperbolic surface** $f : V \to S$.
   There exists $\rho \in \mathrm{Hom}(\pi_1, SO(2,1))$ s.t. developing maps ρ-equivariant

$$\tilde{f} \circ \gamma = \rho_\gamma \tilde{f}, \quad \tilde{f}^\dagger \circ \gamma = \rho_\gamma \tilde{f}^\dagger$$

Geometric Laplacian

Key feature: Geodesic triangulation of a hyperbolic surface $\Longrightarrow$ Edge weight in $\triangle_H$

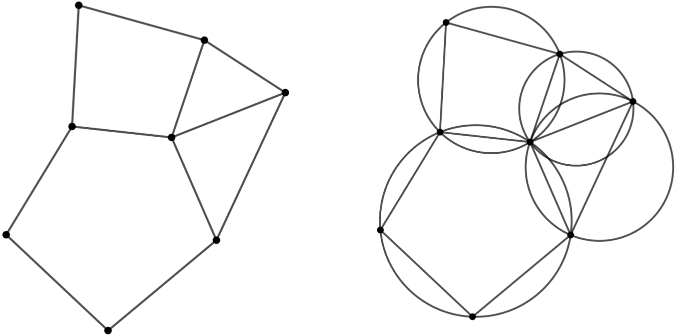
Similar formula for spherical graph Laplacian $\triangle_S$.

Next slides:

1. Compatible with discrete differential geometry: circle patterns
2. Discrete harmonic maps to closed surfaces   Edge weight $\Longrightarrow$ Geometry

Deformation of circle patterns

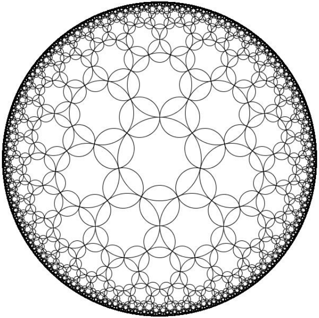
- Circle pattern is a **realization of a planar graph in $\mathbb{C} \cup \infty$ such that the vertices of each face lie on a circle**

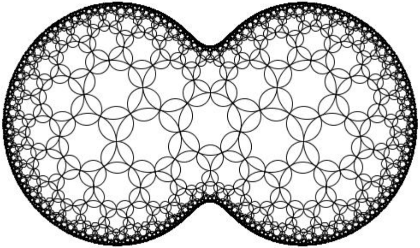


- Deform the realization such that intersection angles of neighboring circles are preserved.

Deformation of circle patterns

Intersection angles of neighboring circles are preserved.





Keywords: Discrete conformal maps, Generalization of Thurston's circle packings.

Theorem (L.-Pinkall 2016)

Given a circle pattern in $\mathbb{R}^2$*, any* $u : V \to \mathbb{R}$ *satisfying* $\triangle_E u = 0$ *corresponds to an infinitesimal deformation preserving intersection angles.*

Given a family of circle patterns parameterized by $t \in (-\epsilon, \epsilon)$, consider the Euclidean radius R_t of each circle

$\implies$ logarithmic derivative of radii R_t with respect to t is discrete harmonic

$$\triangle_E\left(\frac{d}{dt} \log R_t\Big|_{t=0}\right) = 0$$

Theorem (L.-Izmestiev 2024)

Given a circle pattern in $\mathbb{H}^2$*, any* $u : V \to \mathbb{R}$ *satisfying* $\triangle_H u = 2u$ *corresponds to an infinitesimal deformation preserving intersection angles.*

Given a circle pattern in $\mathbb{S}^2$*, any* $u : V \to \mathbb{R}$ *satisfying* $\triangle_S u = -2u$ *corresponds to an infinitesimal deformation preserving intersection angles.*

Remarks: Solutions u yield discrete minimal surfaces (L.2018)

Corollary

Circle patterns on closed hyperbolic surfaces are infinitesimally rigid.

Some more:

Theorem (L.-Izmestiev 2024)

Let $P \subset \mathbb{R}^3$ be a convex polyhedron with outward unit normals $p_1, \dots, p_n$ and the corresponding support numbers $h_1, \dots, h_n$, that is

$$P = \{x \in \mathbb{R}^3 \mid \langle x, p_i \rangle \leq h_i \text{ for all } i\},$$

Then the surface area of P is equal to

$$\frac{1}{2}\langle h, 2h + \triangle_s h \rangle_d .$$

Here $\langle u, v \rangle_d = \sum_i d_i u_i v_i$, and $\triangle_s$ is the discrete spherical Laplacian for the triangulation of $\mathbb{S}^2$ with the vertices $p_1, \dots, p_n$ and the edges dual to the edges of P.

Discrete harmonic map

- (V, E, F) triangulation of a **closed** surface S_g of genus $g \geq 1$
- Arbitrary positive edge weights $c : E \to (0, \infty)$

For any geodesic realization $f : (V, E) \to S_g$ equipped with a constant-curvature metric, define **Dirichlet energy**

$$\mathcal{D}_c(f) = \sum_{ij} c_{ij} \ell_{ij}^2.$$

where ℓ is the geodesic edge length.

1st. minimization problem

Fixing the metric, minimize $\mathcal{D}_c$ over geodesic realizations in the same homotopy class.

Colin de Verdière (1991): A minimizer exists uniquely, called discrete harmonic map.

2nd. minimization problem

Minimize energy $\mathcal{D}_c$ of discrete harmonic maps over all constant-curvature metrics, namely "*Teichmüller space*" Teich(S).

Kajigaya-Tanaka (2021): A minimizer exists uniquely.

Question: How edge weights related to the optimal metric? Little known cases...

Discrete harmonic map

For arbitrary positive edge weights c, minimize

$$\mathcal{D}_c(f) = \sum_{ij} c_{ij}\ell_{ij}^2.$$

Theorem (L.2021)

Suppose $g = 1$. Then a Euclidean metric is optimal for $\mathcal{D}_c \iff$ the discrete harmonic map is a weighted Delaunay triangulation with "cotangent weights" equal to c.

Theorem (L.2024)

Suppose $g \geq 2$. Then a hyperbolic metric is optimal for $\mathcal{D}_c \iff$ the discrete harmonic map is a weighted Delaunay triangulation with

$$c_{ij} = \frac{\tan\left(\frac{\alpha^i_{jk}+\alpha^j_{ki}-\alpha^k_{ij}}{2}\right) + \tan\left(\frac{\alpha^i_{lj}+\alpha^j_{il}-\alpha^l_{ji}}{2}\right)}{2\cosh^2\frac{\ell_{ij}}{2}} \cdot \frac{\sinh \ell_{ij}}{\ell_{ij}}$$

Observations (Torus case)

- (V, E, F) topological triangulation of a torus
- $(\tilde{V}, \tilde{E}, \tilde{F})$ lift of the triangulation to the universal cover
- f geodesic triangulation of a Euclidean torus.
- $\tilde{f} : \tilde{V} \to \mathbb{C}$ developing map of the universal cover with translation periods

$$\tilde{f} \circ \gamma_r - \tilde{f} = \tau_r$$

- Observe $\mathcal{D}_c(f) = \sum_{ij} c_{ij} \ell_{ij}^2 = \sum_{ij} c_{ij} |\tilde{f}_j - \tilde{f}_i|^2$
- We deduce f discrete harmonic map

$\iff$ For each i, we have $\sum_{ij} c_{ij}(\tilde{f}_j - \tilde{f}_i) = 0$

$\iff$ Exists realization of dual graph $\tilde{f}^\dagger : \tilde{F} \to \mathbb{C}$ satisfying $\forall ij \in \tilde{E}$

$$\tilde{f}^\dagger_{ij,l} - \tilde{f}^\dagger_{ij,r} = \sqrt{-1} c_{ij} (\tilde{f}_j - \tilde{f}_i)$$

- $\tilde{f}$ has translation periods

$$\tilde{f}^\dagger \circ \gamma_r - \tilde{f}^\dagger = \tau_r^\dagger$$

Question: $\tilde{f}$ and $\tilde{f}^\dagger$ same translation periods? i.e. $(\tau_1, \tau_2) = (\tau_1^\dagger, \tau_2^\dagger)$?

OBSERVATIONS (TORUS CASE) (CONT)

- $\tilde{f}$ and $\tilde{f}^\dagger$ have the same translation periods, i.e. $(\tau_1, \tau_2) = (\tau_1^\dagger, \tau_2^\dagger)$
  $\iff (\tau_1, \tau_2)$ optimal Euclidean torus, i.e. minimizer of $\mathcal{D}_c$ over Teich(S)
  $\iff (\tau_1, \tau_2)$ eigenvector of the "period matrix"
- Same translation periods $\implies$ $\tilde{f}$ and $\tilde{f}^\dagger$ project to the same Euclidean torus
  f vertices of a weighted Delaunay triangulation
  $f^\dagger$ centers of face circles
  The prescribed edge weight satisfies $c_{ij} = \frac{|f^\dagger_{ij,l} - f^\dagger_{ij,r}|}{|f_j - f_i|}$

Observations (Hyperbolic surfaces)

- f geodesic triangulation of a closed hyperbolic surface (S, h).
- $\tilde{f} : \tilde{V} \to \mathbb{H} \subset \mathbb{R}^{2,1}$ developing map of the universal cover which is ρ-equivariant

$$\tilde{f} \circ \gamma = \rho_\gamma \tilde{f}, \quad \text{where } \rho \in \mathrm{Hom}(\pi_1, SO(2,1))$$

- Observe $\mathcal{D}_c(f) = \sum_{ij} c_{ij} \ell_{ij}^2 = \sum_{ij} c_{ij} \left(\cosh^{-1}(-\langle \tilde{f}_i, \tilde{f}_j \rangle_{\mathbb{R}^{2,1}})\right)^2$
- We deduce f discrete harmonic map

$\iff$ For each i, we have

$$\sum_j \frac{c_{ij}\ell_{ij}}{\sinh \ell_{ij}} (f_i \times f_j) = 0$$

$\iff$ Exists realization of dual graph $\tilde{f}^\dagger : \tilde{F} \to \mathbb{R}^{2,1}$ satisfying $\forall ij \in \tilde{E}$

$$\tilde{f}^\dagger_{ij,l} - \tilde{f}^\dagger_{ij,r} = \frac{c_{ij}\ell_{ij}}{\sinh \ell_{ij}} (f_i \times f_j)$$

- $\tilde{f}$ "almost" ρ-equivariant, i.e. exists translation period $\tau : \pi_1 \to \mathbb{R}^{2,1} \cong so(2,1)$

$$\tilde{f}^\dagger \circ \gamma = \rho_\gamma \tilde{f}^\dagger + \tau_\gamma$$

Question: $\tilde{f}$ and $\tilde{f}^\dagger$ both ρ-equivariant? i.e. $\tau \equiv 0$?

Observations (Hyperbolic surface) (cont)

- $\tilde{f}$ and $\tilde{f}^{\dagger}$ both ρ-equivariant, i.e. $\tau \equiv 0$
  $\iff$ optimal hyperbolic metric is reached, i.e. minimizer of $\mathcal{D}_c$ over Teich(S)
- Suppose for $t \in (-\epsilon, \epsilon)$, $f^{(t)}$ geodesic triangulation of hyperbolic surfaces $(S, h^{(t)})$. Write $\dot{f} := \frac{d}{dt}\tilde{f}^{(t)}|_{t=0}$ and the change of holonomy yields

$$\sigma := \dot{\rho}\rho^{-1} : \pi_1 \to so(2,1)$$

$$\begin{aligned}\frac{d}{dt}\mathcal{D}_c(f^{(t)})|_{t=0} &= \sum_{ij\in E'} \frac{c_{ij}\ell_{ij}}{\sinh \ell_{ij}} \langle f_i \times f_j, f_j \times \dot{f}_j - f_i \times \dot{f}_i \rangle \\ &= \sum_{ij \in E} \langle \tilde{f}^{\dagger}_{ij,l} - \tilde{f}^{\dagger}_{ij,r}, f_j \times \dot{f}_j - f_i \times \dot{f}_i \rangle \\ &= \omega_G(\tau, \sigma)\end{aligned}$$

  where ω_G Weil-Petersson symplectic form over Teich(S), expressed as the cup product in group cohomology $[\tau], [\sigma] \in H^1_{\mathrm{Ad}(\rho)}(\pi_1(S), so(2,1)) \cong T_\rho\left(\mathrm{Hom}(\pi_1, SO(2,1))/\sim\right) \cong T_h\mathrm{Teich}(S)$
- Thus, the translation period $[\tau] \in T_h\mathrm{Teich}(S)$ is the symplectic gradient of $\mathcal{D}_c$

Discrete harmonic map to hyperbolic surfaces

For arbitrary positive edge weights c, minimize

$$\mathcal{D}_c(f) = \sum_{ij} 2c_{ij} \sinh^2 \frac{\ell_{ij}}{2}.$$

Theorem (L.2024)

Suppose $g > 1$. Then a hyperbolic metric is optimal for $\mathcal{D}_c \iff$ the discrete harmonic map is a weighted Delaunay cell decomposition with

$$c_{ij} = \frac{\tan(\frac{\alpha^i_{jk}+\alpha^j_{ki}-\alpha^k_{ij}}{2}) + \tan(\frac{\alpha^i_{lj}+\alpha^j_{il}-\alpha^l_{ji}}{2})}{2\cosh^2 \frac{\ell_{ij}}{2}} \cdot 1$$

SUMMARY

GEOMETRIC LAPLACIAN

Given a geodesic triangulation of a hyperbolic surface, a graph Laplacian with edge weights adapted to the geometry.

Goal: Geometry $\Longleftrightarrow$ Edge weights

Reference:

1. Discrete Laplacians – spherical and hyperbolic. With Ivan Izmestiev. (2024). arXiv:2408.04877
2. Discrete harmonic maps between hyperbolic surfaces. (2024). arXiv: 2405.02205
3. Delaunay decompositions minimizing energy of weighted toroidal graphs. (2022). arXiv: 2203.03846

Thank you!