Discrete isothermic tori

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Joint work with J. Cho (Pohang) and Y. Ogata (Kyoto)

Discrete Geometric Structures 2024

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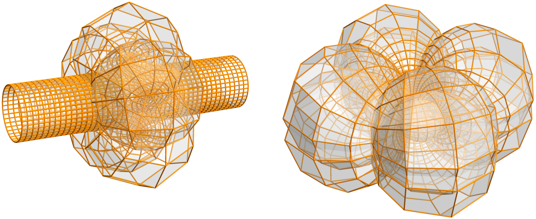

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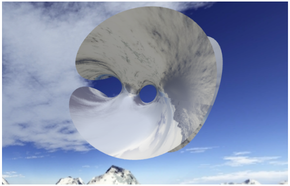
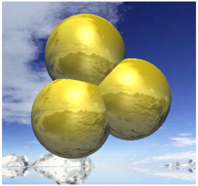
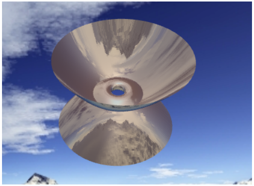

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Model:

$$\mathbb{H} = \mathrm{span}_{\mathbb{R}}\{1, i, j, k\}$$

with

$$i^2 = j^2 = k^2 = ijk = -1\,.$$

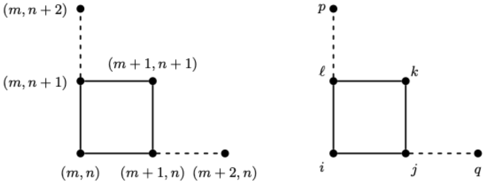
Discrete isothermic surfaces

Definition

A discrete surface $f : \Sigma^2 \to \mathbb{H}$ from a simply connected discrete domain is called discrete isothermic if

$$\mathrm{cr}(f_i, f_j, f_k, f_\ell) := (f_i - f_j)(f_j - f_k)^{-1}(f_k - f_\ell)(f_\ell - f_i)^{-1} = \frac{\mu_{i\ell}}{\mu_{ij}}$$

on every elementary quadrilateral $(ijk\ell)$ for some real–valued function μ, $\mu > 0$ or $\mu < 0$, defined on unoriented edges



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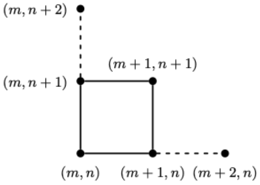

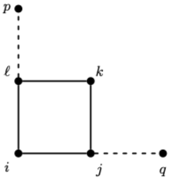

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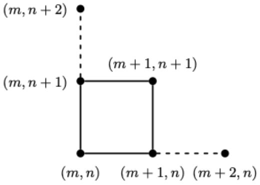

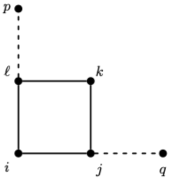

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- The cross-ratios factorising function can be scaled by a real factor.

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$$\mathrm{d}f^*_{ij} := f^*_i - f^*_j = \omega_{ij} \,.$$

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Let $f : \Sigma^2 \to \mathbb{H}$ be a discrete isothermic surface with cross-ratios factorising function μ. A second discrete surface $\hat{f} : \Sigma^2 \to \mathbb{H}$ is called a Darboux transform of f with spectral parameter ν if

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Aim: Describe discrete Darboux transforms via an associated family of discrete flat connections.

The family of discrete flat connections of an isothermic surface

Let $f : \Sigma^2 \to \mathbb{H}$ be discrete isothermic with cross–ratios factorising μ and corresponding dual surface given by $\mathrm{d}f^*_{ij} = \frac{1}{\mu_{ij}}\,\mathrm{d}f^{-1}_{ij}$.

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Define $D(\lambda)_{ji} : \{i\} \times \mathbb{H}^2 \to \{j\} \times \mathbb{H}^2$ with $\lambda \in \mathbb{R}$ on every edge (ij) via

$$D(\lambda)_{ji} := \mathrm{id}_{ji} + \begin{pmatrix} 0 & \mathrm{d}f_{ij} \\ \lambda\,\mathrm{d}f^*_{ij} & 0 \end{pmatrix},$$

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Fact: $\hat{F}$ is a Darboux transform of the associated lift F of f with spectral parameter ν if and only if $\hat{F}$ is $r(\nu)^P$–parallel.

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i.e.,

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Denote by $f^0 : \Sigma^1 \to \mathbb{H}$ any discrete curvature line of f defined on the corresponding simply-connected domain Σ^1.

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Rather than controlling the closing condition on the discrete isothermic surface, we reduce the closing condition to the discrete curvature lines.

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Similarly to the case of discrete isothermic surfaces it is now possible to define a Darboux transformation of discrete polarised curves and solve the closing conditions (see Ogata's talk).

Reduction of the monodromy

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Hence $\hat{f}_{m,n_0+1}$ must also be periodic, that is,

$$\hat{f}_{m,n_0+1} = \hat{f}_{m+M,n_0+1}.$$

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Theorem (Cho–L–Ogata)

Let $f : (\Sigma^2, \frac{1}{\mu}) \to \mathbb{H}$ be a discrete isothermic surface with one family of periodic curvature lines. If $\hat{f} : (\Sigma^2, \frac{1}{\mu}) \to \mathbb{H}$ is a Darboux transform of f, then the corresponding family of curvature lines of $\hat{f}$ are also periodic with the same period if and only if one of the corresponding family of curvature lines is periodic.

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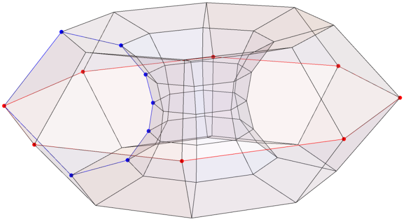

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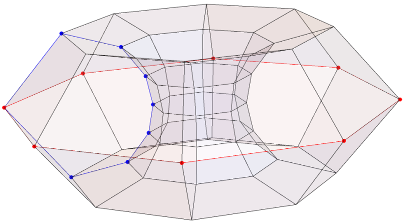
2. Viewing both curves as closed discrete polarised curves, find a spectral parameter ν and an initial condition $\hat{f}_{m_0,n_0}$ to obtain simultaneously closed Darboux transforms $\hat{f}_{m,n_0}$ and $\hat{f}_{m_0,n}$ of f_{m,n_0} and $f_{m_0,n}$, respectively.



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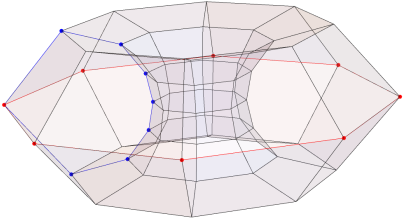
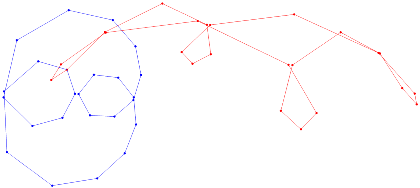
3. Use the spectral parameter ν and the initial condition $\hat{f}_{m_0,n_0}$ to obtain a new periodic discrete isothermic torus $\hat{f}$.





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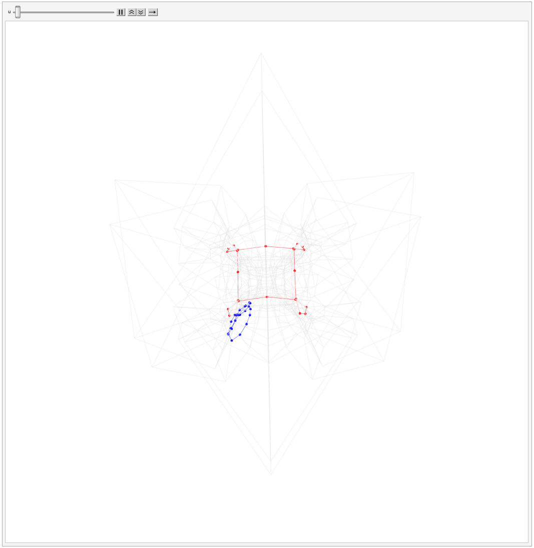

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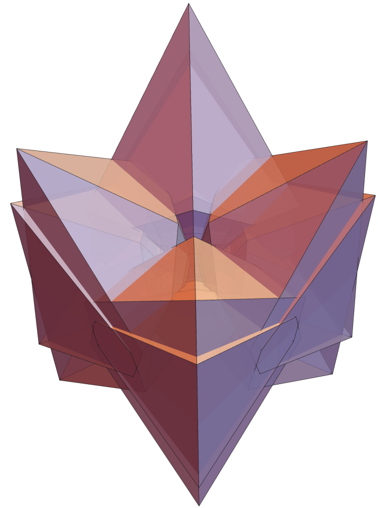

Explicit discrete isothermic bubbletons (Cho–L–Ogata)

In cases of a discrete cylinder

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In the case when the spectral parameter is

$$0 > \nu = \frac{1}{4}(1 - \cot^2\frac{\pi}{M}\tan^2\frac{k\pi}{\rho M}), \qquad k, \rho \in \mathbb{N}$$

every Darboux transform with parameter ν is closed on the ρ–fold cover and we obtain the following closed-form formulae:

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$$\hat{f} = f + ab^{-1} = f + \frac{1}{C}(iT^0 + jT^1)$$

where $c_2 \in \mathbb{C}$ and

$$\begin{aligned}
A_m &= e^{\frac{k\pi \mathbf{i}}{\rho M}m} + e^{-\frac{k\pi \mathbf{i}}{\rho M}m} c_2 \\
B_m &= e^{\frac{k\pi \mathbf{i}}{\rho M}m} \sin\frac{(\rho+k)\pi}{\rho M} + e^{-\frac{k\pi \mathbf{i}}{\rho M}m} \sin\frac{(\rho-k)\pi}{\rho M} c_2 \\
C_{m,n} &= 2\big((N-\sqrt{-\nu})^n + (N+\sqrt{-\nu})^n\big)^2 |B_m|^2 \\
&\quad + \big((N-\sqrt{-\nu})^n - (N+\sqrt{-\nu})^n\big)^2 |A_m|^2 (\cos\tfrac{2\pi}{M} - \cos\tfrac{2k\pi}{\rho M}) \\
T^0_{m,n} &= \frac{1}{\sqrt{-\nu}}\big((N-\sqrt{-\nu})^{2n} - (N+\sqrt{-\nu})^{2n}\big) \cdot \\
&\quad \big(2|B_m|^2 + |A_m|^2(\cos\tfrac{2\pi}{M} - \cos\tfrac{2k\pi}{\rho M})\big) \\
T^1_{m,n} &= -16 e^{\frac{2\pi \mathbf{i}}{M}m} (N-\sqrt{-\nu})^n (N+\sqrt{-\nu})^n \sin\tfrac{\pi}{M} \cos\tfrac{k\pi}{\rho M} A_m \overline{B_m}
\end{aligned}$$

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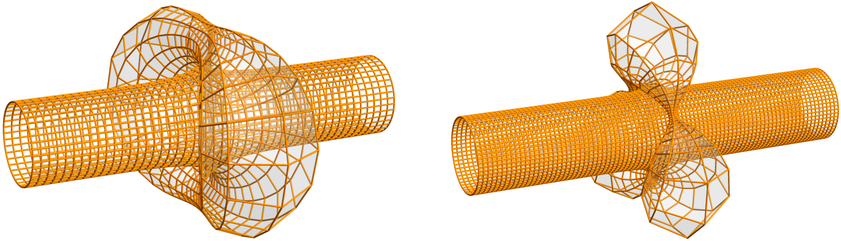

Figure: Discrete isothermic bubbletons with $k = 2$, $\rho = 1$, and $c_2 = -10$ and $k = 3$, $\rho = 1$, $c_2 = -4$

Explicit discrete CMC bubbletons (Cho–L–Ogata)

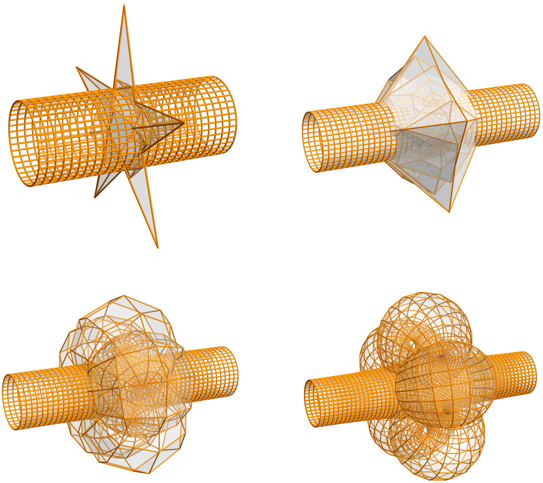

Figure: Discrete cmc bubbletons $k = 5$, with varying number of covers (top row from the left: $\rho = 1, 2$; bottom row from the left: $\rho = 3, 4$).

Explicit discrete isothermic tori (Cho–L—Ogata)

Similarly, we have explicit parametrisations for all Darboux transforms of discrete homogeneous tori

$$f_{m,n} = qe^{\frac{2\pi in}{N}} + jpe^{\frac{2\pi im}{N}}, \qquad p^2 + q^2 = 1$$

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To obtain non–trivial closed Darboux transforms we choose $k_1, k_2, \rho_1, \rho_2 \in \mathbb{N}$ and define p, q by

$$0 < p^2 = \frac{1 - \cot^2 \frac{\pi}{M} \tan^2 \frac{k_1\pi}{\rho_1 M}}{\cot^2 \frac{\pi}{N} \tan^2 \frac{k_2\pi}{\rho_2 N} - \cot^2 \frac{\pi}{M} \tan^2 \frac{k_1\pi}{\rho_1 M}},$$

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Similarly, we have explicit parametrisations for all Darboux transforms of discrete homogeneous tori

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all Darboux transforms are discrete isothermic tori.

Explicit discrete isothermic tori (Cho–L—Ogata)

We obtain explicit parametrisations of Darboux transforms of $f_{m,n}$ which are discrete isothermic tori in S^3, in analogy to the smooth Bernstein's tori (2001).

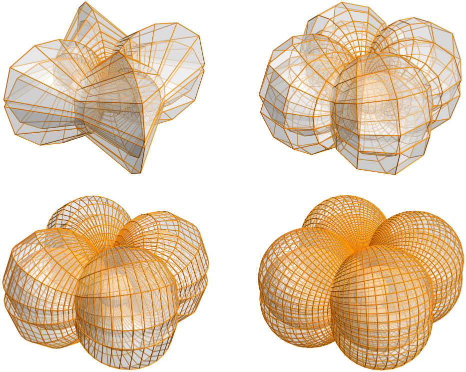

Figure: Discrete isothermic tori in $\mathbb{S}^3$ with $k_1 = 4$, $\rho_1 = 3$, $k_2 = 2$, $\rho_2 = 3$, (on the top left: $M = 12$, $N = 40$; on the top right: $M = 40$, $N = 40$; on the bottom left: $M = 160$, $N = 40$; on the bottom right: $M = 160$, $N = 160$).

Explicit discrete isothermic tori (Cho–L—Ogata)

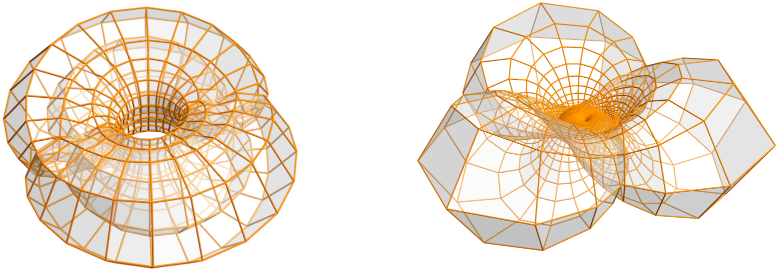

Figure: Discrete isothermic tori in $\mathbb{S}^3$ (on the left: $k_1 = 2$, $\rho_1 = 3$, $k_2 = 3$, $\rho_2 = 2$, $M = 12$, $N = 12$; on the right: $k_1 = 3$, $\rho_1 = 2$, $k_2 = 2$, $\rho_2 = 3$, $M = 60$, $N = 40$).

The smooth case (Cho–Leschke–Ogata)

- The continuum limit $M, N \to \infty$ gives smooth isothermic surfaces in the discussed examples.

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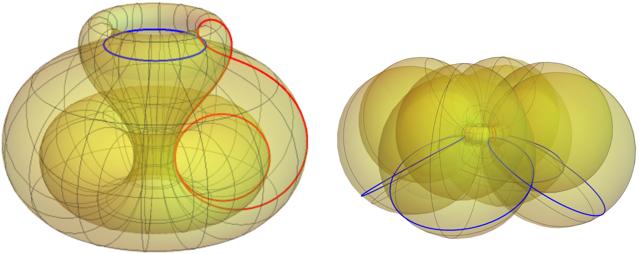
- Constructions carry through in the smooth case.



Thank you!