Discrete moving frames and invariant discrete conservation laws

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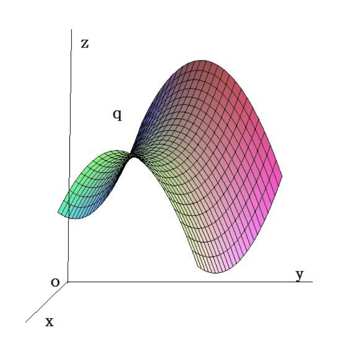
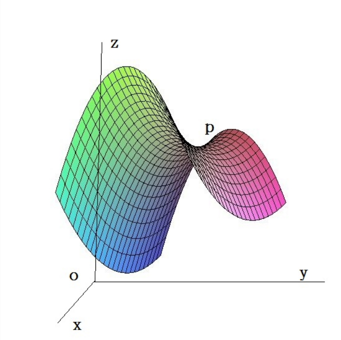
†Based on joint work with Peter Hydon, Elizabeth Mansfield, and Ana Rojo-Echeburúa

Classical moving frames

Equivalence of curves or surfaces





Are these two surfaces equivalent (locally)?

A classical solution: differential invariants of a surface $(x, y, f(x, y))$

The differential invariants with respect to rotations and translations in $\mathbb{R}^3$, i.e., the $SE(3)$ group, are

- Gaussian curvature

$$K_p = \det \operatorname{Hess}_p(f) = (f_{xx} f_{yy} - f_{xy}^2)|_p$$

- Mean curvature

$$H_p = \frac{1}{2} \operatorname{tr} \operatorname{Hess}_p(f) = \frac{f_{xx} + f_{yy}}{2}\Big|_p$$

Example. Consider a curve $C : \mathbb{R} \to \mathbb{R}^2, s \mapsto C(s)$ where s is the arc length parameter.

- The action of $SE(2)$ on the (x, u)-plane $\mathbb{R}^2$, $SE(2) \times \mathbb{R}^2 \to \mathbb{R}^2$, is

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{u} \end{pmatrix} \mapsto R \begin{pmatrix} x \\ u \end{pmatrix} + \mathbf{a},$$

  where $SE(2)$ is the special Euclidean group with elements

$$A = \begin{pmatrix} 1 & 0 \\ \mathbf{a} & R \end{pmatrix},$$

  with $\mathbf{a} \in \mathbb{R}^2$ and

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2).$$

How does a moving frame work: curves on the plane

- Define a map $F : \mathbb{R} \to SE(2)$ (a moving frame) by

$$F : s \mapsto A = \begin{pmatrix} 1 & 0 \\ C(s) & (e_1(s), e_2(s)) \end{pmatrix},$$

where $e_1(s) = C'(s)$, and $e_2(s)$ is orthogonal to $e_1(s)$.

- Pulling back the invariant differential form (the Maurer–Cartan form) $A^{-1}\,\mathrm{d}A$ on $SE(2)$ by F provides the (invariant) differential forms

$$\omega^1 = \mathrm{d}s, \;\; \omega^2 = -\kappa\,\mathrm{d}s, \;\; \omega^3 = \kappa\,\mathrm{d}s,$$

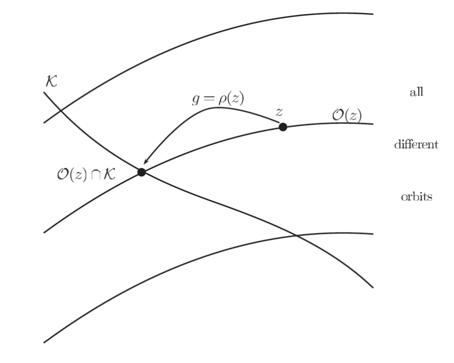
where the invariant (curvature) $\kappa(s) = \|C''(s)\|$ satisfies

$$\frac{\mathrm{d}e_1(s)}{\mathrm{d}s} = \kappa e_2(s).$$

Remark. $(e_1(s), e_2(s))$ is a moving frame: a basis that is moving along the curve.

Modern moving frames

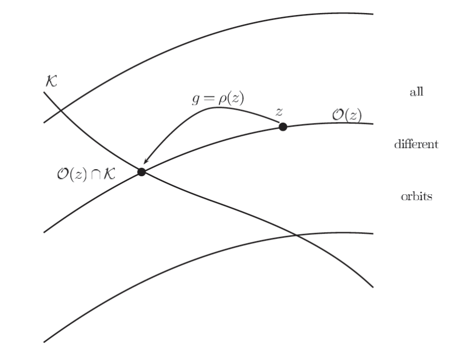
Moving frames: modern definition, Fels and Olver (1998, 1999)



- A regular and free group action $G \times M \to M$
- Orbits $\mathcal{O}(z) = \{\widetilde{z} = g \circ z \mid \forall g \in G\}$ for any given $z \in M$

Definition. A moving frame is a map $\rho : M \to G$, such that $\rho(z)$ is G-equivalent, namely $\rho(g \circ z) = \rho(z) \cdot g^{-1}$.

Calculation of a moving frame



- Given a cross section $\mathcal{K}$: $\psi_i(z) = 0$, $i = 1, \ldots, r = \dim G$, the moving frame $\rho : M \to G$ satisfies $\rho(z) \circ z = k$.
- The normalisation equations will yield $\rho(z)$:

$$\psi_i(g \circ z) = 0, \quad i = 1, \ldots, \dim G.$$

Invariants

Definition. A function $f(z)$ is (an) invariant if $f(g \circ z) = f(z)$.

- G-equivalent, $\rho(g \circ z) = \rho(z) \cdot g^{-1}$, implies

$$\rho(g \circ z) \circ (g \circ z) = \rho(z) \circ z.$$

  Denote the invariant w.r.t. z as $I(z) = \rho(z) \circ z$.

- Replacement rule: The invariantisation of an invariant $f(z)$ is $f(\rho(z) \circ z) = f(I(z))$.

Example. Alternatively, the action of $SE(2)$ on the plane $\mathbb{R}^2 \ni (x, u)$ can be written as follows:

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}.$$

Choose the normalisation equations

$$\widetilde{x} = 0, \quad \widetilde{u} = 0, \quad \text{and} \quad 0 = \widetilde{u}_x \left(= \frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\widetilde{x}} = \frac{\mathrm{d}\widetilde{u}}{\mathrm{d}x} \Big/ \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}x} \right).$$

The moving frame ρ is given in parametric form

$$a = -\frac{x + uu_x}{\sqrt{1+u_x^2}}, \quad b = -\frac{u - xu_x}{\sqrt{1+u_x^2}}, \quad \text{and} \quad \theta = \arctan u_x.$$

Example: $SE(2)$ action on $\mathbb{R}^2$

The Euclidean curvature (differential invariant) is obtained

$$\kappa := \widetilde{u_{xx}}\Big|_{g=\rho} = \frac{u_{xx}}{(1+u_x^2)^{3/2}},$$

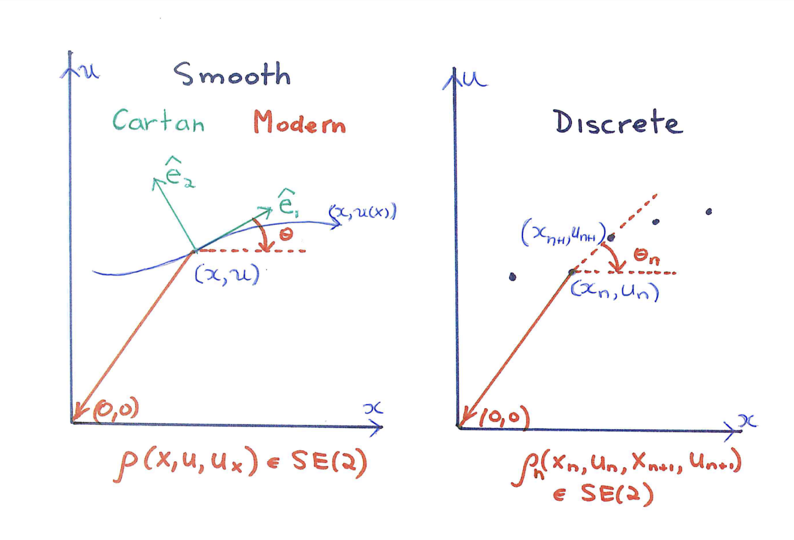
which is consistent with

$$\kappa = \frac{\det(r', r'')}{\|r'\|^3}$$

with $r(x) = (x, u(x))^T$.

Remark. Higher order differential invariants can be computed using either the invariantisation of prolonged group actions, or using the invariantised total derivative

$$D_s = D_{\widetilde{x}}\Big|_{g=\rho} = \frac{1}{\sqrt{1+u_x^2}} D_x, \text{ and } \mathrm{d}s = \sqrt{1+u_x^2}\,\mathrm{d}x.$$


Smooth
Cartan
Modern
$\hat{e}_2$
$\hat{e}_1$
$(x, u(x))$
θ
(x, u)
$(0,0)$
$\rho(x, u, u_x) \in SE(2)$
Discrete
(x_{n+1}, u_{n+1})
θ_n
(x_n, u_n)
$(0,0)$
$\rho_n(x_n, u_n, x_{n+1}, u_{n+1}) \in SE(2)$

Discrete case: OΔEs

The shift operator S is defined as

$$\mathrm{S} : n \mapsto n + 1.$$

An induced operator on functions is

$$\mathrm{S} : f(n) \mapsto f(n + 1).$$

Notations:

- local coordinates $z = (n, u_0 := u(n))$, where $n \in \mathbb{Z}$ and $u \in \mathbb{R}$
- independent variable n is invariant

Remark. A geometric setting for discrete equations can be found in Mansfield et al. (2019), and LP and Hydon (2022, 2023).

Example: scaling

Example. Consider the following scaling transformation

$$g : u_0 \mapsto \widetilde{u}_0 = e^{\varepsilon} u_0.$$

- Assuming $u_0 > 0$, the moving frame is determined by choosing a section, for instance, $\widetilde{u}_0 = 1$:

$$\rho_0 : \quad \varepsilon = \ln u_0.$$

- This gives the difference (fundamental) invariants

$$I_{0,j} := \widetilde{u}_j \Big|_{g=\rho_0} = \frac{u_j}{u_0}, \quad \forall j,$$

and syzygies (relation between invariants):

$$\mathrm{S} I_{0,j} = \frac{u_{j+1}}{u_1} = I_{1,j+1} \left(= \widetilde{u}_{j+1} \Big|_{g=\rho_1} \right) = \frac{I_{0,j+1}}{I_{0,1}}.$$

Consider the following invariant discrete variational problem

$$\mathscr{L}^{\triangle}[u] = \sum_{n=0}^{N} L^{\triangle}(u_0, u_1),$$

where

$$L^{\triangle}(u_0, u_1) = \frac{1}{2}\left(\frac{u_1}{u_0}\right)^2 = \frac{1}{2}\left(I_{0,1}\right)^2.$$

- The invariant difference Euler–Lagrange equation reads

$$\left(I_{0,1}\right)^2 - \frac{1}{\left(I_{0,-1}\right)^2} = 0.$$

- The corresponding (invariant) conservation law, obtained from Noether's theorem, is

$$\frac{1}{\left(I_{0,-1}\right)^2} = \text{const. noting that } \mathrm{S}\, I_{0,-1} = \frac{1}{I_{0,1}}.$$

Discrete Maurer–Cartan invariants, Mansfield et al. (2019)

- $\rho_k = \mathrm{S}^k \rho_0$ where ρ_k is the moving frame determined by a cross section at $n+k$
- Fundamental invariants: $I_{k,j} = \rho_k \circ u_j$
- (Right) discrete Maurer–Cartan group elements/invariants

$$K_k = \rho_{k+1} \cdot \rho_k^{-1}$$

  - Discrete syzygy

$$I_{k+1,j} = K_k \circ I_{k,j}$$

  - (Differential-difference) syzygy

$$\frac{\mathrm{d}}{\mathrm{d}\tau} K_k = (\mathrm{S} N_k)\, K_k - K_k N_k, \text{ where } N_k = \left(\frac{\mathrm{d}}{\mathrm{d}\tau} \rho_k\right) \rho_k^{-1}$$

Invariant Noether's conservation laws, LP (2013), Mansfield et al. (2019)

- The invariant EL equations $\mathcal{H}^*\mathbf{E}_{\boldsymbol{\kappa}}(L^{\triangle}(n, \boldsymbol{\kappa}, \boldsymbol{\kappa}_1, \ldots)) = 0$, where the difference Euler operator reads $\mathbf{E}_{\boldsymbol{\kappa}} = \mathrm{S}^j_{-1} \frac{\partial}{\partial \boldsymbol{\kappa}_j}$, are obtained by using syzygies of invariants $\boldsymbol{\kappa}$ (the $I_{0,1}$ in the example, related to K_0) and $\boldsymbol{\sigma}$ (related to N_0),

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{\kappa} = \mathcal{H}\boldsymbol{\sigma},$$

  where $\mathcal{H}$ is a (pre-)Hamiltonian operator.
- Invariant CLs are obtained via the invariant Noether's theorem based on adjoint representation of moving frames, i.e., the tangent map $Ad_\rho : \mathfrak{g} \to \mathfrak{g}$ induced by the conjugation $h \mapsto ghg^{-1}$, by identifying the variation parameter τ as each parameter of the symmetry group.

Invariant variational integrators

Variational integrators, Veselov (1988, 1991), Marsden & West (2001)

Numerical methods derived from discrete variational calculus

- ordinary difference ($u_0 = u(n)$, etc.):

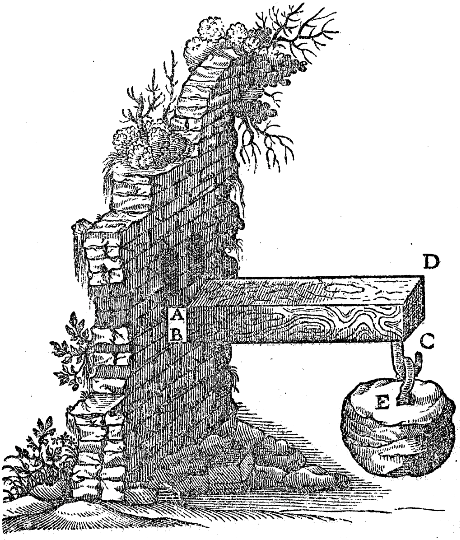
$$\mathscr{L}^\triangle[u] = \sum_n L^\triangle(u_0, u_1, \ldots)$$

- partial difference ($u_{0,0} = u(i,j)$, etc.):

$$\mathscr{L}^\triangle[u] = \sum_{i,j} L^\triangle(u_{0,0}, u_{1,0}, u_{0,1}, \ldots)$$

  - Preservation of symplectic structure: better long-term behavior

Invariant variational integrators: Elasticity, Galileo (1638)



(From the *Discorsi*, Leiden 1638.)

Invariant variational integrators: Euler's variational approach

Euler (1744): *ut, inter omnes curvas ejusdem longitudinis, quæ non solum per puncta A & B transeant, sed etiam in his punctis a rectis positione datis tangantur, definiatur ea in qua sit valor hujus expressionis* $\int \frac{\mathrm{d}s}{RR}$ *minimus.*

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- Elastica in the variational formulation

$$\int \kappa^2 \, \mathrm{d}s,$$

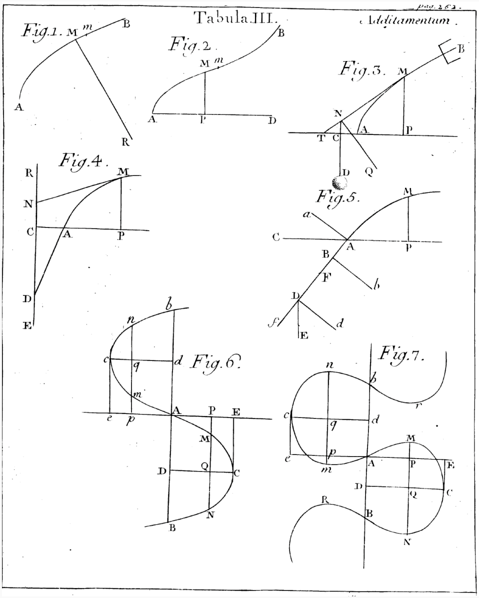
  where the curvature $\kappa = 1/R$ and acr length s of a curve $(x, u(x))$, i.e., $SE(2)$ invariants, are

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \quad \mathrm{d}s = \sqrt{1+u_x^2}\,\mathrm{d}x$$

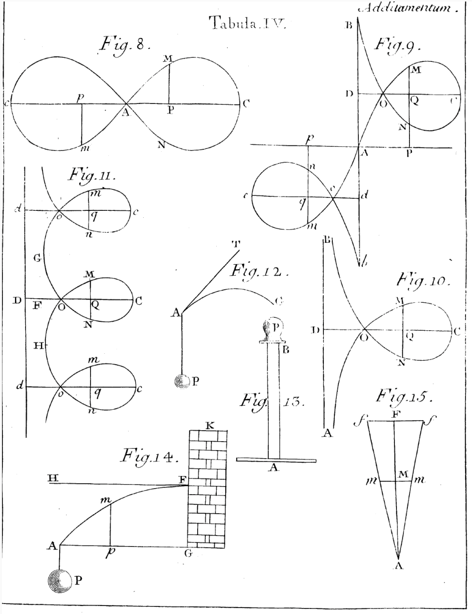
- The invariant Euler–Lagrange equation reads

$$\kappa_{ss} + \frac{1}{2}\kappa^3 = 0$$

Invariant variational integrators: Euler's elastica



Invariant variational integrators: Euler's elastica



Invariant variational integrator of Euler's elastica, Mansfield et al. (2019)

Discrete generating invariants determined by the discrete moving frame about $SE(2)$ acting on (x_j, u_j):

- discrete analogue of arc length $\mathrm{d}s$

$$\ell = |(x_1, u_1) - (x_0, u_0)|$$

- discrete curvature

$$\overline{\kappa} = \ell^{-1} \sin h_\theta$$

  where $h_\theta = \theta_1 - \theta_0$ with

$$\sin\theta_0 = -\frac{u_1 - u_0}{\ell}, \quad \cos\theta_0 = \frac{x_1 - x_0}{\ell}, \quad \dots$$

The discrete invariant variational problem corresponding to $\int \kappa^2 \,\mathrm{d}s$ is

$$\sum \ell^{-1} \sin^2 h_\theta$$

Invariant variational integrator of Euler's elastica, Mansfield et al. (2019)

- Discrete invariant EL equations (OΔEs):

$$\begin{aligned}
&\left(\mathrm{S}_{-1}\left(\ell^{-1}\sin h_\theta\right)(\mathrm{S}_{-1}^2-\mathrm{S}_{-1})\right)\mathbf{E}_{h_\theta}(L^\triangle)\\
&\qquad\qquad + \left(\mathrm{S}_{-1}(\cos h_\theta)\ \mathrm{S}_{-1} - \mathrm{id}\right)\mathbf{E}_\ell(L^\triangle) = 0,\\
&\left(\ell^{-1}(\mathrm{S}_{-1}-\mathrm{id}) - \mathrm{S}_{-1}\left(\ell^{-1}\cos h_\theta\right)(\mathrm{S}_{-1}^2-\mathrm{S}_{-1})\right)\mathbf{E}_{h_\theta}(L^\triangle)\\
&\qquad\qquad + \left(\mathrm{S}_{-1}(\sin h_\theta)\ \mathrm{S}_{-1}\right)\mathbf{E}_\ell(L^\triangle) = 0,
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{E}_{h_\theta}(L^\triangle) &= \frac{\partial L}{\partial h_\theta} = \ell^{-1}\sin(2h_\theta),\\
\mathbf{E}_\ell(L^\triangle) &= \frac{\partial L^\triangle}{\partial \ell} = -\ell^{-2}\sin^2 h_\theta.
\end{aligned}$$

Invariant variational integrator of Euler's elastica, Mansfield et al. (2019)

- Discrete invariant CLs corresponding to $SE(2)$:

$$
(\,V_1 \;\; V_2 \;\; V_3)\begin{pmatrix} R_{\theta_0} & JR_{\theta_0}(x_0, u_0)^T \\ 0 & 1 \end{pmatrix} = (c_1 \;\; c_2 \;\; c_3),
$$

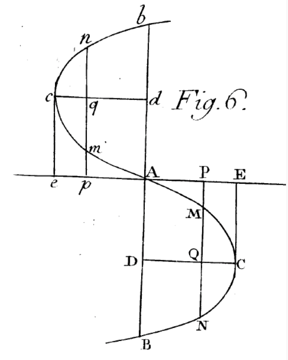
where

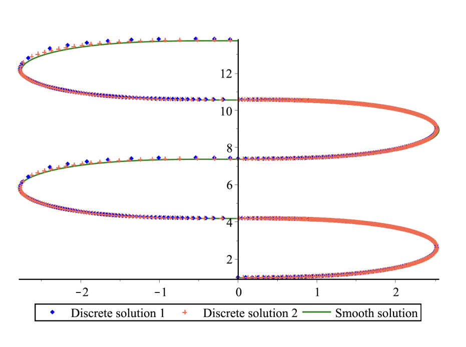
$$
R_{\theta_0} = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
$$

and

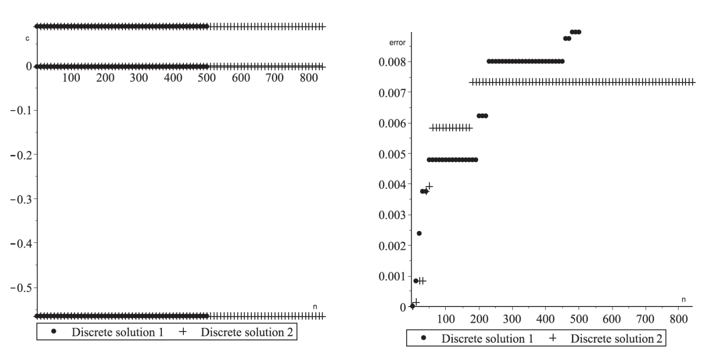
$$
\begin{aligned}
V_1 &= \mathrm{S}_{-1}(\cos h_\theta \mathbf{E}_\ell(L^\triangle)) + \left(\mathrm{S}_{-1}(\ell^{-1}\sin h_\theta)(\mathrm{S}_{-1}^2 - \mathrm{S}_{-1})\right)\mathbf{E}_{h_\theta}(L^\triangle),\\
V_2 &= \mathrm{S}_{-1}(\sin h_\theta \mathbf{E}_\ell(L^\triangle)) - \left(\mathrm{S}_{-1}(\ell^{-1}\cos h_\theta)(\mathrm{S}_{-1}^2 - \mathrm{S}_{-1})\right)\mathbf{E}_{h_\theta}(L^\triangle),\\
V_3 &= -\,\mathrm{S}_{-1}\left(\mathbf{E}_{h_\theta}(L^\triangle)\right).
\end{aligned}
$$

Invariant variational integrator of Euler's elastica





Right: numerical solutions obtained from the invariant Euler–Lagrange equations



Left: conserved quantities are constants. Right: error of solution

Summary

Summary

Moving frames, differential and difference invariants, invariant variational calculus, symmetry-preserving (and hence conservation law-preserving) variational integrators

- "In contrast" to Ge and Marsden (1988): a symplectic integrator, possibly after reduction so that only the conservation of energy remains, cannot exactly preserve the smooth energy without computing the exact solution.
- Practical applications, such as reassembly of jigsaw puzzles, signature of geometric objects and detection

Thank you for your attention!

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