Discrete maximal surfaces from s-embeddings

w/ N. Affolter, F. Dellinger, Ch. Müller, N. Smeenk

Denis Polly

TU Wien

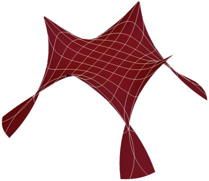
Discrete and geometric structures 2024
Monday 2$^{\text{nd}}$ September, 2024

Weierstrass-type representation [Kobayashi, 1983]

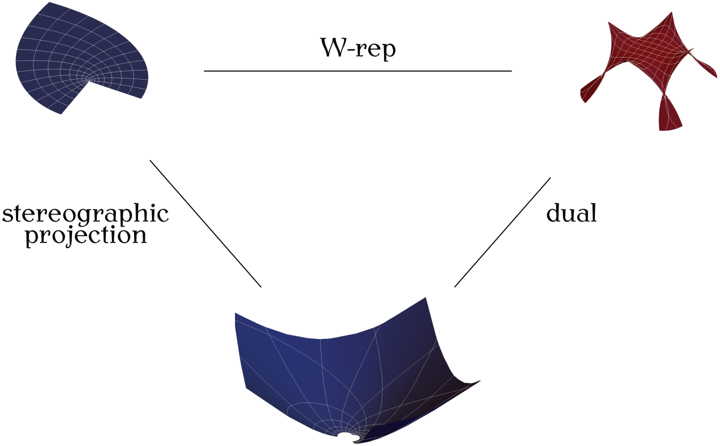
For any holomorphic function ψ, the surface

$$x(z) = \operatorname{Re} \int_{z_0}^{z} \begin{pmatrix} 1 + \psi^2(w) \\ i\left(1 - \psi^2(w)\right) \\ 2\psi(w) \end{pmatrix} \frac{dw}{\psi_z(w)},$$

has zero mean curvature and is thus a *maximal surface* in Minkowski space $\mathbb{R}^{2,1}$.



derivation



what we will see

Goal

special s-embeddings in the plane ⟷ discrete maximal surfaces

(statistical mechanics) (discrete differential geometry)

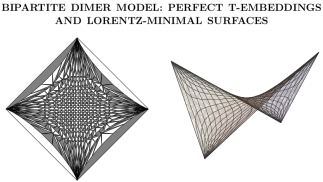


image from [Chelkak et al., 2023]

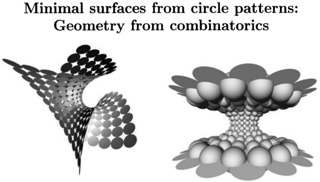
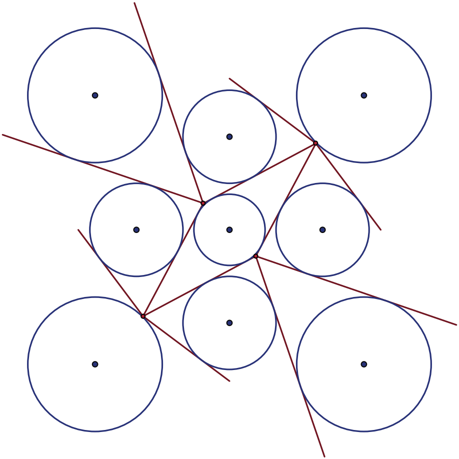


image from [Bobenko et al., 2006]

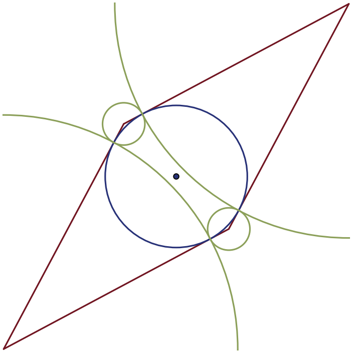
s-embeddings

Definition. An *s-embedding* is a map $s : \mathbb{Z}^2 \to \mathbb{R}^2$ such that the image of each face has an incircle.



s-embeddings

Pitot's theorem. A quadrilateral has an incircle if and only if the sums of opposite side lengths are equal.



Corollary. For an s-embedding s, the 1-form

$$\alpha = \begin{pmatrix} ds \\ \pm|ds| \end{pmatrix},$$

is closed.

Interlude: Minkowski space

We equip $\mathbb{R}^3$ with the metric

$$(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

and call $\big(\mathbb{R}^3, (\cdot, \cdot)\big) \cong \mathbb{R}^{2,1}$ *Minkowski space.*

- A vector v is called *spacelike, timelike* or *isotropic* if (v, v) is positive, negative or vanishes respectively.
- A plane is called *spacelike, timelike* or *isotropic* if the metric induced on it has signature $(++), (+-)$ or $(+0)$ respectively.

Interlude: Minkowski space

We equip $\mathbb{R}^3$ with the metric

$$(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

and call $\big(\mathbb{R}^3, (\cdot,\cdot)\big) \cong \mathbb{R}^{2,1}$ *Minkowski space.*

- A *sphere* in $\mathbb{R}^{2,1}$ is given by the equation

$$(x - c, x - c) = r.$$

  We call a sphere *spacelike, timelike* or *null* if $r < 0, r > 0$ or $r = 0$ respectively. We call c the *center* of the sphere. Spacelike spheres have spacelike tangent planes, timelike spheres have timelike tangent planes and null spheres have isotropic tangent planes.

Interlude: Minkowski space

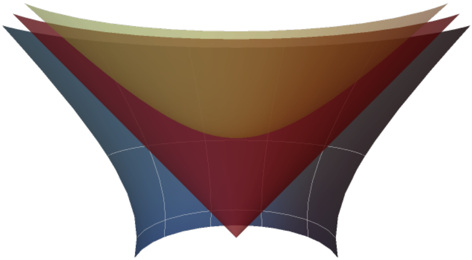
We equip $\mathbb{R}^3$ with the metric

$$(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

and call $\left(\mathbb{R}^3, (\cdot, \cdot)\right) \cong \mathbb{R}^{2,1}$ *Minkowski space.*

- A *sphere* in $\mathbb{R}^{2,1}$ is given by the equation

$$(x - c, x - c) = r.$$

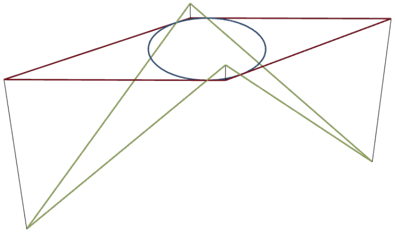


The Lorentz lift

Definition. For an s-embedding s, the map $s^{\mathcal{L}}$ satisfying

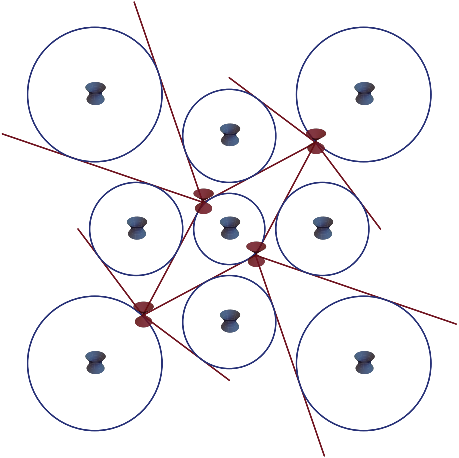
$$ds^{\mathcal{L}} = \begin{pmatrix} ds \\ \pm|ds| \end{pmatrix},$$

is called the *Lorentz lift* of s.



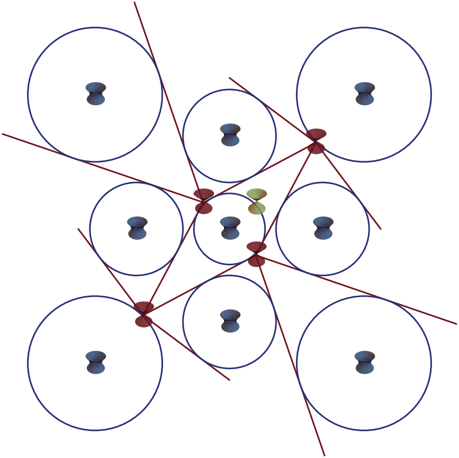
The Lorentz lift comes with null spheres at each vertex and timelike spheres at each face → *null congruence*.

isothermic nets



Schematic of a null congruence

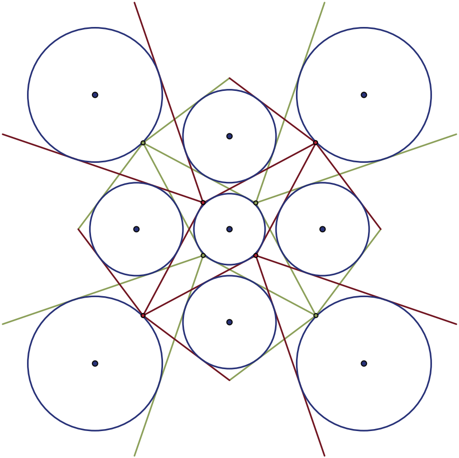
isothermic nets



Schematic of a null congruence

isothermic s-embeddings

Definition. An *isothermic s-embedding* is an s-embedding such that the other tangents also constitute an s-embedding (for the same incircles).



s-isothermic nets

Definition. A (spacelike) *s-isothermic net h* is a map

$$h_\bullet : \mathbb{Z}^2_\bullet \to \text{circles in } \mathbb{R}^{2,1},$$
$$h_\circ : \mathbb{Z}^2_\circ \to \text{timelike spheres in } \mathbb{R}^{2,1},$$
$$h_\Box : F(\mathbb{Z}^2) \to \mathbb{R}^{2,1},$$

such that around each face f

1. spheres touch in $h_\Box(f)$, and
2. spheres intersect circles orthogonally.

Theorem

s-isothermic nets are precisely the Lorentz lifts of isothermic s-embeddings.

discrete maximal surfaces

Definition. The (discrete) dual of an s-isothermic net h is obtained by integrating

$$\pm \frac{R(w)^{-1} + R(w')^{-1}}{R(w) + R(w')} dm_\circ(w, w').$$

Facts:

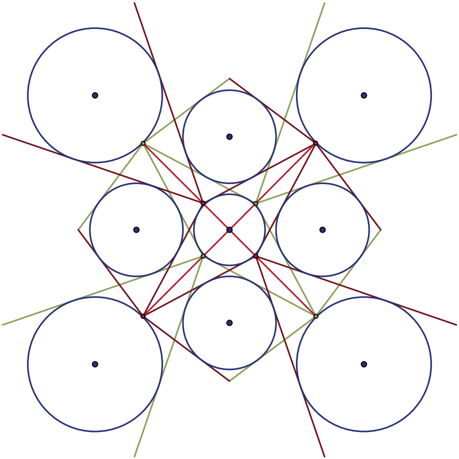
- the dual is also an s-isothermic net
- duals are edge-parallel

Definition. An s-isothermic net is called *maximal* if the circles of the Christoffel dual are contained in the space-like unit sphere.

Maximal s-isothermic nets are in bijection to orthogonal hyperbolic circle patterns (Minkowski version of [Bobenko et al., 2006]).

Koebe s-embeddings

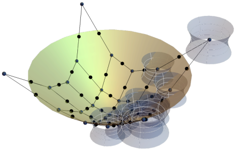
Definition. A *Koebe s-embedding* is an isothermic s-embedding such that the connecting lines of vertices intersect in one point.

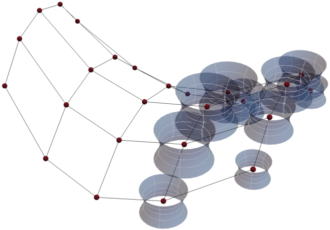


somewhat maximal s-embeddings

Theorem

The Lorentz lift $s^{\mathcal{L}}$ of a Koebe s-embedding lifts to an s-isothermic net such that the circles are contained in a spacelike sphere (generically). The duals of $s^{\mathcal{L}}$ are maximal s-isothermic nets. All maximal s-isothermic nets arise this way.





somewhat maximal s-embeddings

Theorem

The Lorentz lift $s^{\mathcal{L}}$ of a Koebe s-embedding lifts to an s-isothermic net such that the circles are contained in a spacelike sphere (generically). The duals of $s^{\mathcal{L}}$ are maximal s-isothermic nets. All maximal s-isothermic nets arise this way.

Further results and considerations:

- W-type representation for maximal s-isothermic nets
- associated families
- more connections to Ising model (X-variables)
- more sphere geometries
- more nicer pictures

The last slide

Thank you for your attention!

References

Bobenko, A. I., Hoffmann, T., and Springborn, B. A. (2006).
Minimal Surfaces from Circle Patterns: Geometry from Combinatorics.
Annals of Mathematics, 164(1):231–264.

Chelkak, D., Laslier, B., and Russkikh, M. (2023).
Dimer model and holomorphic functions on t-embeddings of planar graphs.
Proceedings of the London Mathematical Society, 126(5):1656–1739.

Kobayashi, O. (1983).
Maximal Surfaces in the 3-Dimensional Minkowski space L^3.
Tokyo Journal of Mathematics, 06(2):297–309.