Solving Euclidean problems by isotropic initialization

Caigui Jiang[1], Olimjoni Pirahmad[2], Helmut Pottmann[2], Mikhail Skopenkov[2], Bolun Wang[2], Khusrav Yorov[2]

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Discrete Geometric Structures 2024
Vienna Technical University
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Overview

1. Euclidean problems
2. Isotropic geometry
3. Flexible nets
4. Asymptotic nets with a constant angle
5. Asymptotic-geodesic webs

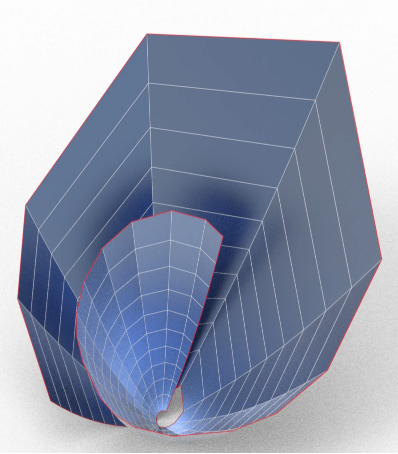
Euclidean problems

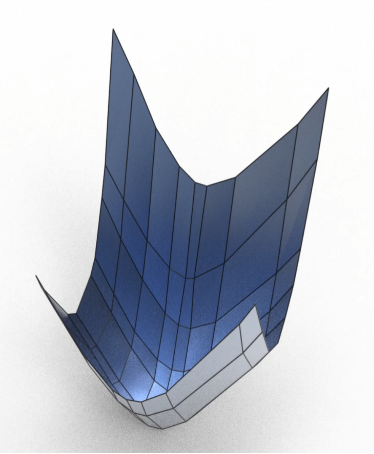
Transformable design

Chuck Hoberman

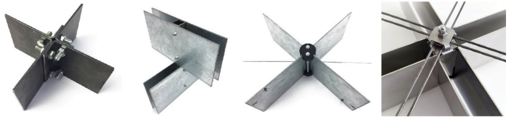
HAPPHO

New quad-mesh mechanisms aka flexible nets

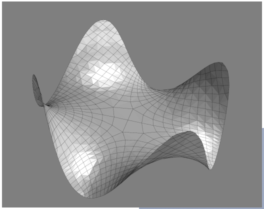


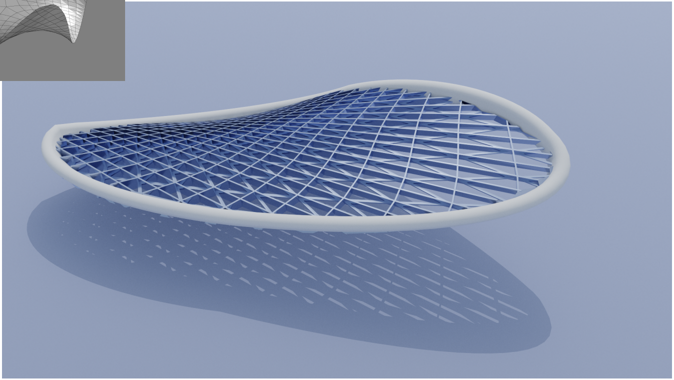


Freeform architecture



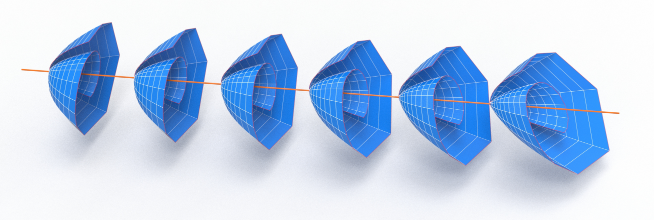
New asymptotic nets with a constant angle and webs





A general approach to problems in Euclidean geometry

1. First solve the problem in *isotropic geometry* (structure-preserving simplification of Euclidean geometry)



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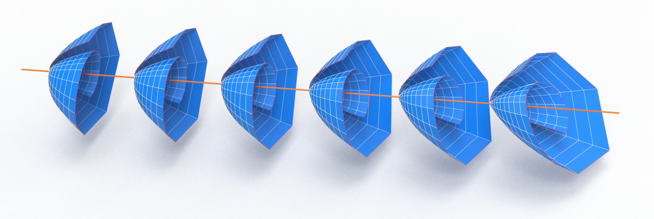
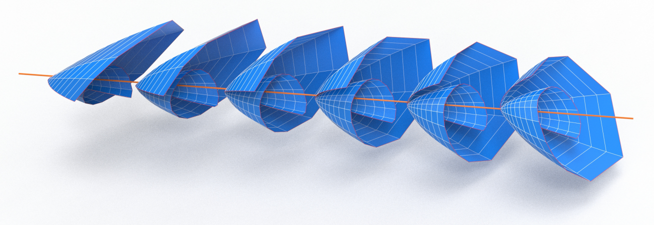
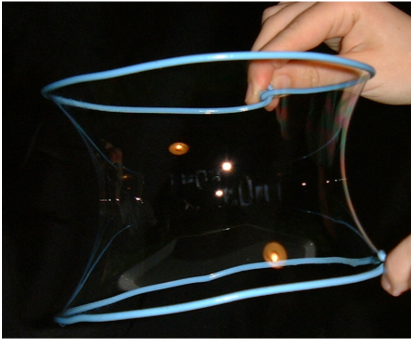


2. Optimize to Euclidean solutions (and get insight)



Plateau's Problem (1760). Prove the existence of a minimal surface with a given boundary.



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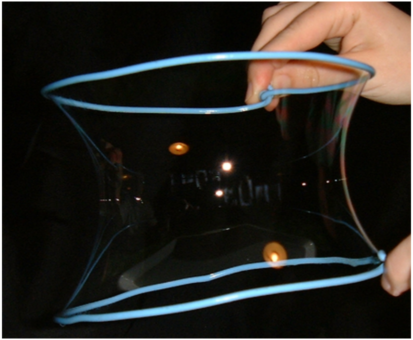
Plateau's Problem (1760). Prove the existence of a minimal surface with a given boundary.
Partial Solution (Müntz, 1911): by deformation of a graph of a harmonic function (=*isotropic minimal surface*)



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Isotropic geometry

Isotropic space I^3, distances, and angles

- The *isotropic distance* between (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

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- The *isotropic congruence transformations* are special affine transformations preserving the isotropic distance:
$$x' = a + x\cos\phi \pm y\sin\phi,$$
$$y' = b \mp x\sin\phi + y\cos\phi,$$
$$z' = c + c_1 x + c_2 y + z.$$

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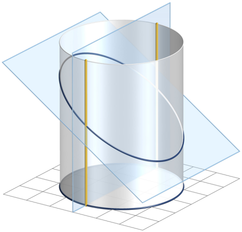
- The *isotropic angle* between non-isotropic planes is the difference of their slopes (= tangent of inclination angle)

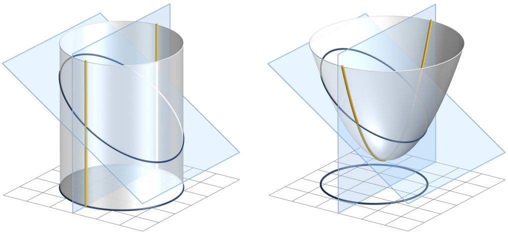
- An *isotropic sphere* (*of cylindrical type*) is the set of points at a constant isotropic distance from a given point



Isotropic spheres

- An *isotropic sphere* (*of cylindrical type*) is the set of points at a constant isotropic distance from a given point
- An *isotropic sphere of parabolic type* (surface of constant normal curvature) is

$$2z = a(x^2 + y^2) + bx + cy + d, \qquad a \neq 0$$



Metric duality

- The *metric duality* is the polarity with respect to *isotropic unit sphere* $2z = x^2 + y^2$:

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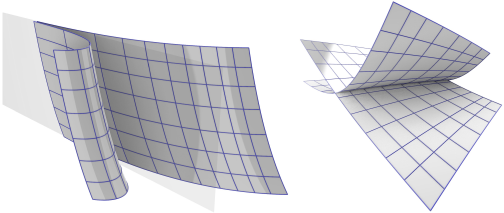
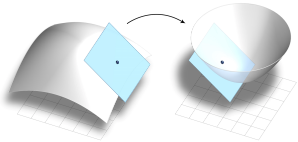
- Parallel points $\mapsto$ parallel planes
- Surface $\mapsto$ the set of points dual to tangent planes



Isotropic Gaussian curvature

- The *isotropic Gauss map* takes a point of the surface to the point of the isotropic unit sphere such that the tangent planes at the two points are parallel



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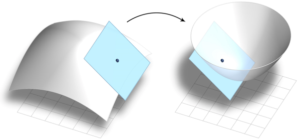
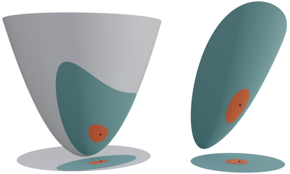


- The *total isotropic Gaussian curvature*
  $=$ the *isotropic area* of the Gaussian image (if it is 1–1)
  $=$ the oriented area of the top view of the Gaussian image
  $=$ the oriented area of the top view of the metric dual



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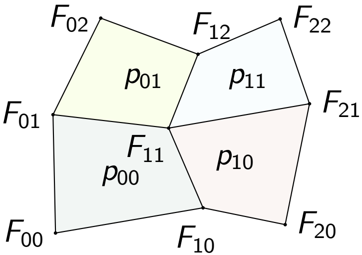
- A nondegenerate definition of an *isotropic isometry* was invented only recently by **Müller–Pottmann'24**
- **Difficulty:** preservation of distances is insufficient!
- **Insight: Gauss' Theorem Egregium.** Euclidean isometries of a surface preserve Gaussian curvature.
- An *isotropic isometry* is a map preserving both isotropic distances and the isotropic Gaussian curvature.

Flexible nets

14:15 Olimjoni Pirahmad
14:30 Alisher Aikyn

- An *$m \times n$ net* is a collection of $(m+1)(n+1)$ points $F_{ij} \in I^3$ indexed by two integers $0 \leq i \leq m$ and $0 \leq j \leq n$ such that $F_{ij}, F_{i+1,j}, F_{i+1,j+1}, F_{i,j+1}$ are consecutive vertices of a convex quadrilateral p_{ij} for all $0 \leq i < m$, $0 \leq j < n$.



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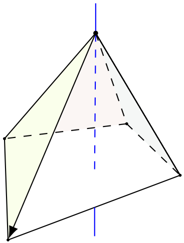
- A convex polyhedral angle in I^3 is *admissible* if the isotropic line through its vertex intersects its interior.



Dual-convex $m \times n$ nets

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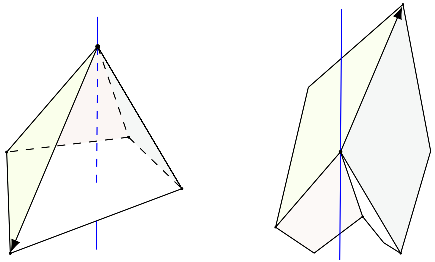
- An $m \times n$ net is *dual-convex* if the planes of the four consecutive faces around each non-boundary vertex are the planes of the four consecutive flat angles of some admissible 4-hedral angle



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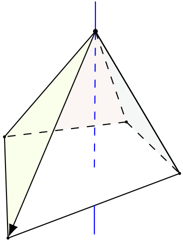
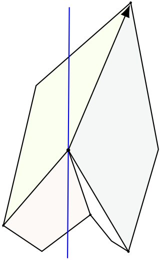
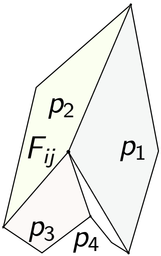
- An $m \times n$ net is *dual-convex* if the planes of the four consecutive faces around each non-boundary vertex are the planes of the four consecutive flat angles of some admissible 4-hedral angle $\Leftrightarrow$ the dual p^*_{kl} has convex faces

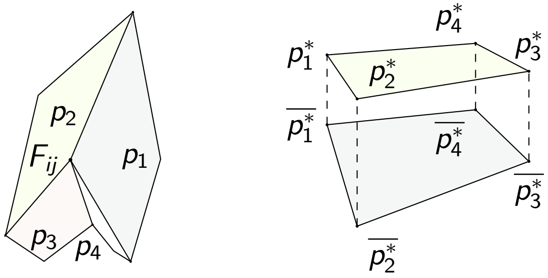




Let F_{ij} be a non-boundary vertex and p_1, p_2, p_3, p_4 be the consecutive faces around it.



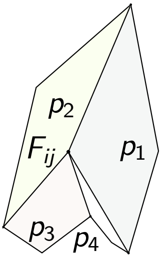
Let F_{ij} be a non-boundary vertex and p_1, p_2, p_3, p_4 be the consecutive faces around it. Let $\overline{p_j^*}$ be the top view of the metric dual of p_j.

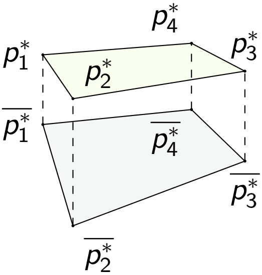


Discrete isotropic Gaussian curvature

Let F_{ij} be a non-boundary vertex and p_1, p_2, p_3, p_4 be the consecutive faces around it. Let $\overline{p_j^*}$ be the top view of the metric dual of p_j. The *total isotropic Gaussian curvature at the vertex* F_{ij}, or the *curvature at* F_{ij}, is the oriented area

$$\Omega(F_{ij}) := \mathrm{Area}(\overline{p_1^*}\,\overline{p_2^*}\,\overline{p_3^*}\,\overline{p_4^*}) = \frac{1}{2}\sum_{j=1}^{4}\det(\overline{p_j^*}, \overline{p_{j+1}^*}).$$





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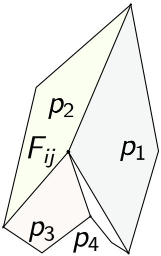
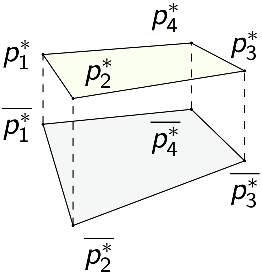
(Different from **Bobenko–Pottmann–Wallner'10**)





Flexible nets in Euclidean geometry

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Problem. Construct examples of flexible $m \times n$ nets.

Examples:

- **Voss nets'1888**
- **Graf–Sauer's T-nets'1931**
- classification of all flexible 3×3 nets **(Izmestiev'2017)**
- combination to larger nets **(He–Guest'2020)**
- nonplanar faces **(Nawratil'2023, Aikyn et al' 2024)**

A dual-convex $m \times n$ net F_{ij} is *flexible in I^3* if contained in a continuous family of $m \times n$ non-isotropically-congruent nets $F_{ij}(t)$, where $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, satisfying

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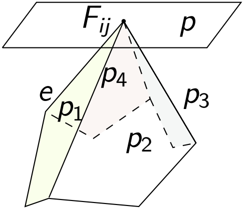
Problem. Find all flexible $m \times n$ nets in I^3.

Flexible nets in isotropic geometry: an auxiliary notion

- F_{ij} = a non-boundary vertex of a dual-convex $m \times n$ net
- e = an edge emanating from the vertex
- p_1, p_2, p_3, p_4 = the consecutive faces around the vertex such that $p_1 \cap p_4 = e$
- p = the plane spanned by the lines $p_1 \cap p_3$ and $p_2 \cap p_4$
- $\angle(p_1, p)$ = the isotropic angle between p_1 and p
- the *opposite ratio of F_{ij} with respect to e* is

$$\frac{\angle(p_3, p)}{\angle(p_1, p)} \cdot \frac{\angle(p_4, p)}{\angle(p_2, p)}.$$



Flexible nets in isotropic geometry: classification

Theorem (Pirahmad–Pottmann–S.'24+)

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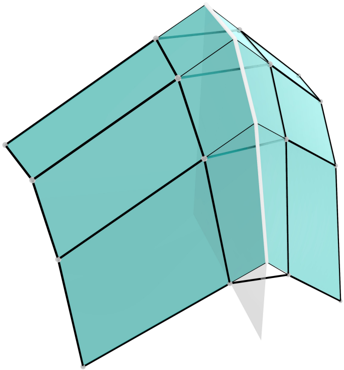

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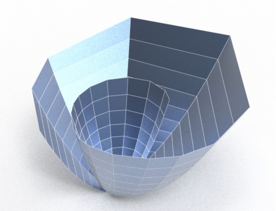

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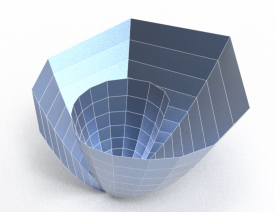
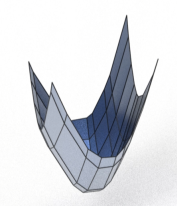
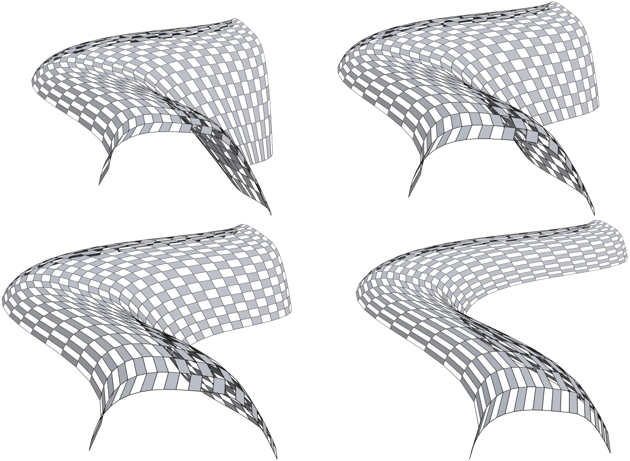
(ii) *any two non-boundary vertices joined by an edge have equal opposite ratios with respect to it.*





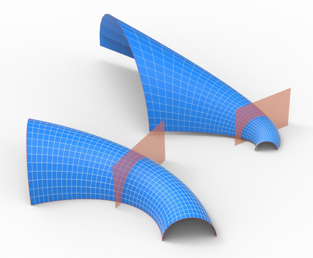
Flexible nets in isotropic geometry: classification

Proof: by the metric duality, the theorem reduces to the classification of *area-preserving Combescure transformations*



Flexible nets in isotropic geometry: example

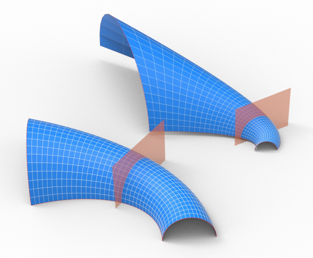
A *generalized T-net* is a dual-convex $m \times n$ net with planar parameter lines, where one family of parameter lines lie in isotropic planes.



Flexible nets in isotropic geometry: example

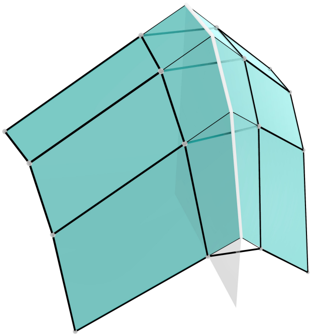
A *generalized T-net* is a dual-convex $m \times n$ net with planar parameter lines, where one family of parameter lines lie in isotropic planes.

- a particular case of $m \times n$ nets with planar parameter lines **(Bobenko–Rörig'19, Fairley'23, Wang et al.'22)**
- a particular case of metric duals of *double cone-nets* **(Kilian–Müller–Tervooren'23)**



A *generalized T-net* is a dual-convex $m \times n$ net with planar parameter lines, where one family of parameter lines lie in isotropic planes.

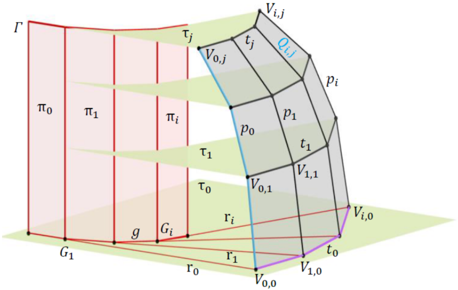
- a particular case of class (i) $\Rightarrow$ flexible in I^3
- generate the whole class (i) by "truncation of edges"



Flexible nets in isotropic geometry: example

A *generalized T-net* is a dual-convex $m \times n$ net with planar parameter lines, where one family of parameter lines lie in isotropic planes.

- generalize flexible *T-nets* by **Graf–Sauer'31** (parameter lines lie in two orthogonal families of planes)



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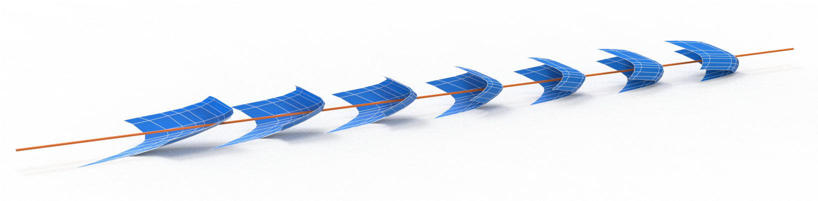
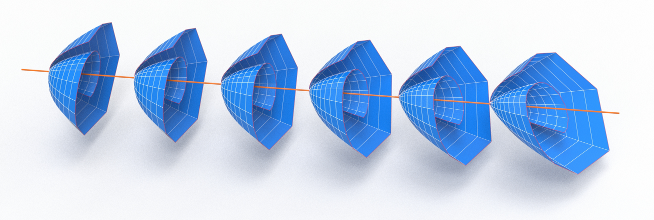
Flexible nets in isotropic geometry: example (ii)



From isotropic to Euclidean flexible nets

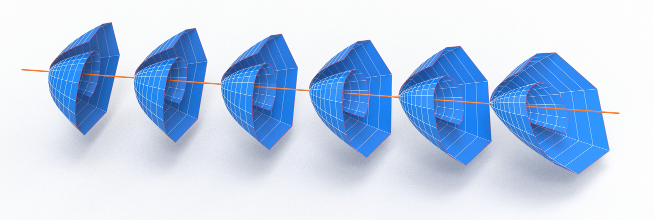
Use a flexible net in I^3 to initialize numerical optimization towards a Euclidean flexible net:



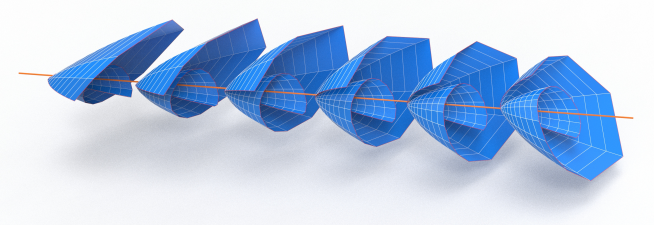
Isotropic

From isotropic to Euclidean flexible nets

Use a flexible net in I^3 to initialize numerical optimization towards a Euclidean flexible net:



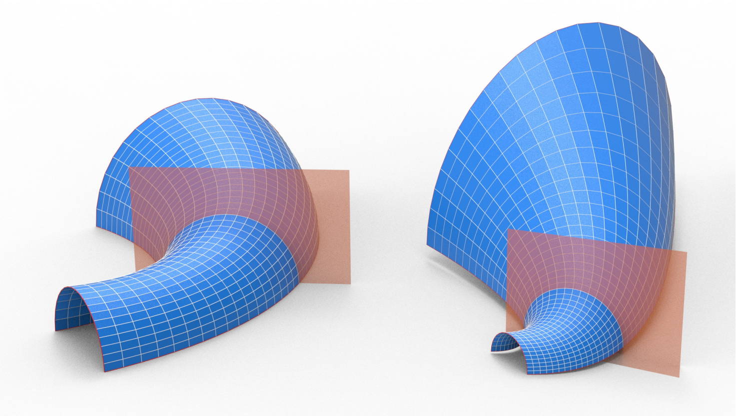
Isotropic



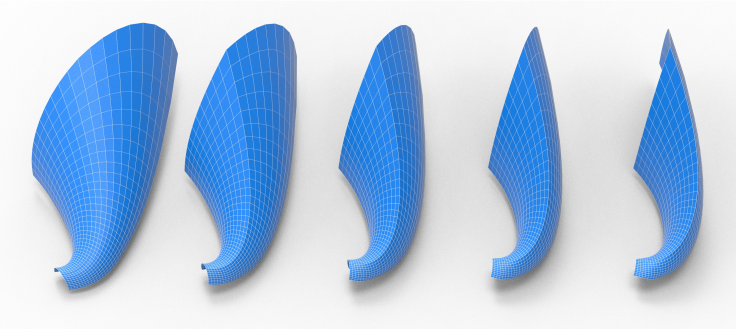
Euclidean

From isotropic to Euclidean flexible nets

A projective transformation preserving the isotropic direction:



Flexion of the transformed net:



Asymptotic nets
with a constant angle

Examples of CRPC surfaces

Problem. Construct examples of surfaces with a constant angle between asymptotic curves

Examples of CRPC surfaces

Problem. Construct examples of surfaces
with a constant angle between asymptotic curves
$\Leftrightarrow$ a constant negative ratio of principal curvatures

Problem. Construct examples of *CRPC surfaces*

$\Leftrightarrow$ a constant angle between asymptotic curves

$\Leftrightarrow$ a constant negative ratio of principal curvatures

Problem. Construct examples of *CRPC surfaces*
$\Leftrightarrow$ a constant angle between asymptotic curves
$\Leftrightarrow$ a constant negative ratio of principal curvatures

Examples:

- *Minimal* surfaces: the angle is right, the ratio is -1
- *Rotational* CRPC surfaces **(Hopf'51)**:

$$z = \int \frac{dr}{\sqrt{r^{-2a}-1}} = \frac{r^{1+a}}{1+a}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}+\frac{1}{2a}; \frac{3}{2}+\frac{1}{2a}; r^{2a}\right)$$

- *Helical* CRPC surfaces **(Liu–Pirahmad–Wang–Michels–Pottmann'23)**

. . . and no other ones are known!

Examples of isotropic CRPC surfaces

Problem. Construct examples of *isotropic CRPC surfaces*
$\Leftrightarrow$ a constant *isotropic* angle between asymptotic curves
$\Leftrightarrow$ a constant negative ratio of *isotropic* principal curvatures

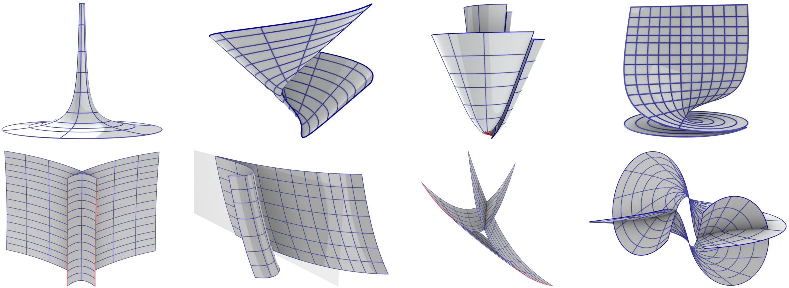
Examples:

- *Isotropic minimal* surfaces: the angle is right, ratio is -1
- *Isotropic rotational* CRPC surfaces:

$$z = \int \frac{dr}{\sqrt{r^{-2a} \cancel{-1}}} = \frac{r^{1+a}}{1+a} \cancel{{}_2F_1\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2a}; \frac{3}{2} + \frac{1}{2a}; r^{2a}\right)}$$

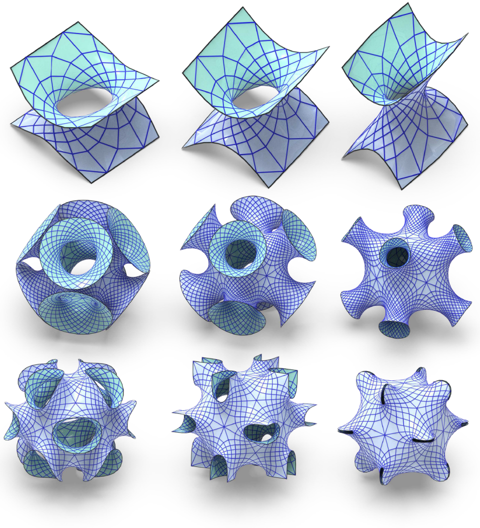
- *Isotropic Helical, ruled, channel, translational (with planar parameter lines), Voss* CRPC surfaces **(Yorov–S.–Pottmann'23)**

Problem. Construct examples of *isotropic CRPC surfaces*
$\Leftrightarrow$ a constant *isotropic* angle between asymptotic curves
$\Leftrightarrow$ a constant negative ratio of *isotropic* principal curvatures



- *Isotropic Helical, ruled, channel, translational (with planar parameter lines), Voss* CRPC surfaces **(Yorov–S.–Pottmann'23)**

Changing the angle between asymptotic curves

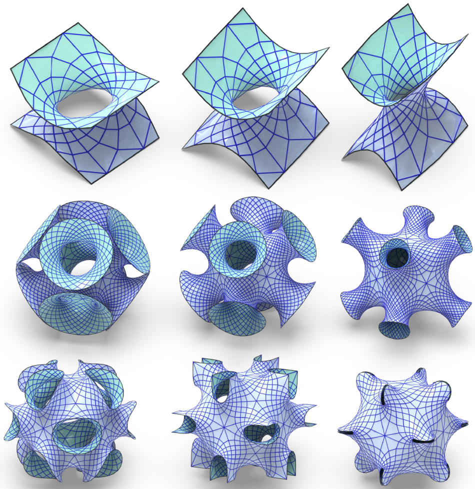


SEPTEMBER

Discrete surfaces with a constant ratio of principal curvatures. These surfaces are obtained by changing the right angle between asymptotic directions in discrete minimal surfaces (middle column) to another constant angle. Surprisingly, this is possible while keeping the combinatorics of the asymptotic net.

Contributors. *Hui Wang, Helmut Pottmann*

Changing the angle between asymptotic curves



These surfaces are obtained by changing the right angle between asymptotic directions in discrete minimal surfaces (middle column) to another constant angle.

Changing the isotropic angle between asymptotic curves

Theorem (Yorov–S.'24+)

Each isotropic minimal surface Φ^0 *with an analytic boundary and without flat points (even on the boundary), which is a graph of a C^1 function in a Jordan domain,* **is contained in a** *unique real analytic* **family of surfaces Φ^s with** *the same boundary and* **the ratio of isotropic principle curvatures equal $s-1$.**

Remark. If Φ^0 has a non-degenerate flat point (where $K = K_x = K_y = 0$ but $K_{xx}K_{yy} - K_{xy}^2 \neq 0$), then Φ^0 is *not* contained in a C^4 family of surfaces Φ^s as in the theorem.

Proof: the Schauder estimates for the Poisson equation

$\Rightarrow$ insight for the Euclidean case!

Conjecture (Yorov–S., in progress)

Each *Euclidean* **minimal surface** Φ^0 *with an analytic boundary and without flat points (even on the boundary), which is a graph of a C^1 function in a Jordan domain,* **is contained in a** *unique real analytic* **family of surfaces** Φ^s **with** *the same boundary and* **the ratio of** *Euclidean* **principle curvatures equal** $s-1$**.**

Remark. If Φ^0 has a non-degenerate flat point (where $K = K_x = K_y = 0$ but $K_{xx}K_{yy} - K_{xy}^2 \neq 0$), then Φ^0 is *not* contained in a C^4 family of surfaces Φ^s as in the theorem.

Proof: *the same with more technicalities*

The isotropic case gives even more!

Changing angle: explicit 2nd order approximation

If $f^0\colon D \to \mathbb{R}$ is harmonic in a domain D, then the graph of

$$f(w) := 2\mathrm{Re}\, g(w) + |h(w)|^2\varepsilon + \mathrm{Re}\, h(w)^2 \log|g''(w)|\varepsilon^2/2$$

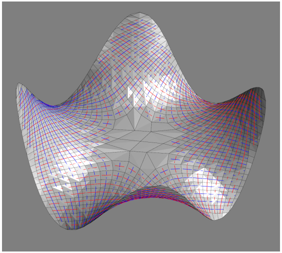
is a "2nd order approximation" to an isotropic CRPC surface.

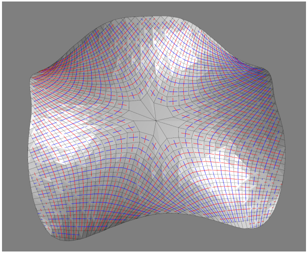
Here:

- $w = x + iy$ is the complex coordinate in the xy-plane
- $g(w)$ is an analytic function such that $f^0(w) = 2\mathrm{Re}\, g(w)$ (computed from the boundary values of f^0 using a conformal mapping of D to the unit circle and the Schwarz integral formula)
- $h(w)$ is an antiderivative of $\sqrt{g''(w)}$ (computed approximately by truncating the power series)
- $f(w)$ can then be optimized to a *Euclidean* CRPC surface with a given boundary

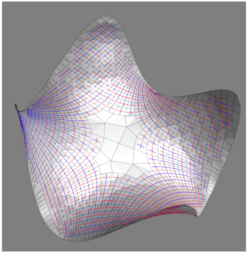
Changing angle: numerical results

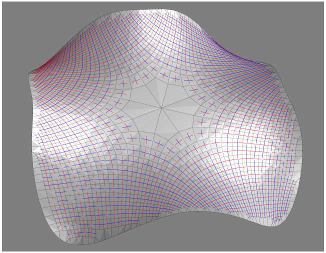
a 2nd order approximation to an isotropic CPRC surface





Euclidean CPRC surface



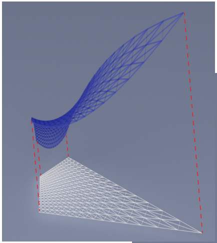


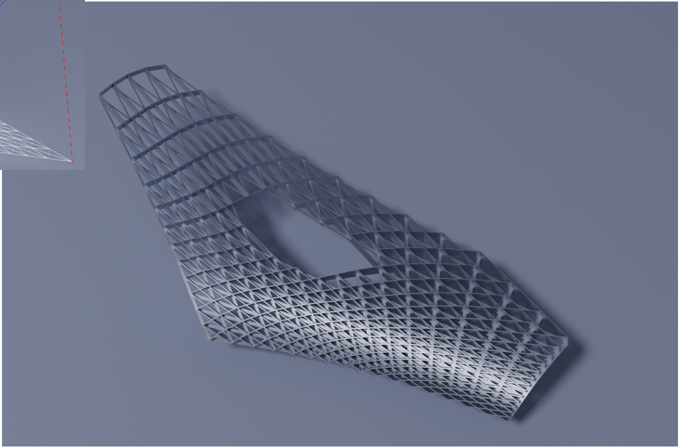
Asymptotic-geodesic webs

Today 14:00 Khusrav Yorov

Asymptotic and geodesic webs: numerical results

A Euclidean AGAG web by optimization of an isotropic one





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THANKS!