Polyhedral surfaces in homogeneous 3-manifolds

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joint work with François Fillastre (U Montpellier)

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Convex bodies in model 3-spaces

Theorem (Cauchy, Alexandrov 1942)

Every Euclidean cone-metric on the topological 2-sphere S^2 *with singular curvatures* > 0 *can be realized on the boundary of a convex polyhedron* $C \subset \mathbb{E}^3$, *unique up to ambient isometry.*

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- Singular curvatures > 0 means that the total angles at the vertices of gluing are $\leq 2\pi$.
- The edges of the gluing may have nothing to do with the edges of the polyhedral realization.

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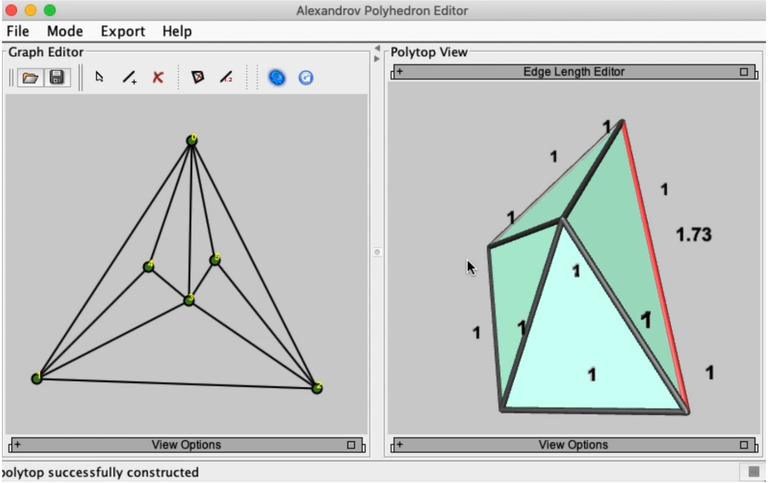

- Alexandrov (1942): non-constructive proof;
- Volkov (1955); Bobenko–Izmestiev (2008): constructive proofs;
- Sechelmann: implementation of the Bobenko–Izmestiev proof.

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Theorem (Weyl, Lewy, Nirenberg, Alexandrov, Pogorelov, Cohn-Vossen, Herglotz)

Every smooth Riemannian metric on S^2 of curvature > 0 can be realized on the boundary of a smooth convex body $C \subset \mathbb{E}^3$, unique up to ambient isometry.

The same for convex bodies in $\mathbb{S}^3$, $\mathbb{H}^3$.

Hyperbolic 3-manifolds

- A “generic” closed 3-manifold can be endowed with a hyperbolic metric.
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Theorem (Labourie 1992, Schlenker 2006)

Let M be admissible. Every smooth Riemannian metric on ∂M of curvature > -1 can be realized by a hyperbolic metric on M with smooth strictly convex boundary, the realization is unique up to isotopy.

Hyperbolic 3-manifolds

Theorem (P. 2022)

Let M be admissible. Every hyperbolic cone-metric metric on ∂M with singular curvatures > 0 can be realized by a hyperbolic metric on M with "weakly polyhedral" convex boundary. If the realization is controllably polyhedral, then it is unique up to isotopy.

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- Polyhedral boundary = locally modeled on polyhedra in $\mathbb{H}^3$.
- "Weakly polyhedral" = partially "crumpled".
- A generic cone-metric has a controllably polyhedral realization.

Convex cocompact hyperbolic 3-manifolds

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Corollary

For every pair d_1, d_2 of metrics on S with curvature > -1 there exists a unique convex cocompact hyperbolic metric on $\hat{M} = S \times \mathbb{R}$ and a unique pair of convex embeddings of d_1, d_2 in the respective ends of $\hat{M}$.

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GHMC (2+1)-spacetimes

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- Case of curvature zero: parametrized by $T^*\mathcal{T}(S)$ and by future-/past-completeness.
- $h \in \mathcal{T}(S)$ encodes the asymptotic geometry "at infinity".

GHMC (2+1)-spacetimes

Theorem I (Fillastre-P. 2023)

Let d be a Euclidean cone-metric on a surface S with singular curvatures < 0, and h be a hyperbolic metric on S. Then there exists a unique future-complete GHMC (2+1)-spacetime of curvature 0 with asymptotic geometry given by h, containing a unique convex polyhedral Cauchy surface with the induced metric d.

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Theorem II (Fillastre-P. 2023)

Let d_1, d_2 be two Euclidean cone-metrics on S with singular curvatures < 0. Then there exists a unique pair of GHMC (2+1)-spacetime of curvature 0 with the same holonomy, future- and past-complete, containing unique convex polyhedral Cauchy surfaces with the induced metrics d_1 and d_2 respectively.

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Smooth analogues by Smith (2020).

The end

Thank you!

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